Effective Field Theories and Higgs Physics

Lecture 2

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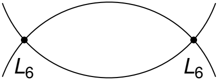
Reminder

$$L = L_{SM} + \frac{1}{\Lambda^2} L_6 + \left[\frac{1}{\Lambda^4} L_8 + \ldots \right]_{\text{drop}}$$

One-loop graph



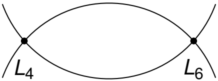
$$A \sim \frac{1}{\Lambda^4} \left[c_6^2 \left(\frac{1}{\epsilon} + \text{finite}_1 \right) + c_8 \left(\frac{1}{\epsilon} + \text{finite}_2 \right) \right]_{\text{drop}}$$

Need L_8 to absorb divergences.

Drop both the loop and L_8 to order $1/\Lambda^2$.

Warning: Keep all operators of given dimension

$$L = L_{SM} + \left[\frac{1}{\Lambda^2} L_6\right]_{\text{keep some}} + \left[\frac{1}{\Lambda^2} L_6 + \ldots\right]_{\text{drop those I don't like}}$$



$$A \sim \frac{1}{\Lambda^2}\left[c_6\left(\frac{1}{\epsilon} + \text{finite}_1\right) + c_6\left(\frac{1}{\epsilon} + \text{finite}_2\right)_{\text{drop}}\right]$$

Need L_6 to absorb divergences, and same order as terms that have been retained.

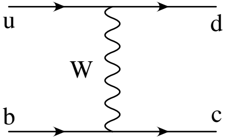
Setting some $c_6 \to 0$ is scheme-dependent, e.g. $\mu^2 \to \overline{\mu}^2 4\pi e^{-\gamma}$

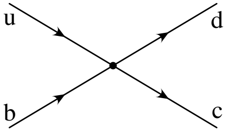
Toy Model (Integral)

Rather than do an explicit EFT example, look at a simple integral which illustrates what happens.

Tree-level:

$$-\frac{1}{k^2 - M^2} = c_1 \frac{1}{M^2} + c_2 \frac{k^2}{M^4} + \ldots \qquad c_i = 1$$



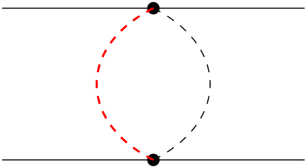


Toy Model (Integral)

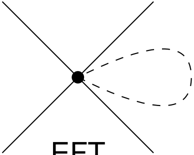
One Loop:

$$I_F = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)(k^2 - M^2)}$$

Integral arises as a one-loop graph in a field theory, has some couplings in front.



Full



EFT

Expanding does not commute with loop integration

Do the integral exactly in $d = 4 - 2\epsilon$ dimensions:

$$
\begin{aligned}
I_F &= \mu^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)(k^2 - M^2)} \\
&= \frac{i}{16\pi^2} \left[\frac{1}{\epsilon} + \frac{m^2 \log(m^2/\mu^2) - M^2 \log(M^2/\mu^2)}{M^2 - m^2} + 1 \right]
\end{aligned}
$$

Relatively simple because only 2 denominators. Three denominators gives Spence functions (dilogs).

Expand, do the integral term by term, and then sum up the result:

$$
\begin{aligned}
I_{\text{eft}} &= \mu^{2\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)} \left[-\frac{1}{M^2} - \frac{k^2}{M^4} - \ldots \right] \\
&= \frac{i}{16\pi^2} \left[-\frac{1}{\epsilon} \frac{m^2}{M^2 - m^2} + \frac{m^2}{M^2 - m^2} \log \frac{m^2}{\mu^2} - \frac{m^2}{M^2 - m^2} \right]
\end{aligned}
$$

Points to Note

- Missing the non-analytic terms in M.
- The $1/\epsilon$ terms do not agree, they are cancelled by counterterms which differ in the full and EFT.
- The two theories have different anomalous dimensions.
- The term non-analytic in the IR scale, $\log(m^2)$ agrees in the two theories. This is the part which must be reproduced in the EFT.
- The analytic parts are local, and can be included as matching contributions to the Lagrangian.
- Sum $\log M^2/m^2$ terms using RG evolution.

