Resurgence, Trans-series and Non-perturbative Physics

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UK Theory Meeting, December 16, 2014

GD & M. Ünsal, 1210.2423, 1210.3646, 1306.4405, 1401.5202

GD, lectures at CERN 2014 Winter School

also with: G. Başar, A. Cherman, D. Dorigoni, R. Dabrowski: 1306.0921, 1308.0127, 1308.1108, 1405.0302, 1412.xxxx

Physical Motivation

- infrared renormalon puzzle in asymptotically free QFT
  (i) IR renormalons $\Rightarrow$ perturbation theory ill-defined
  (ii) $\mathcal{I}\bar{\mathcal{I}}$ interactions $\Rightarrow$ instanton-gas ill-defined
- non-perturbative physics without instantons: physical meaning of non-BPS saddle configurations

Bigger Picture

- non-perturbative definition of nontrivial QFT in continuum
- analytic continuation of path integrals
- dynamical and non-equilibrium physics from path integrals
- "exact" asymptotics in QFT and string theory

Mathematical Motivation

Resurgence: 'new' idea in mathematics (Écalle, 1980; Stokes, 1850)

- explore implications for physics

resurgence = unification of perturbation theory and non-perturbative physics

- perturbation theory generally $\Rightarrow$ divergent series
- series expansion $\longrightarrow$ *trans-series* expansion
- trans-series 'well-defined under analytic continuation'
- perturbative and non-perturbative physics entwined
- applications: ODEs, PDEs, fluids, QM, Matrix Models, QFT, String Theory, ...
- philosophical shift:

  view semiclassical expansions as potentially exact

Resurgent Trans-Series

- trans-series expansion in QM and QFT applications:

$$f(g^2) = \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{k-1} \underbrace{c_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}} \underbrace{\left(\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k}-\text{instantons}} \underbrace{\left(\ln\left[\pm\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zero-modes}}$$

- J. Écalle (1980): set of functions closed under:

(Borel transform) + (analytic continuation) + (Laplace transform)

- *trans-monomial elements*: g^2, $e^{-\frac{1}{g^2}}$, $\ln(g^2)$, are familiar
- "multi-instanton calculus" in QFT
- new: trans-series coefficients $c_{k,l,p}$ highly correlated
- new: analytic continuation under control
- new: exponentially improved asymptotics

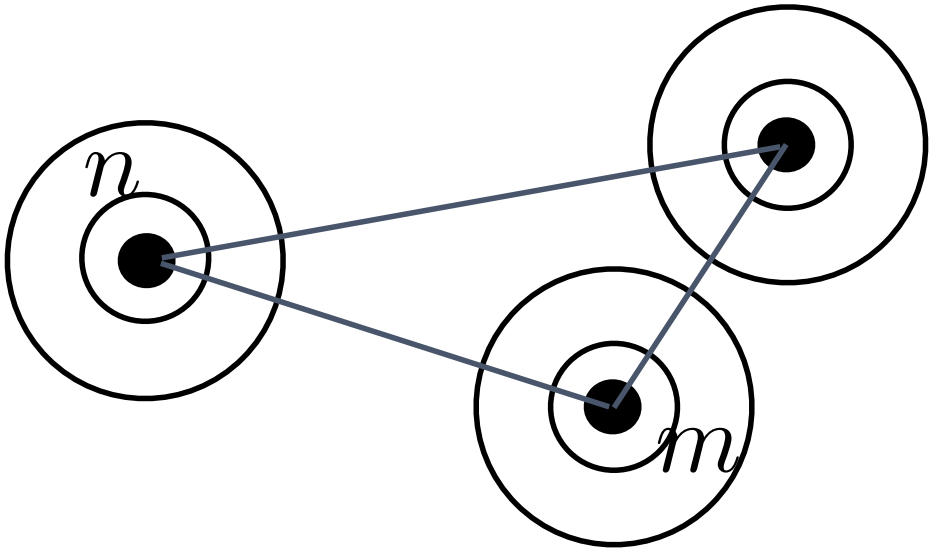
No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms

G. H. Hardy, Divergent Series, 1949

Resurgence

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



recap: rough basics of Borel summation

(i) divergent, alternating:

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, g^{2n} = \int_0^\infty dt\, e^{-t}\, \frac{1}{1+g^2\, t}$$

(ii) divergent, non-alternating:

$$\sum_{n=0}^{\infty} n!\, g^{2n} = \int_0^\infty dt\, e^{-t}\, \frac{1}{1-g^2\, t}$$

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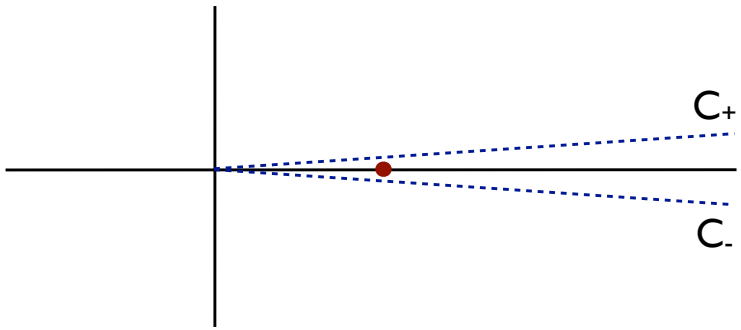
(ii) divergent, non-alternating:

$$\sum_{n=0}^{\infty} n!\, g^{2n} = \int_0^\infty dt\, e^{-t}\, \frac{1}{1-g^2\, t}$$

$\Rightarrow$ ambiguous imaginary non-pert. term: $\pm\frac{i\,\pi}{g^2}\, e^{-1/g^2}$

avoid singularities on $\mathbb{R}^+$: *lateral* Borel sums:



$\theta = 0^\pm \longrightarrow$ non-perturbative ambiguity: $\pm\mathrm{Im}[\mathcal{B}f(g^2)]$

challenge: use physical input to resolve ambiguity

Borel summation in practice (physical applications)

direct quantitative correspondence between:

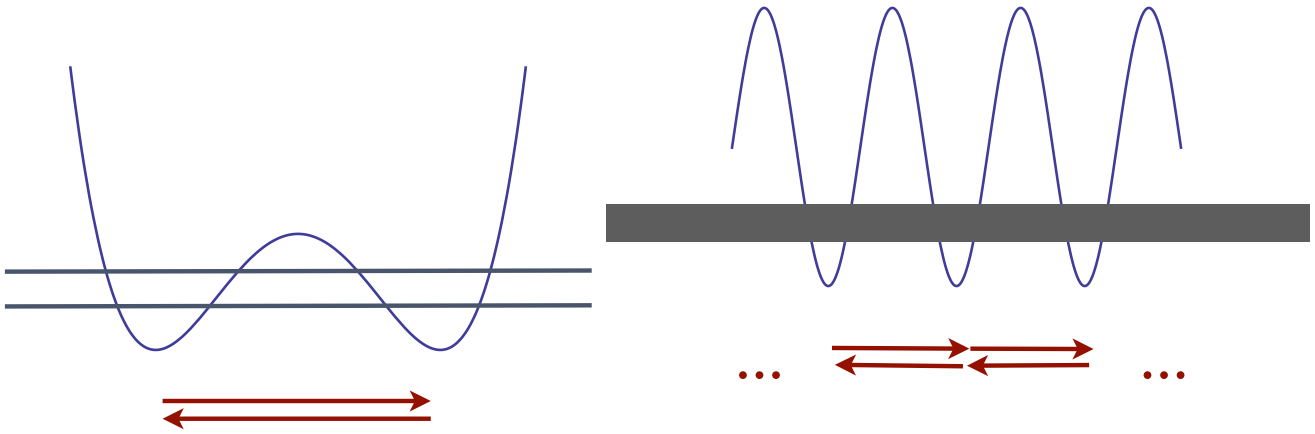
rate of growth $\leftrightarrow$ Borel poles $\leftrightarrow$ non-perturbative exponent

non-alternating factorial growth: $c_n \sim b^n \, n!$

positive Borel singularity: $t_c = \dfrac{1}{b\, g^2}$

non-perturbative exponent: $\pm i \, \dfrac{\pi}{b\, g^2} \, \exp\left[- \left(\dfrac{1}{b\, g^2} \right) \right]$

Analogue of IR Renormalon Problem in QM



- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

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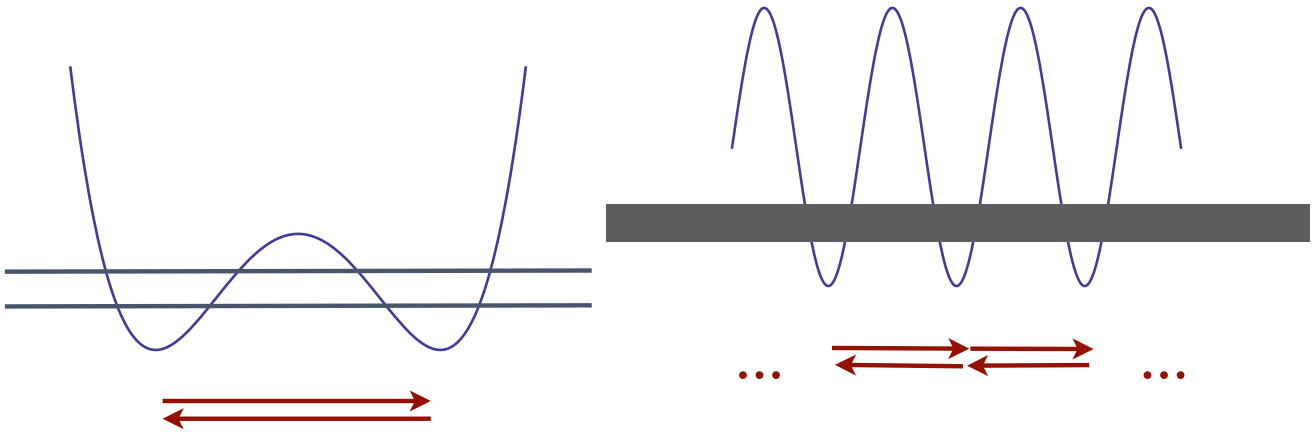


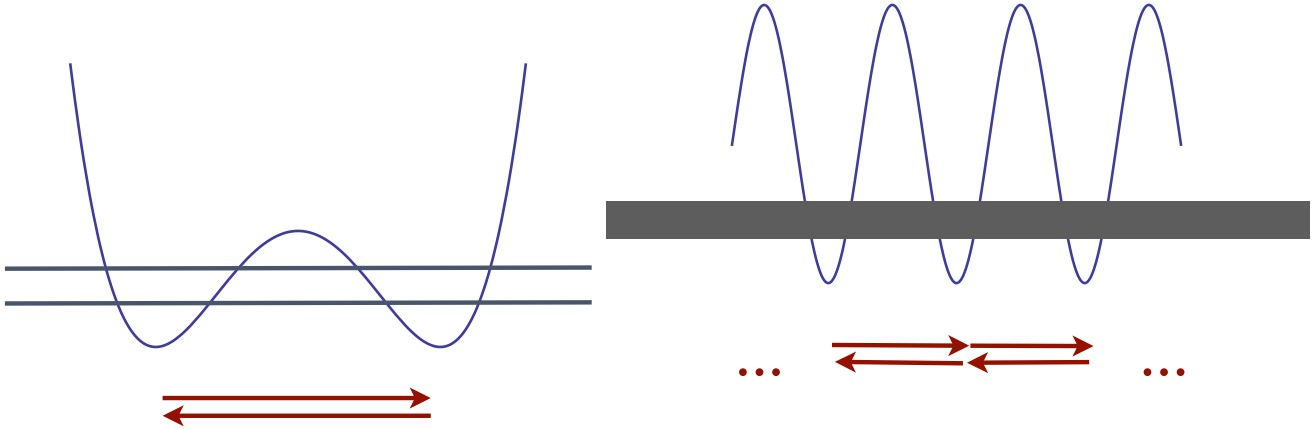
- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel-summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a 2-instanton effect

"Bogomolny/Zinn-Justin mechanism" Bogomolny 1980; Zinn-Justin 1980



- degenerate vacua: double-well, Sine-Gordon, ...

  1. perturbation theory non-Borel-summable: ill-defined/incomplete

  2. instanton gas picture ill-defined/incomplete: $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract

- regularize *both* by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

"resurgence" $\Rightarrow$ cancellation to all orders

Towards Resurgence in QFT

- resurgence $\equiv$ analytic continuation of trans-series
- effective actions, partition functions, ..., have natural integral representations with resurgent asymptotic expansions
- analytic continuation of external parameters: temperature, chemical potential, external fields, ...
- e.g., magnetic $\leftrightarrow$ electric; de Sitter $\leftrightarrow$ anti de Sitter, ...
- matrix models, large N, strings (Mariño, Schiappa, Aniceto, ...)
- soluble QFT: Chern-Simons, ABJM, $\rightarrow$ matrix integrals

- asymptotically free QFT ?

QFT: Renormalons

QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

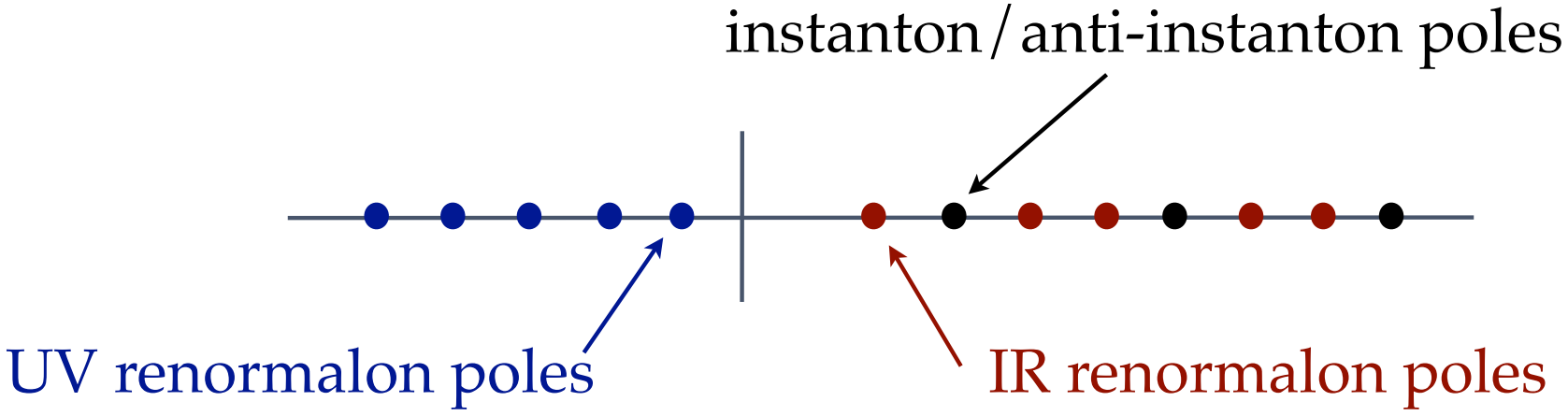
QFT: new physical effects occur, due to running of couplings with momentum

- faster source of divergence: "renormalons"
- both positive and negative Borel poles

IR Renormalon Puzzle in Asymptotically Free QFT

perturbation theory: $\longrightarrow \quad \pm\, i\, e^{-\frac{2S}{\beta_0\, g^2}}$

instantons on $\mathbb{R}^2$ or $\mathbb{R}^4$: $\longrightarrow \quad \pm\, i\, e^{-\frac{2S}{g^2}}$



appears that BZJ cancellation cannot occur

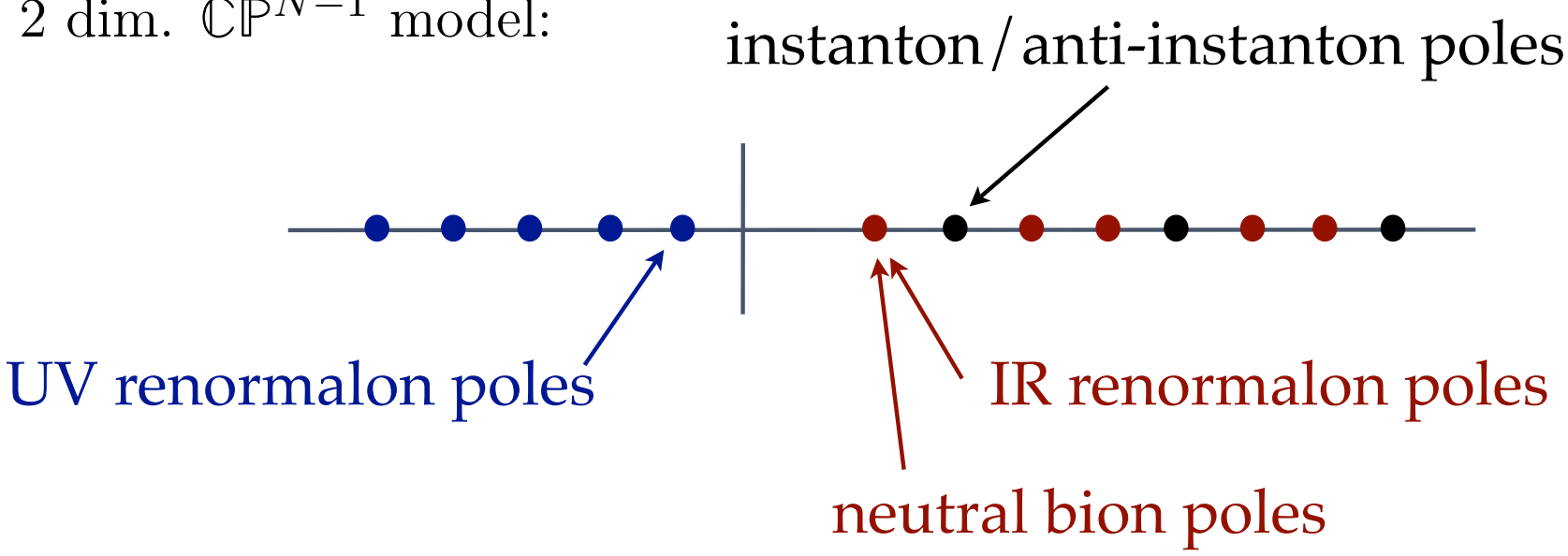
asymptotically free theories remain inconsistent