No Non-Analytic Terms in M

$$\log \frac{m^2}{M^2} = \log \frac{m^2}{\mu^2} - \log \frac{M^2}{\mu^2}$$

$$I_F = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} + \frac{m^2}{M^2 - m^2}\log\frac{m^2}{\mu^2} - \frac{M^2}{M^2 - m^2}\log\frac{M^2}{\mu^2} + 1\right]$$

$$I_{\text{eft}} = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon}\,\frac{m^2}{M^2 - m^2} + \frac{m^2}{M^2 - m^2}\log\frac{m^2}{\mu^2} - \frac{m^2}{M^2 - m^2}\right]$$

$1/\epsilon$ terms are different

$$I_F = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{M^2}{M^2-m^2}\log\frac{M^2}{\mu^2} + 1\right]$$

$$I_{eft} = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon}\,\frac{m^2}{M^2-m^2} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{m^2}{M^2-m^2}\right]$$

Each theory has its own counterterms (renormalization).

Different anomalous dimensions

Full theory:

$$\frac{1}{\epsilon}$$

The amplitude has an anomalous dimensions

EFT:

$$-\frac{1}{\epsilon}\,\frac{m^2}{M^2-m^2} = -\frac{1}{\epsilon}\,\frac{m^2}{M^2} - \frac{1}{\epsilon}\,\frac{m^4}{M^4} + \ldots$$

Each EFT order in $1/M$ has its own anomalous dimension.

Non-analytic Terms in m Agree

$$I_F = \frac{i}{16\pi^2}\left[\frac{1}{\epsilon} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{M^2}{M^2-m^2}\log\frac{M^2}{\mu^2} + 1\right]$$

$$I_{eft} = \frac{i}{16\pi^2}\left[-\frac{1}{\epsilon}\,\frac{m^2}{M^2-m^2} + \frac{m^2}{M^2-m^2}\log\frac{m^2}{\mu^2} - \frac{m^2}{M^2-m^2}\right]$$

The EFT reproduces the complete low-energy limit of the full theory, including all the dependence on low energy (IR) scales.

If there are multiple IR scales $m_1, m_2, \ldots$, reproduces the complete m_i/m_j dependence.

Matching

Infinite parts cancelled by counterterms.

The difference between the finite parts of the two results is

$$
\begin{aligned}
I_F - I_{eft} &= \frac{i}{16\pi^2}\left[\log\frac{\mu^2}{M^2} + \frac{m^2\log(\mu^2/M^2)}{M^2-m^2} + \frac{M^2}{M^2-m^2}\right] \\
&= \frac{i}{16\pi^2}\left[\left(\log\frac{\mu^2}{M^2}+1\right) + \frac{m^2}{M^2}\left(\log\frac{\mu^2}{M^2}+1\right) + \ldots\right]
\end{aligned}
$$

The terms in parentheses are matching coefficients to terms of order 1, order $1/M^2$, etc. from integrating out heavy particle. They are analytic in m.

Note:

$$
\log\frac{m}{M} \to -\log\frac{M}{\mu} + \log\frac{m}{\mu}
$$

with the first part in the matching, and the second part in the EFT.

Summing Large Logs

The full theory has $\log M^2/m^2$ terms. At higher orders, get

$$\alpha_s^n \log^n M^2/m^2$$

- If $M \gg m$, perturbation theory breaks down as $\alpha_s \log M/m \sim 1$.
- Full theory involves two widely separated scales.
- Calculations become very difficult at higher orders.

Divide one calculation into two calculations, each involving one scale.

- Each calculation much easer since it involves a single scale
- For the matching to be accurate, want $\mu = M$.
- For the EFT to be accurate, want $\mu = m$.

- For the matching use $\mu = M$. $\log M/\mu$ small
- For the EFT calculation, pick $\mu = m$. $\log m/\mu$ small
- Use the EFT renormalization group to convert the Lagrangian from $\mu = M$ to $\mu = m$.
- RG perturbation theory valid as long as α_s small. Do not need α_s log to be small.

RG Improved Perturbation Theory

$$\mu \frac{d}{d\mu} c = \left[\frac{g^2}{16\pi^2} \gamma_0 + \mathcal{O}\left(\frac{g^4}{(16\pi^2)^2} \right) \right] c$$

$$\mu \frac{d}{d\mu} g = -b_0 \frac{g^3}{16\pi^2} + \mathcal{O}\left(\frac{g^5}{(16\pi^2)^2} \right)$$

with solution

$$\frac{c(\mu_1)}{c(\mu_2)} = \left[\frac{\alpha(\mu_1)}{\alpha(\mu_2)} \right]^{-\gamma_0/(2b_0)}, \qquad \alpha = \frac{g^2}{4\pi}$$

Correction can be big (factors of two or more), but perturbation theory is valid as long as $\alpha/(4\pi)$ is small.