't Hooft, 1980; David, 1981

IR Renormalon Puzzle in Asymptotically Free QFT

resolution: there is another problem with the non-perturbative instanton gas analysis (Argyres, Ünsal 1206.1890; GD, Ünsal, 1210.2423)

- scale modulus of instantons
- spatial compactification and principle of continuity
- 2 dim. $\mathbb{CP}^{N-1}$ model:



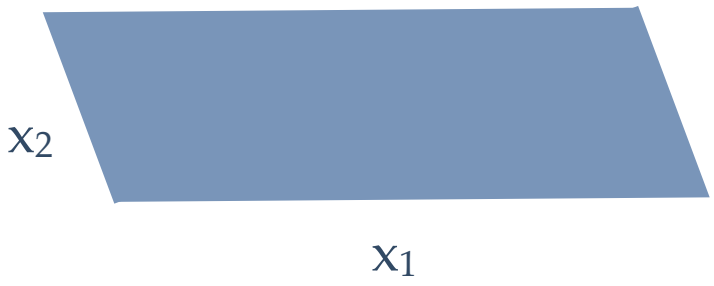
cancellation occurs ! (GD, Ünsal, 1210.2423, 1210.3646)

semiclassical realization of IR renormalons

Topological Molecules in Spatially Compactified Theories

$\mathbb{CP}^{N-1}$: regulate scale modulus problem with (spatial) compactification

$$\mathbb{R}^2 \to S_L{}^1 \times \mathbb{R}^1$$



Euclidean time

instantons fractionalize Kraan/van Baal; Lee/Yi; Bruckmann; Brendel et al

Topological Molecules in Spatially Compactified Theories

temporal compactification: information about deconfined phase

$$\mathbb{R}^1 \xrightarrow[\text{high T}]{\mathbb{R}^1 \times S_\beta{}^1 \quad \text{low T}} \mathbb{R}^2$$

spatial compactification: semi-classical small L regime continuously connected to large L:

principle of continuity

$$\mathbb{R}^1 \xrightarrow[\text{“continuity”}]{S_L{}^1 \times \mathbb{R}^1} \mathbb{R}^2$$

SUSY: Seiberg-Witten 1996, Davies-Hollowood-Khoze-Mattis 1999, ...

non-SUSY: Kovtun-Ünsal-Yaffe, 2007; Ünsal, 2009, ...

$\mathbb{CP}^{N-1}$ Model

- 2d sigma model analogue of 4d Yang-Mills
- asymptotically free: $\beta_0 = N$ (independent of N_f)
- instantons, theta vacua, fermion zero modes, ...
- divergent perturbation theory (non-Borel summable)
- renormalons (both UV and IR)
- large-N analysis
- non-perturbative mass gap: $m_g = \mu\, e^{-4\pi/(g^2 N)}$
- couple to fermions, SUSY, ...
- analogue of center symmetry

Fractionalized Instantons in $\mathbb{CP}^{N-1}$ on $S^1 \times \mathbb{R}^1$

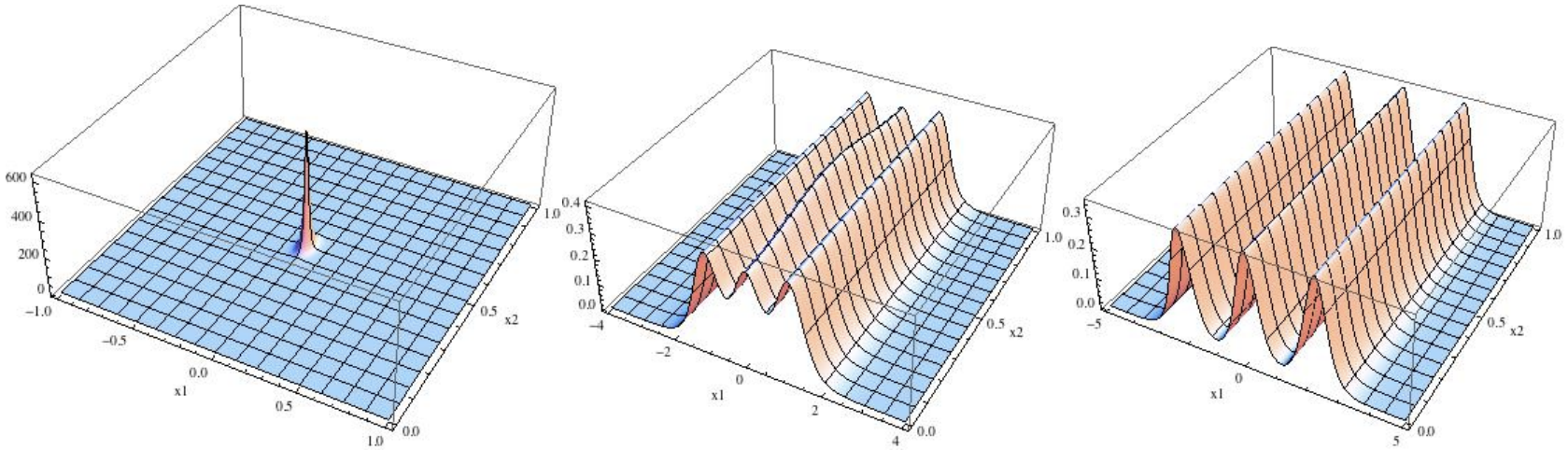
$\mathbb{Z}_N$ twisted instantons fractionalize Bruckmann, 2007; Brendel et al, 2009

- *spatial* compactification $\Rightarrow$ $\mathbb{Z}_N$ twist:

$$v_{\text{twisted}} = \begin{pmatrix} 1 \\ \left(\lambda_1 + \lambda_2\, e^{-\frac{2\pi}{L} z}\right)\, e^{\frac{2\pi}{L}\,\mu_2\, z} \end{pmatrix}$$

(twist in x_2)+(holomorphicity)$\Rightarrow$ fractionalization along x_1

$$\Rightarrow \qquad S_{\text{inst}} \longrightarrow \frac{S_{\text{inst}}}{N} = \frac{S_{\text{inst}}}{\beta_0}$$



Fractionalized Bions in $\mathbb{CP}^{N-1}$ GD/Ünsal, 1210.2423

bions: topological molecules of instantons/anti-instantons

- "orientation" dependence of $\mathcal{I}\bar{\mathcal{I}}$ interaction:
- charged bions: repulsive bosonic interaction

$$\mathcal{B}_{ij} = [\mathcal{I}_i \bar{\mathcal{I}}_j] \sim e^{-S_i(\varphi) - S_j(\varphi)} e^{i\sigma(\alpha_i - \alpha_j)}$$

- neutral bions: attractive bosonic interaction

$$\mathcal{B}_{ii} = [\mathcal{I}_i \bar{\mathcal{I}}_i] \sim e^{-2S_i(\varphi)}$$

- instanton/anti-instanton amplitude is ambiguous:

$$[\mathcal{I}_i \bar{\mathcal{I}}_i]_\pm = \left(\ln \left(\frac{g^2 N}{8\pi} \right) - \gamma \right) \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}} \pm i\pi \frac{16}{g^2 N} e^{-\frac{8\pi}{g^2 N}}$$

Perturbation Theory in Spatially Compactified $\mathbb{CP}^{N-1}$

- small radius limit $\longrightarrow$ effective QM Hamiltonian

$$H^{\text{zero}} = \frac{g^2}{2} P_\theta^2 + \frac{\xi^2}{2g^2} \sin^2\theta + \frac{g^2}{2\sin^2\theta} P_\phi^2 \quad , \qquad \xi = \frac{2\pi}{N}$$

- Born-Oppenheimer approximation: drop high ϕ-sector modes

effective Mathieu equation:

$$-\frac{1}{2}\psi'' + \frac{\xi^2}{2g^2}\sin^2(g\theta)\psi = E\,\psi$$

- Stone-Reeve (Bender-Wu methods):

$$\mathcal{E}(g^2) = \sum_{n=0}^{\infty} a_n (g^2)^n, \qquad a_n \sim -\frac{2}{\pi}\left(\frac{N}{8\pi}\right)^n n! \left(1 - \frac{5}{2n} + \dots\right)$$

- non-Borel-summable!

BZJ cancellation in Spatially Compactified $\mathbb{CP}^{N-1}$ (1210.2423)

- perturbative sector: lateral Borel summation

$$B_{\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_{\pm}} dt\, B\mathcal{E}(t)\, e^{-t/g^2} = \mathrm{Re}\, B\mathcal{E}(g^2) \mp i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

- non-perturbative sector: bion-bion amplitudes

$$\left[\mathcal{I}_i\bar{\mathcal{I}}_i\right]_{\pm} = \left(\ln\left(\frac{g^2\, N}{8\pi}\right) - \gamma\right)\frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}} \pm i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

exact cancellation !

explicit application of resurgence to nontrivial QFT

Graded Resurgence Triangle

saddle points labelled by instanton/anti-instanton number:

$$
\begin{array}{ccccccccc}
 & & & & [0] & & & & \\
 & & & [\mathcal{I}] & & [\bar{\mathcal{I}}] & & & \\
 & & [\mathcal{I}^2] & & [\mathcal{I}\,\bar{\mathcal{I}}] & & [\bar{\mathcal{I}}^2] & & \\
 & [\mathcal{I}^3] & & [\mathcal{I}^2\,\bar{\mathcal{I}}] & & [\mathcal{I}\,\bar{\mathcal{I}}^2] & & [\bar{\mathcal{I}}^3] & \\
[\mathcal{I}^4] & & [\mathcal{I}^3\,\bar{\mathcal{I}}] & & [\mathcal{I}^2\,\bar{\mathcal{I}}^2] & & [\mathcal{I}\,\bar{\mathcal{I}}^3] & & [\bar{\mathcal{I}}^4] \\
\ddots & & & & \vdots & & & & \ddots
\end{array}
$$

cancellations can only occur within columns

Graded Resurgence Triangle and Extended SUSY

extended SUSY: no superpotential; no bions, no condensates

$$
\begin{array}{ccccccccc}
 & & & & [0] & & & & \\
 & & & [\mathcal{I}] & & [\bar{\mathcal{I}}] & & & \\
 & & [\mathcal{I}^2] & & 0 & & [\bar{\mathcal{I}}^2] & & \\
 & [\mathcal{I}^3] & & 0 & & 0 & & [\bar{\mathcal{I}}^3] & \\
[\mathcal{I}^4] & & 0 & & 0 & & 0 & & [\bar{\mathcal{I}}^4] \\
\ddots & & & & \vdots & & & & \ddots
\end{array}
$$

$\Rightarrow$ perturbative expansions must be Borel summable !

GD, Shifman, Ünsal, ...

The Bigger Picture

Q: should we expect resurgent behavior in QM and QFT ?

QM uniform WKB $\Rightarrow$

(i) trans-series structure is generic
(ii) all multi-instanton effects encoded in perturbation theory

(GD, Ünsal, 1306.4405, 1401.5202)

Q: what is behind this resurgent structure ?

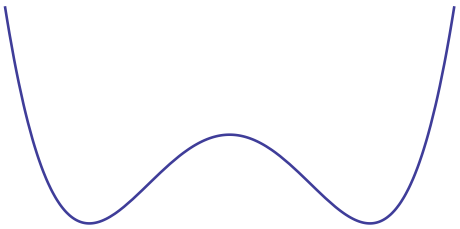
- basic property of all-orders steepest descents integrals

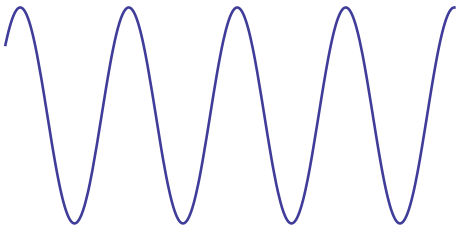
Q: could this extend to (path) functional integrals ?

Uniform WKB and Resurgent Trans-Series for Eigenvalues

(GD, Ünsal, 1306.4405, 1401.5202)

$$-\frac{d^2}{dx^2}\psi + \frac{V(g\,x)}{g^2}\psi = E\,\psi \to -g^4\frac{d^2}{dy^2}\psi(y) + V(y)\psi(y) = g^2\,E\,\psi(y)$$





- weak coupling: degenerate harmonic classical vacua
- non-perturbative effects: $g^2 \leftrightarrow \hbar \quad \Rightarrow \quad \exp\left(-\frac{c}{g^2}\right)$
- approximately harmonic

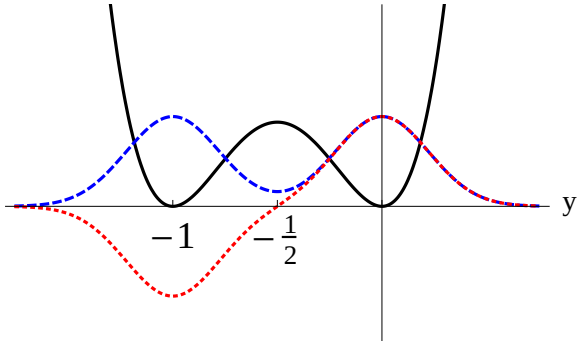
$\Rightarrow$ uniform WKB with parabolic cylinder functions

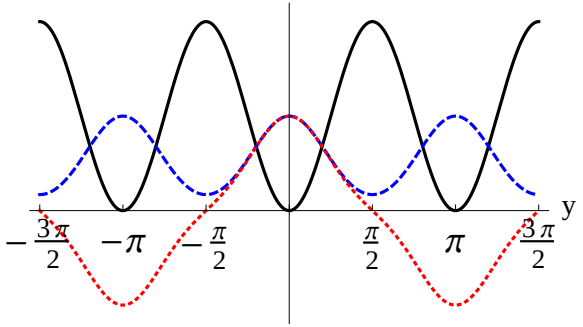
Uniform WKB and Resurgent Trans-Series for Eigenvalues

- uniform WKB ansatz (parameter ν): $\psi(y) = \dfrac{D_\nu\left(\frac{1}{g}u(y)\right)}{\sqrt{u'(y)}}$
- perturbative expansion for E and $u(y)$:

$$E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$$

- $\nu = N$: usual perturbation theory (not Borel summable)
- global analysis $\Rightarrow$ boundary conditions:





- midpoint $\sim \frac{1}{g}$; non-Borel summability $\Rightarrow \quad g^2 \to e^{\pm i\,\epsilon}\, g^2$

Uniform WKB and Resurgent Trans-Series for Eigenvalues

$$D_\nu(z) \sim z^\nu\, e^{-z^2/4}\,(1+\dots) + e^{\pm i\pi\nu}\,\frac{\sqrt{2\pi}}{\Gamma(-\nu)}\, z^{-1-\nu}\, e^{z^2/4}\,(1+\dots)$$

$\longrightarrow$ exact quantization condition

$$\frac{1}{\Gamma(-\nu)}\left(\frac{e^{\pm i\pi}\,2}{g^2}\right)^{-\nu} = \frac{e^{-S/g^2}}{\sqrt{\pi\, g^2}}\,\mathcal{F}(\nu, g^2)$$

$\Rightarrow$ ν is only exponentially close to N (here $\xi \equiv \frac{e^{-S/g^2}}{\sqrt{\pi\, g^2}}$):

$$\begin{aligned}\nu \;&=\; N + \frac{\left(\frac{2}{g^2}\right)^N \mathcal{F}(N, g^2)}{N!}\,\xi \\ &-\frac{\left(\frac{2}{g^2}\right)^{2N}}{(N!)^2}\left[\mathcal{F}\frac{\partial \mathcal{F}}{\partial N} + \left(\ln\left(\frac{e^{\pm i\pi}\,2}{g^2}\right) - \psi(N+1)\right)\mathcal{F}^2\right]\xi^2 + O(\xi^3)\end{aligned}$$

- insert: $E = E(\nu, g^2) = \sum_{k=0}^{\infty} g^{2k} E_k(\nu)$ $\Rightarrow$ trans-series

Connecting Perturbative and Non-Perturbative Sector

trans-series form for energy eigenvalues arises from the (resurgent) analytic continuation properties of the parabolic cylinder functions

generic and universal

Zinn-Justin/Jentschura: generate *entire trans-series* from

(i) perturbative expansion $E = E(N, g^2)$
(ii) single-instanton fluctuation function $\mathcal{F}(N, g^2)$
(iii) rule connecting neighbouring vacua (parity, Bloch, ...)

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in fact ... (GD, Ünsal, 1306.4405, 1401.5202)

$$\mathcal{F}(N, g^2) = \exp\left[S \int_0^{g^2} \frac{dg^2}{g^4} \left(\frac{\partial E(N, g^2)}{\partial N} - 1 + \frac{\left(N + \frac{1}{2}\right) g^2}{S} \right)\right]$$

implication: perturbation theory encodes everything !