Radiative Corrections: Operator Mixing

$$L = -\frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* \ (c_1 O_1 + c_2 O_2)$$

$$O_1 = (\bar{c}^\alpha \gamma^\mu P_L b_\alpha) \left(\bar{d}^\beta \gamma_\mu P_L u_\beta \right) \qquad c_1 = 1 + \mathcal{O}(\alpha_s)$$

$$O_2 = (\bar{c}^\alpha \gamma^\mu P_L b_\beta) \left(\bar{d}^\beta \gamma_\mu P_L u_\alpha \right) \qquad c_2 = 0 + \mathcal{O}(\alpha_s)$$

$$\mu \frac{d}{d\mu} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{\alpha_s}{4\pi} \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Can integrate by finding the eigenvalues and eigenvectors of γ.

Matching

$$I_M = \frac{i}{16\pi^2}\left[\left(\log\frac{\mu^2}{M^2}+1\right)+\frac{m^2}{M^2}\left(\log\frac{\mu^2}{M^2}+1\right)+\ldots\right]$$

We computed the matching from $I_F - I_{eft}$.

But there is an easier way which does not involve computing the two scale integral I_F.

I_M is analytic in m. Therefore, we can compute

$$I_F(m=0) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{(k^2)(k^2-M^2)}$$

$$\frac{\partial I_F}{\partial m^2}(m=0) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{(k^2)^2(k^2-M^2)}$$

Keep only the finite terms. More and more IR divergent.

Summary of EFT procedure

1. Write down the most general EFT Lagrangian with coefficients c_i.
2. Compute the EFT Lagrangian $c_i(\mu = M)$ by expanding in $1/M$ around $M = \infty$.
3. Compute the EFT coefficients δc_i by matching:
   Expand in powers of m around $m = 0$
4. RG improve the EFT result by running c_i from $\mu = M$ to $\mu = m$.
5. Compute EFT graphs in terms of $c_i(\mu = m)$ using L_{eft}.

We have added the two contributions from expanding in $1/M$ and expanding in m.

$$\frac{1}{k^2 - m^2} \frac{1}{k^2 - M^2}$$

This gives I_F, not $2I_F$.

LSZ Reduction Formula (Very Important)

One can compute an S-matrix element

$$_{\text{out}}\langle q_1, \ldots, q_m | p_1, \ldots, p_n \rangle_{\text{in}}$$

from the $r = m + n$ point Green's function

$$\langle 0 | T \left\{ \phi_1(x_1) \ldots \phi_r(x_r) \right\} | 0 \rangle$$

as long as

$$\langle p | \phi(x) | 0 \rangle \neq 0$$

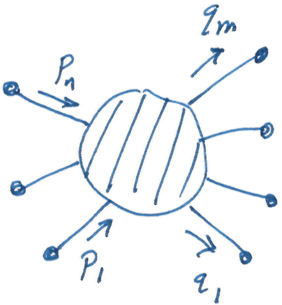
Any field is okay, as long as it can produce the particle.

Basic Objects

What you compute using Feynman diagrams:

Momentum space Green's function:

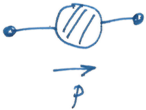
$$G(q_1,\ldots,q_m;p_1,\ldots,p_n) = \prod_{i=1}^{m} \int \mathrm{d}^4 y_i \; e^{iq_i\cdot y_i} \prod_{j=1}^{n} \int \mathrm{d}^4 x_j \; e^{-ip_j\cdot x_j}$$
$$\times \langle 0 | T\left\{\phi_1(y_1)\ldots\phi_m(y_m)\phi_1(x_1)\ldots\phi_n(x_n)\right\}|0\rangle$$



Wavefunction Renormalization

Propagator:

$$D_i(p) = \int \mathrm{d}^4x \; e^{ip\cdot x} \langle 0 | T\{\phi_i(x)\phi_i(0)\} | 0 \rangle$$



$$D_i(p) \sim \frac{i\,\mathcal{R}_i}{p^2 - m_i^2 + i\epsilon} + \ldots$$

so that the (finite) wavefunction renormalization factor is

$$\lim_{p^2 \to m_i^2} \left(p^2 - m_i^2\right) D_i(p) = i\,\mathcal{R}_i$$

The production of a particle of mass m gives a pole in the two-point function.