Connecting Perturbative and Non-Perturbative Sector

e.g. double-well potential: $B \equiv N + \frac{1}{2}$

$$E(N, g^2) = B - g^2\left(3B^2 + \frac{1}{4}\right) - g^4\left(17B^3 + \frac{19}{4}B\right) - g^6\left(\frac{375}{2}B^4 + \frac{459}{4}B^2 + \frac{131}{32}\right) - \dots$$

- non-perturbative function ($\mathcal{F} \sim (\dots)\exp[-A/2]$):

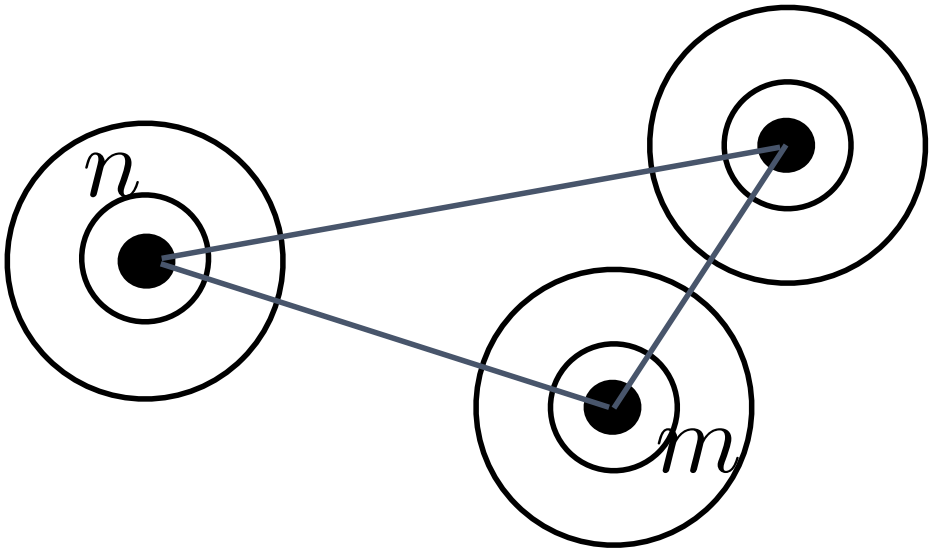
$$A(N, g^2) = \frac{1}{3g^2} + g^2\left(17B^2 + \frac{19}{12}\right) + g^4\left(125B^3 + \frac{153B}{4}\right) + g^6\left(\frac{17815}{12}B^4 + \frac{23405}{24}B^2 + \frac{22709}{576}\right) +$$

- simple relation:

$$\frac{\partial E}{\partial B} = -3g^2\left(2B - g^2\frac{\partial A}{\partial g^2}\right)$$

Connecting Perturbative and Non-Perturbative Sector

all orders of multi-instanton trans-series are encoded in perturbation theory of fluctuations about perturbative vacuum



why ? turn to path integrals ... (also: QFT)

All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals:
  hyperasymptotics (Berry/Howls 1991, Howls 1992)

$$I^{(n)}(g^2) = \int_{C_n} dz\, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}}\, e^{-\frac{1}{g^2} f_n}\, T^{(n)}(g^2)$$

- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, g^{2r}$$

All-Orders Steepest Descents: Darboux Theorem

- universal resurgent relation between different saddles:

$$T^{(n)}(g^2) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - g^2 v/(F_{nm})} T^{(m)}\left(\frac{F_{nm}}{v}\right)$$

- exact resurgent relation between fluctuations about n^{th} saddle and about neighboring saddles m

$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \ldots \right.$$

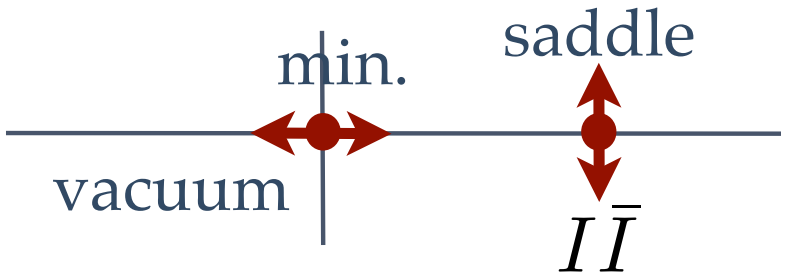
- universal factorial divergence of fluctuations (Darboux)
- fluctuations about different saddles explicitly related !

All-Orders Steepest Descents: Darboux Theorem

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(g^2) = \int_0^\pi dz\, e^{-\frac{1}{g^2}\sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$
\begin{aligned}
T_r^{(0)} &= \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)} \\
&\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1 - \frac{\frac{1}{4}}{(r-1)} + \frac{\frac{9}{32}}{(r-1)(r-2)} - \frac{\frac{75}{128}}{(r-1)(r-2)(r-3)} + \right.
\end{aligned}
$$

All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$
\begin{aligned}
T_r^{(0)} &= \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)} \\
&\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1 - \frac{\frac{1}{4}}{(r-1)} + \frac{\frac{9}{32}}{(r-1)(r-2)} - \frac{\frac{75}{128}}{(r-1)(r-2)(r-3)} + \right.
\end{aligned}
$$

- low order coefficients about saddle z_1:

$$
T^{(1)}(g^2) \sim i\sqrt{\pi}\left(1 - \frac{1}{4}\,g^2 + \frac{9}{32}\,g^4 - \frac{75}{128}\,g^6 + \ldots\right)
$$

- fluctuations about the two saddles are explicitly related

Resurgence in Path Integrals: "Functional Darboux Theorem"

could something like this work for path integrals?

"functional Darboux theorem" ?

- multi-dimensional case is already non-trivial and interesting

Pham (1965); Delabaere/Howls (2002)

- Picard-Lefschetz theory

- do a computation to see what happens ...

Resurgence in Path Integrals

- periodic potential: $V(x) = \frac{1}{g^2}\,\sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n!\left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \ldots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\,e^{-2\frac{1}{2g^2}}\left(1 - \frac{5}{2}\cdot g^2 - \frac{13}{8}\cdot g^4 - \ldots\right)$$

Resurgence in Path Integrals

- periodic potential: $V(x) = \frac{1}{g^2}\,\sin^2(g\,x)$
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- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n!\left(1 - \frac{53}{6}\cdot\frac{1}{3}\cdot\frac{1}{n} - \frac{1277}{72}\cdot\frac{1}{3^2}\cdot\frac{1}{n(n-1)} - \ldots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\,e^{-2\frac{1}{6g^2}}\left(1 - \frac{53}{6}\cdot g^2 - \frac{1277}{72}\cdot g^4 - \ldots\right)$$

Resurgence in Path Integrals

fluctuations about the instanton/anti-instanton saddle are determined by those about the vacuum saddle

Analytic Continuation of Path Integrals: Lefschetz Thimbles

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k\, e^{-\frac{i}{g^2}\, S_{\text{imag}}[A_k]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2}S_{\text{real}}[A]}$$

Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

Analytic Continuation of Path Integrals: Lefschetz Thimbles

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Lefschetz thimble = "functional steepest descents contour"

remaining path integral has real measure:
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact resurgent analysis

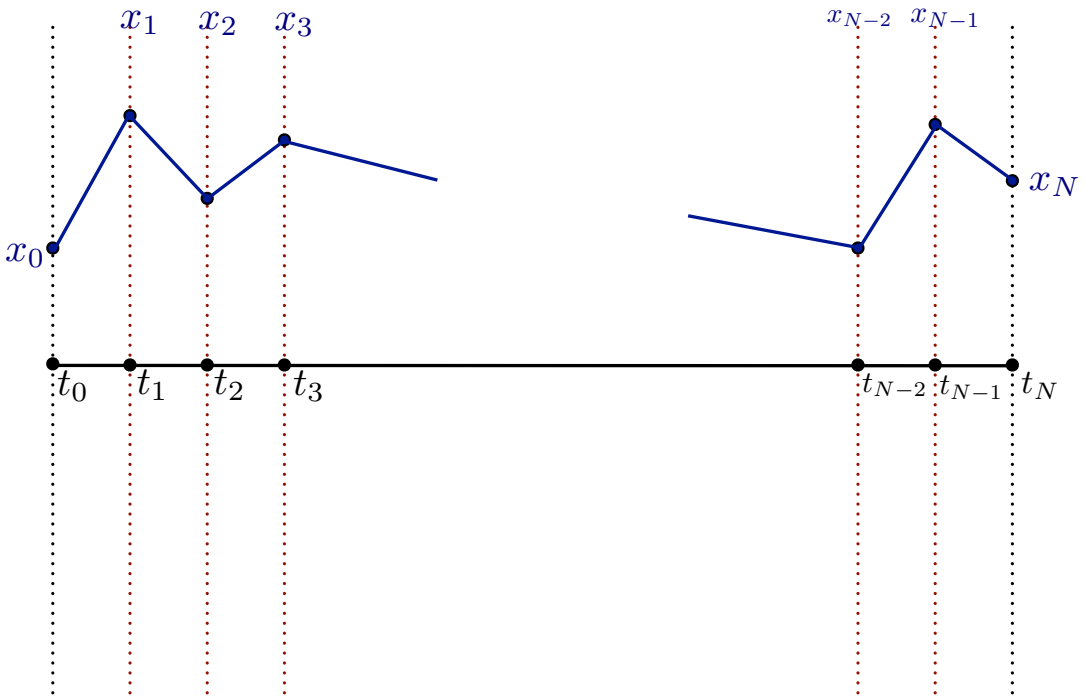
resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

Stokes phenomenon: intersection numbers $\mathcal{N}_k$ can change with phase of parameters

Analytic Continuation of Path Integrals

- brute-force approach:



Analytic Continuation of Path Integrals

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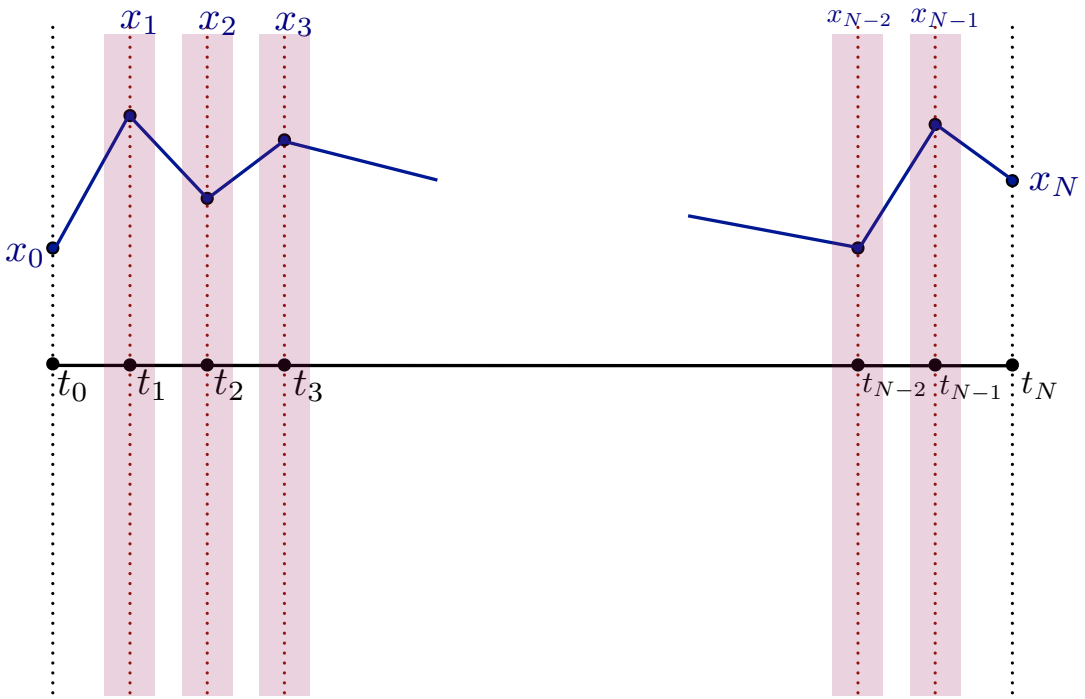

Analytic Continuation of Path Integrals

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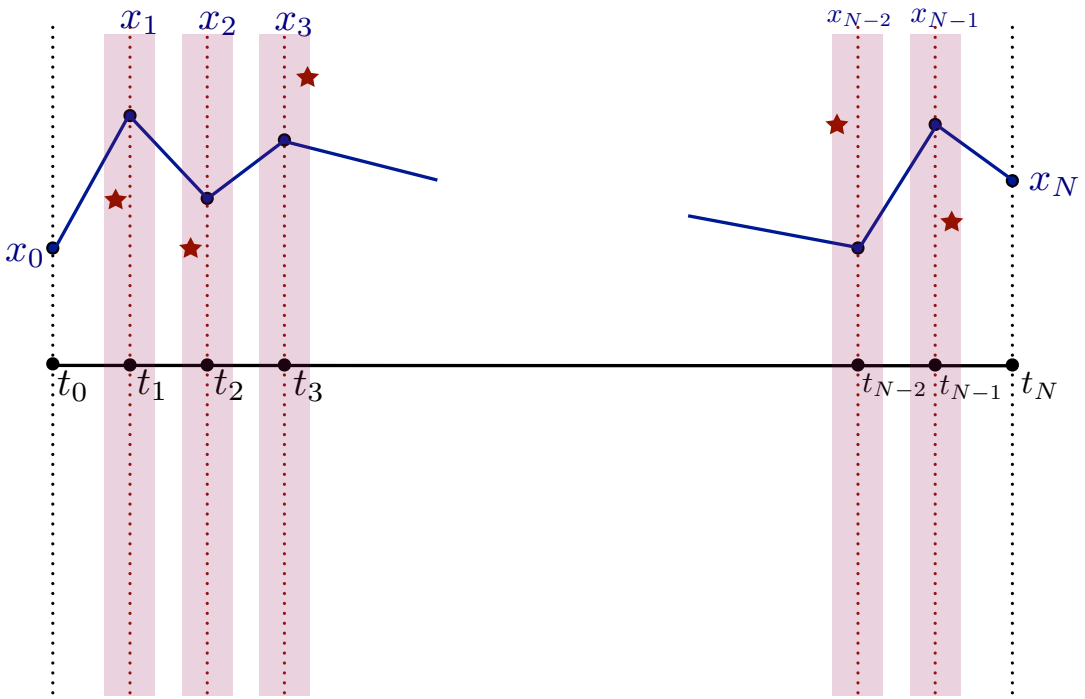

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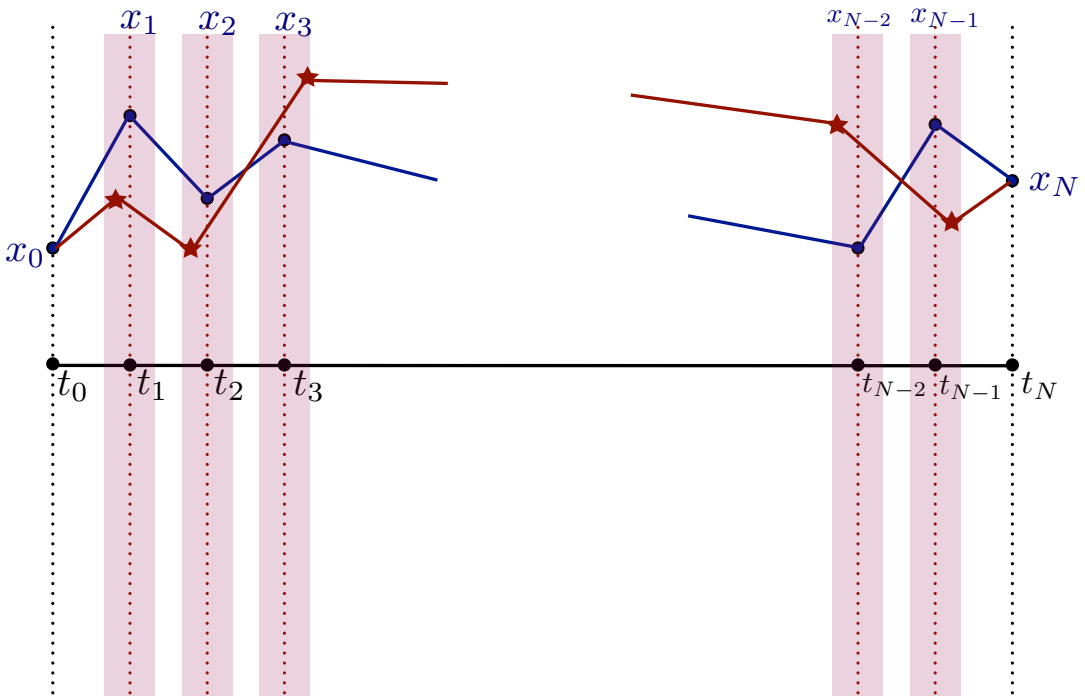


Thimbles from Gradient Flow

gradient flow to generate steepest descent thimble:

$$\frac{\partial}{\partial \tau} A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

- keeps $Im[S]$ constant, and $Re[S]$ is monotonic

$$\frac{\partial}{\partial \tau}\left(\frac{S-\bar{S}}{2i}\right) = -\frac{1}{2i}\int\left(\frac{\delta S}{\delta A}\frac{\partial A}{\partial \tau} - \overline{\frac{\delta S}{\delta A}}\,\overline{\frac{\partial A}{\partial \tau}}\right) = 0$$

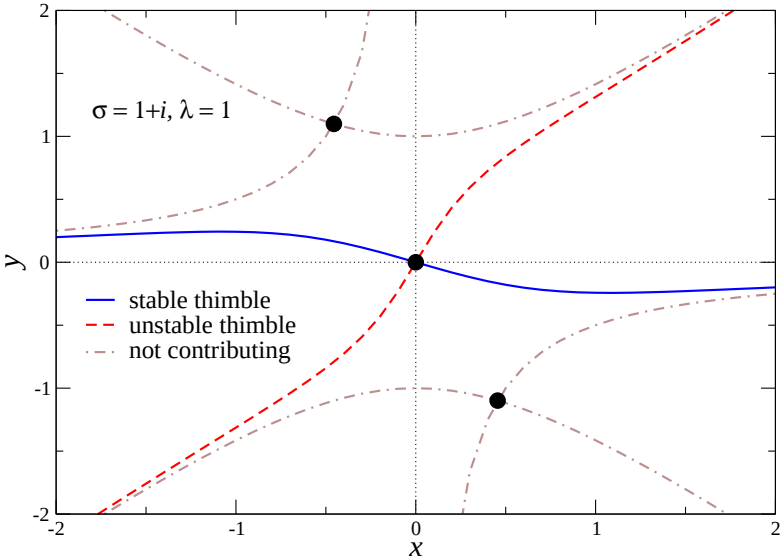
$$\frac{\partial}{\partial \tau}\left(\frac{S+\bar{S}}{2}\right) = -\int\left|\frac{\delta S}{\delta A}\right|^2$$