LSZ Result

Pick out poles in the Green's function:

$$\lim_{q_i^2 \to m_i^2} \lim_{p_j^2 \to m_j^2} \prod_{i=1}^{m} \left(q_i^2 - m_i^2\right) \prod_{j=1}^{n} \left(p_j^2 - m_j^2\right) G(q_1, \ldots, q_m; p_1, \ldots, p_n)$$

$$= \prod_{i=1}^{m} \left(i\sqrt{\mathcal{R}_i}\right) \prod_{j=1}^{n} \left(i\sqrt{\mathcal{R}_j}\right) \quad {}_{\text{out}}\langle q_1, \ldots, q_m | p_1, \ldots, p_n \rangle_{\text{in}}$$

i.e. the $n + m$ particle pole of the Green's function give the S-matrix upto wavefunction normalization factors.

The only complication for fermions and gauge bosons is that one has to contract with spinors $u(p, s)$, $v(p, s)$ and polarization vectors ϵ_μ.

(Careful about unstable particles)

Field Redefinitions

Field redefinitions do not change the S-matrix

$$\langle 0 | T\left\{\phi(x_1)\ldots\phi(x_r)\right\}|0\rangle = \frac{\int D\phi\ \phi(x_1)\ldots\phi(x_r)\ e^{iS(\phi)}}{\int D\phi\ e^{iS(\phi)}}$$

which is computed from

$$Z[J] = \int D\phi\ e^{i\int L(\phi)+J\phi}$$

The numerator and denominator are

$$\frac{\delta}{i\,\delta J(x_1)}\cdots\frac{\delta}{i\,\delta J(x_r)}Z[J], \qquad\qquad Z[J],$$

evaluated at $J=0$.

Make a local field redefinition,

$$\phi'(x) = F(\phi(x)) \qquad \text{e.g. } \phi'(x) = \phi(x) + c_1 \partial^2 \phi(x) + c_2 \phi(x)^3$$

local means all fields at the same x, and a finite number of derivatives.

Change in Lagrangian $L' \leftrightarrow L$:

$$L'(\phi'(x)) = L'(F(\phi(x)) = L(\phi(x))$$

Compute using L or L':

$$Z[J] = \int D\phi \; e^{i \int L(\phi) + J\,\phi} \qquad Z'[J'] = \int D\phi' \; e^{i \int L'(\phi') + J'\,\phi'}$$

which allow us to determine

$$\langle 0 | T \{ \phi(x_1) \ldots \phi(x_r) \} | 0 \rangle \qquad \langle 0 | T \left\{ \phi'(x_1) \ldots \phi'(x_r) \right\} | 0 \rangle$$

Change variables in an integral:

$$Z'[J'] = \int D\phi' \; e^{i\int L'(\phi') + J'\,\phi'} = \int D\phi \left|\frac{\delta\phi'}{\delta\phi}\right| \; e^{i\int L(\phi) + J'\,F(\phi)}$$

The Jacobian $\left|\dfrac{\delta\phi'}{\delta\phi}\right| = 1$ in dim reg up to anomalies

- Computing $Z[J]$ gives Green's functions of ϕ with Lagrangian $L(\phi)$
- Computing $Z'[J']$ gives Green's functions of ϕ' with Lagrangian $L'(\phi')$
- Computing $Z'[J']$ gives Green's functions of $F(\phi)$ with Lagrangian $L(\phi)$
- The Green's functions of ϕ and $F(\phi)$ are different but, as long as $\langle p|F(\phi)|0\rangle \neq 0$, both give the same S-matrix.
- $L'(\phi')$ and $L(\phi)$ give the same S-matrix, i.e. field redefinitions do not change the S-matrix.

Lagrangians which are related by field redefinitions are equivalent, and give the same S-matrix. [note that Green's functions, and individual diagrams, can be different.]

In field theory courses, we study renormalizable Lagrangians with operators of dimension $\leqslant 4$. So the only field redefinitions allowed are

$$\phi_i' = C_{ij}\, \phi_j$$

But we already use this freedom to put the kinetic terms in standard form

$$\frac{1}{2} \partial_\mu \phi_i \, \partial^\mu \phi^i$$

With higher dimension operators, there is a lot more freedom to make field redefinitions.

Equations of Motion

H.D. Politzer, Nucl. Phys. B172 (1980) 349.

Special case of a field redefinition ($\epsilon \ll 1$):

$$\phi(x) = \phi'(x) + \epsilon\, f(\phi'(x))$$

$$S(\phi) = S(\phi') + \epsilon\, f(\phi') \frac{\delta S(\phi')}{\delta \phi'} + \ldots = S(\phi') + \epsilon\, \theta(\phi') + \ldots$$

where θ is the equation of motion operator:

$$E(\phi) = -\partial^2 \phi - m^2 \phi \qquad\qquad \theta(\phi) = f(\phi) E(\phi)$$

$$_{\text{out}}\langle q_1, \ldots, q_m | \theta(z) | p_1, \ldots, p_n \rangle_{\text{in}} = 0$$

because the $\imath^{\text{th}}$ term does not have a pole in $1/(p_i^2 - m_i^2)$. Note that θ can contribute to Green's functions.

This is a special case of field redefinitions — one can eliminate an EOM operator using a field redefinition.

$$L(\phi) + \epsilon f(\phi) E(\phi) \to L(\phi) + \mathcal{O}(\epsilon^2) \qquad \phi \to \phi + \epsilon f(\phi)$$

Dropping EOM terms only true to order ϵ. Field redefinitions can be used to higher order in ϵ, and is the proper way to implement EOM.

One cannot use EOM in the leading order term. Otherwise

$$\overline{\psi}\, i \not{D}\, \psi \to 0$$

using the fermion EOM.

EOM and total derivatives removes, e.g.

$$\left(\phi^\dagger D^2 \phi\right)\left(\phi^\dagger \phi\right), \; D^\mu \left(\phi^\dagger D_\mu \phi\right)\left(\phi^\dagger \phi\right)$$

$O(N)$ Linear Sigma Model

Let ϕ be an N component real scalar field,

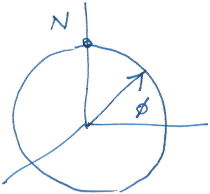
$$\phi = (\phi_1, \ldots, \phi_N)$$

The most general Lagrangian with terms of dimension $\leqslant 4$ is

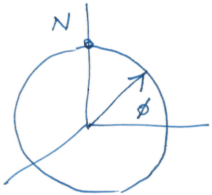
$$L = \frac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} m^2 \phi \cdot \phi - \frac{\lambda}{4}\left(\phi \cdot \phi\right)^2$$

The theory has an $O(N)$ symmetry,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \to g \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}$$



Broken Symmetry



$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \to g \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix}$$

Where $g \in G$ is an $N \times N$ orthogonal matrix, so that

$$\phi \cdot \phi \to \phi g^T \cdot g \phi = \phi \cdot \phi \qquad\qquad g^T g = 1$$

$G = O(N)$ is the symmetry group of the sigma model.

Broken Symmetry

$\lambda \geq 0$ for stability of the scalar potential, but m^2 can have any sign.

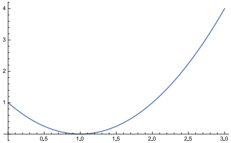
For $m^2 > 0$, the ground state is $\phi = 0$, and the $O(N)$ symmetry is unbroken, because

$$g\,\phi = \phi \qquad \text{for} \qquad \phi = 0$$

For $m^2 < 0$, the Lagrangian can be rewritten as

$$L = \frac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi - \frac{\lambda}{4}(\phi \cdot \phi - v^2)^2, \qquad m^2 = -\lambda v^2$$

and the potential is minimized at $\phi \cdot \phi = v^2$.



Broken Symmetry

A given ϕ can be rotated into another ϕ' provided $\phi \cdot \phi = \phi' \cdot \phi'$. These are called orbits, and form the manifold $S^{N-1} \subset \mathbb{R}^N$.

A field configuration $\phi(x)$ can be rotated to $g\phi(x)$, where g is a constant. (Global rotation)

The ground state of the theory has $\phi \cdot \phi = v^2$, and we have a set of equivalent vacua $\sim S^{N-1}$.

Rotate ϕ using g into a standard form

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} = v\phi_0 \qquad\qquad \phi_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

the North pole of the sphere S^{N-1}. This is a choice of ground state.