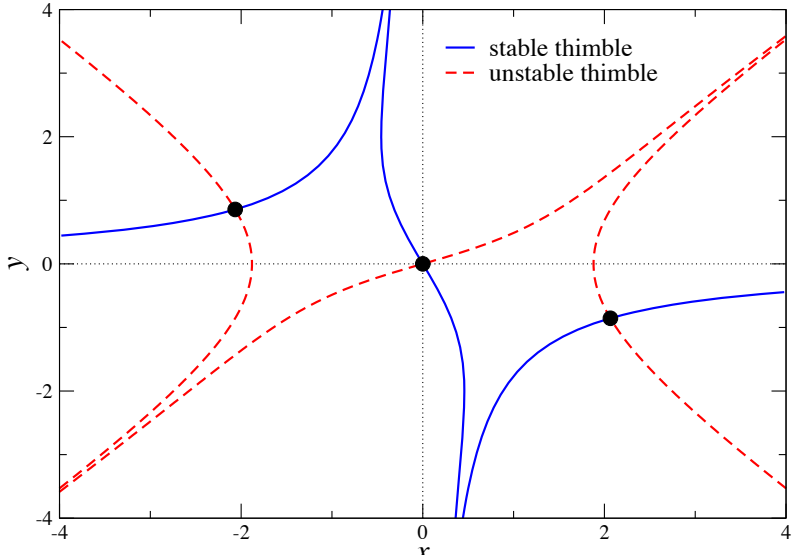
- Chern-Simons theory (Witten 2001)
- comparison with complex Langevin (Aarts 2013, ...)
- lattice (Tokyo/RIKEN, Aurora, 2013): Bose-gas ✓

Thimbles, Gradient Flow and Resurgence

$$Z = \int_{-\infty}^{\infty} dx \, \exp\left[-\left(\frac{\sigma}{2}\, x^2 + \frac{x^4}{4}\right)\right]$$

(Aarts, 2013; GD, Unsal, ...)

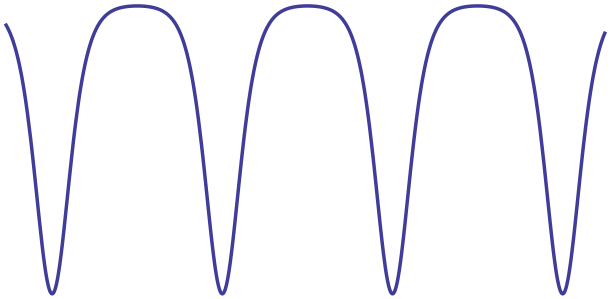




- contributing thimbles change with phase of σ
- need all three thimbles for $Re[\sigma] < 0$
- integrals along thimbles are related (resurgence)
- resurgence: preferred unique "field" choice

Ghost Instantons: Analytic Continuation of Path Integrals

(Başar, GD, Ünsal, arXiv:1308.1108)



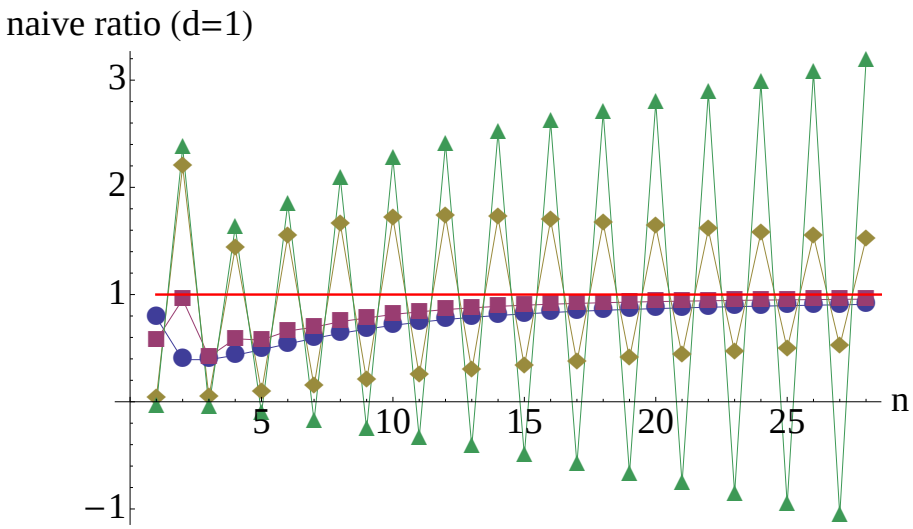
periodic elliptic potential

$$V(x) = \frac{1}{g^2}\,\mathrm{sd}^2(g\,x|m)$$

- large order growth of perturbation theory

(Bender/Wu, Bogomolny, Zinn-Justin, Lipatov, ...)

$$a_n(m) \sim -\frac{16}{\pi} n!\,\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}}$$



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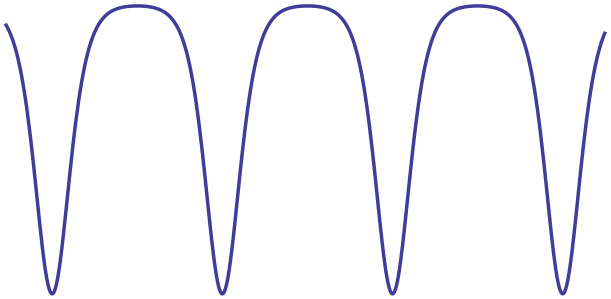

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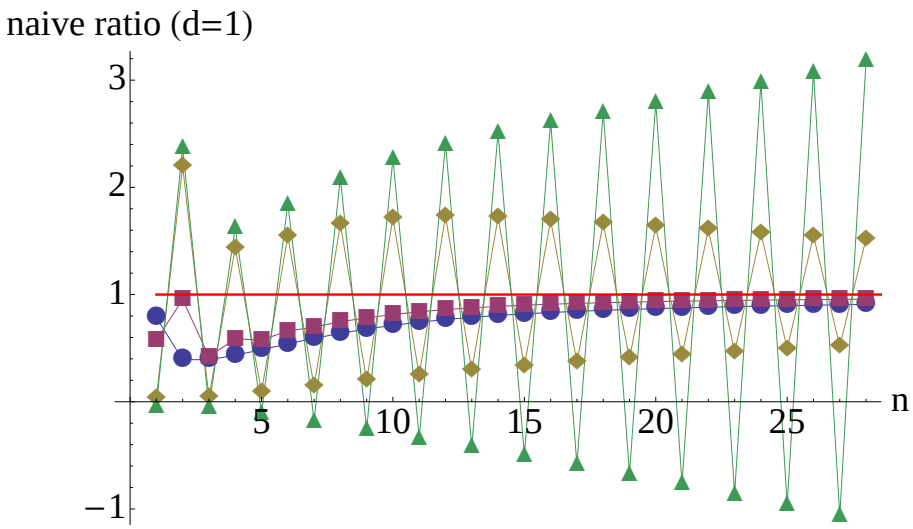
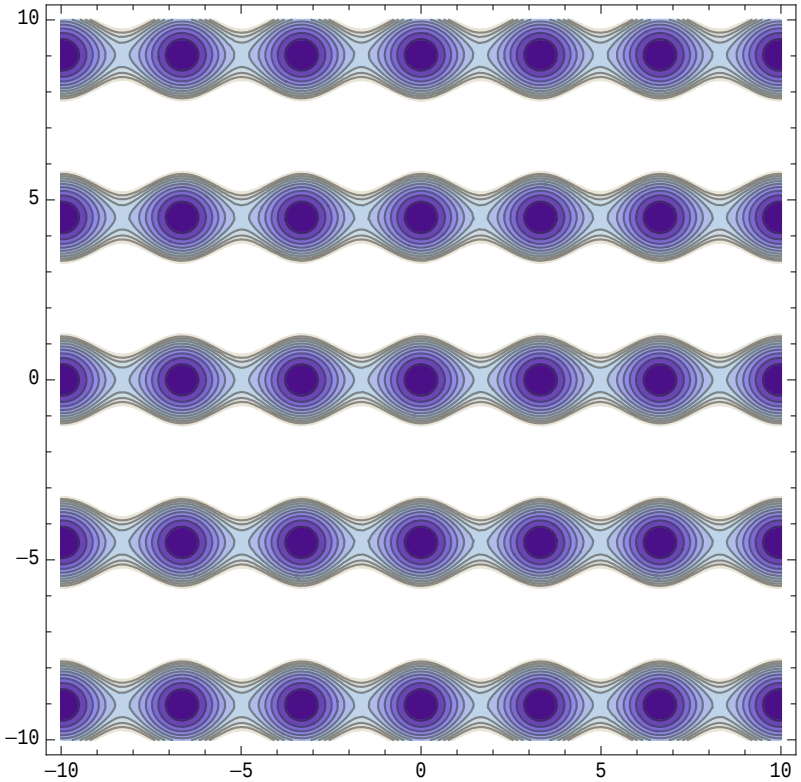


fails miserably !

Ghost Instantons: Analytic Continuation of Path Integrals

$$\mathcal{Z}(g^2|m) = \int \mathcal{D}x\, e^{-S[x]} = \int \mathcal{D}x\, e^{-\int d\tau \left(\frac{1}{4}\dot{x}^2 + \frac{1}{g^2}\,\mathrm{sd}^2(g\,x|m)\right)}$$

- doubly periodic potential: *real* & *complex* instantons



instanton actions:

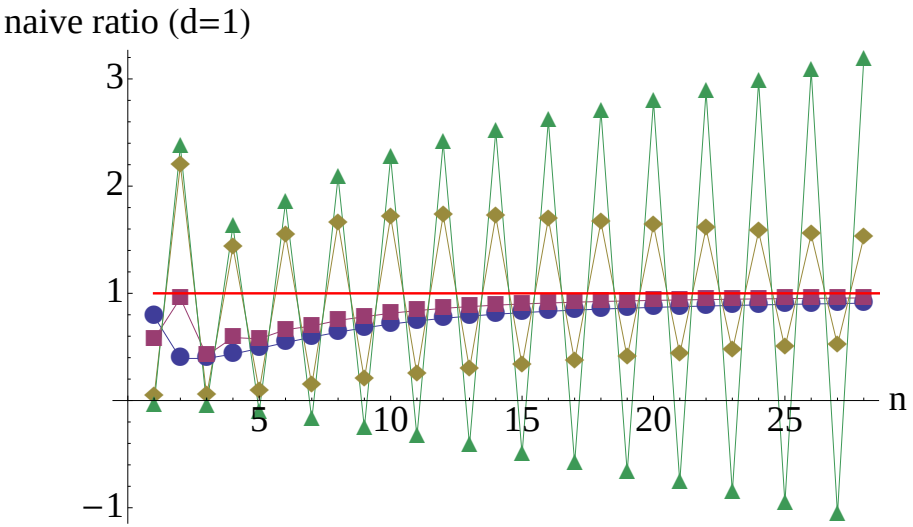
$$S_{\mathcal{I}}(m) = \frac{2\arcsin(\sqrt{m})}{\sqrt{m(1-m)}}$$

$$S_{\mathcal{G}}(m) = \frac{-2\arcsin(\sqrt{1-m})}{\sqrt{m(1-m)}}$$

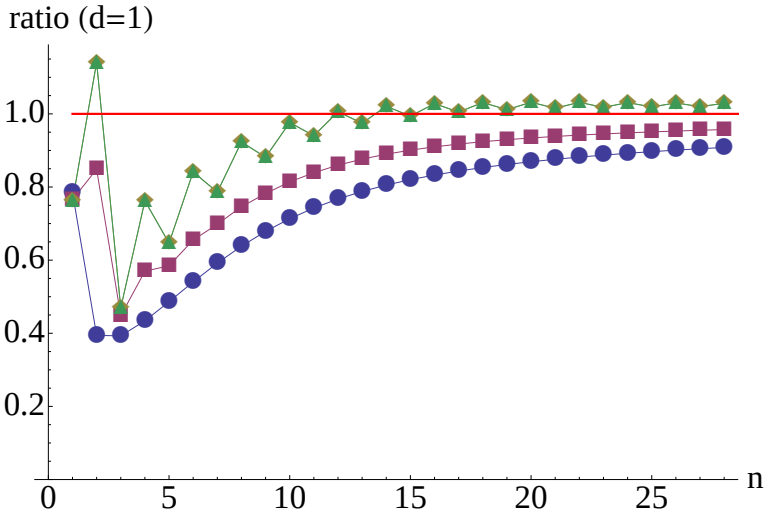
Ghost Instantons: Analytic Continuation of Path Integrals

- large order growth of perturbation theory:

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$



without ghost instantons

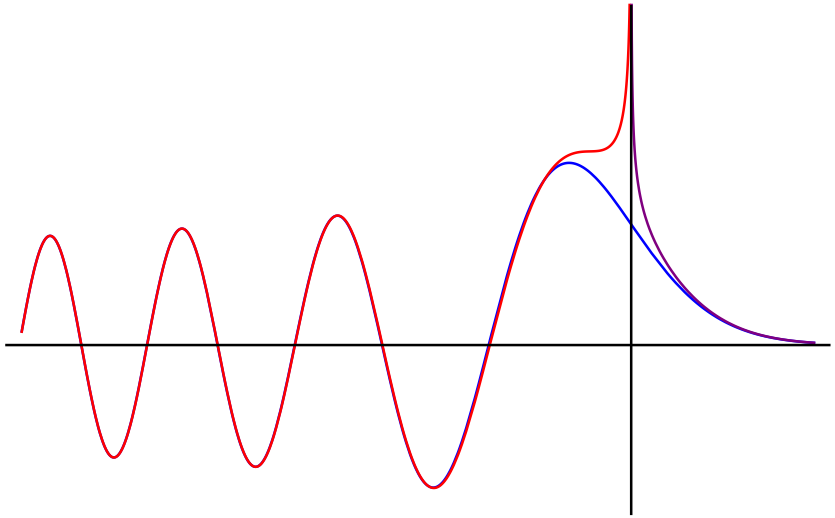


with ghost instantons

- complex instantons directly affect perturbation theory, even though they are not in the original path integral measure

The Stokes Phenomenon and the Airy Function

- supernumerary rainbows



$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}\,t^3+x\,t\right)}\,dt \sim \begin{cases} \dfrac{e^{-\frac{2}{3}\,x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}} & , \quad x \to +\infty \\[2ex] \dfrac{\sin\left(\frac{2}{3}\,(-x)^{3/2}+\frac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}} & , \quad x \to -\infty \end{cases}$$

Non-perturbative Physics Without Instantons

Dabrowski, GD, arXiv:1306.0921, Cherman, Dorigoni, GD, Ünsal, 1308.0127

Yang-Mills, $\mathbb{CP}^{N-1}$, $O(N)$, PCM, ... all have non-BPS solutions with finite action

(Din & Zakrzewski, 1980; Uhlenbeck 1985; Sibner, Sibner, Uhlenbeck, 1989)

- "unstable": negative modes of fluctuation operator
- what do these mean physically ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from lateral Borel sums of perturbation theory

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

Non-perturbative Physics Without Instantons: Principal Chiral Model (Cherman, Dorigoni, GD, Ünsal, 1308.0127)

$$S[U] = \frac{1}{2g^2} \int d^2x \, \mathrm{tr} \, \partial_\mu U \partial^\mu U^\dagger \qquad , \quad U \in SU(N)$$

- non-Borel-summable perturbation theory due to IR renomalons
- but, the theory has no instantons !

resolution: non-BPS saddle point solutions to 2nd-order classical Euclidean equations of motion: "unitons"

$$\partial_\mu \left(U^\dagger \partial_\mu U \right) = 0 \qquad \text{(Uhlenbeck 1985)}$$

- have negative fluctuation modes: saddles, not minima
- fractionalize on cylinder $\longrightarrow$ BZJ cancellation

Resurgence and Localization

(Mariño, 1104.0783; Kallen, Mariño, 1308.6485; Aniceto, Russo, Schiappa, 1410.5834)

- certain protected quantities in especially symmetric QFTs can be reduced to matrix models $\Rightarrow$ resurgent asymptotics

- 3d Chern-Simons on $\mathbb{S}^3 \to$ matrix model

$$Z_{CS}(N,g) = \frac{1}{\mathrm{vol}(U(N))} \int dM \exp\left[-\frac{1}{g}\,\mathrm{tr}\left(\frac{1}{2}\,(\ln M)^2\right)\right]$$

- ABJM: $\mathcal{N}=6$ SUSY CS, $G = U(N)_k \times U(N)_{-k}$

$$Z_{ABJM}(N,k) = \sum_{\sigma\in S_N} \frac{(-1)^{\epsilon(\sigma)}}{N!} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi k} \frac{1}{\prod_{i=1}^{N} 2\mathrm{ch}\left(\frac{x_i}{2}\right)\,\mathrm{ch}\left(\frac{x_i - x_{\sigma(i)}}{2k}\right)}$$

- $\mathcal{N}=4$ SUSY Yang-Mills on $\mathbb{S}^4$

$$Z_{SYM}(N,g^2) = \frac{1}{\mathrm{vol}(U(N))} \int dM \exp\left[-\frac{1}{g^2}\,\mathrm{tr}M^2\right]$$

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative analysis
- trans-series 'encode' all information; expansions about different saddles are intimately related
- there is extra 'magic' in perturbation theory

- matrix models, large N, strings, SUSY QFT
- IR renormalon puzzle in asymptotically free QFT
- multi-instanton physics from perturbation theory
- $\mathcal{N} = 2$ and $\mathcal{N} = 2^*$ SUSY gauge theory (Basar, GD, 1412.xxxx)

- fundamental property of steepest descents
- moral: go complex and consider all saddles, not just minima