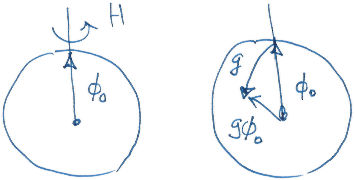
Broken Symmetry

What happens to the $O(N)$ transformations g? There is a subgroup $H \subset G$ of transformations such that

$$h \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix} \qquad\qquad h\phi_0 = \phi_0$$

In this example $H = SO(N-1)$, and is the unbroken symmetry.

The transformations in $G - H$ rotate ϕ, and take you from one vacuum to an equivalent vacuum related by $O(N)$ symmetry. They are the broken generators.

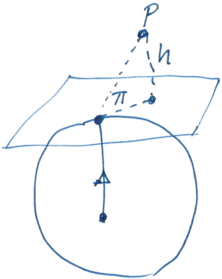


Linear Realization

Let

$$\phi = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_{N-1} \\ v+h \end{bmatrix}, \qquad \boldsymbol{\pi} = (\pi_1, \ldots \pi_{N-1})$$

warning h is the Higgs field and h is also an element of the unbroken group H



Linear Realization

$$L = \frac{1}{2}\partial_\mu h \cdot \partial^\mu h + \frac{1}{2}\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi} - \frac{1}{4}\lambda \left(h^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi} + 2hv\right)^2$$

$$= \frac{1}{2}\partial_\mu h \cdot \partial^\mu h + \frac{1}{2}\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi}$$

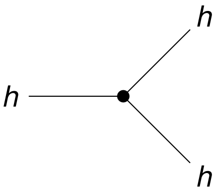
$$- \frac{1}{4}\lambda \left(h^4 + 2h^2(\boldsymbol{\pi}\cdot\boldsymbol{\pi}) + (\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2 + 4vh^3 + +4vh(\boldsymbol{\pi}\cdot\boldsymbol{\pi}) + 4h^2v^2\right)$$

$$m_\pi^2 = 0 \qquad\qquad m_h^2 = 2\lambda v^2$$

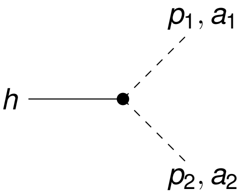
Massless Goldstone bosons and a massive Higgs.

Theory at energies $\ll m_h^2$ is a theory of Goldstone bosons only.

Cubic Vertices



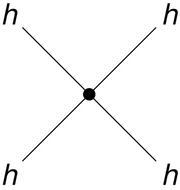
$-6\lambda v$



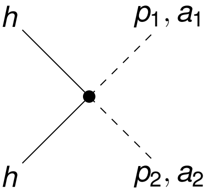
$-2\lambda v$

Theory has two parameters, v and λ.

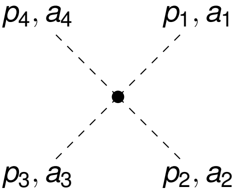
Quartic Vertices



$$-6\lambda$$



$$-2\lambda$$



$$-2\lambda(\delta_{a_1 a_2}\delta_{a_3 a_4} + \delta_{a_1 a_3}\delta_{a_2 a_4} + \delta_{a_1 a_4}\delta_{a_2 a_3})$$

Symmetry Transformation

$$\phi(x) \to g\,\phi(x) \qquad\qquad \phi = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_{N-1} \\ v + h \end{bmatrix}$$

A linear transformation involving v, $h(x)$, $\pi(x)$.

Mixes radial and angular coordinates.

$$m_\pi^2 = 0 \qquad\qquad m_h^2 = 2\lambda v^2$$

At energies much smaller than m_h, one should have an EFT that describes only the dynamics of the Goldstone bosons with derivative couplings, and not mix h, π.

Nonlinear Realizations

Nonlinear realizations focus on the dynamics of the Goldstone bosons.

The general theory was worked out in: Coleman, Wess and Zumino Phys Rev 117 (1969) 2239

The crucial ingredient is invariance of the S-matrix under field redefinitions.

Nonlinear Realization

Look at the $O(N)$ Lagrangian, but write it in a different way. The nonlinear version makes the broken symmetry clear.

We have

$$G = O(N) \qquad \dim G = \dim O(N) = \frac{N(N-1)}{2}$$

broken to

$$H = O(N-1) \qquad \dim H = \dim O(N-1) = \frac{(N-1)(N-2)}{2}$$

Number of massless particles is $N - 1$.

$$\dim G/H = \dim G - \dim H = N - 1 = \dim S^{N-1}$$

This is not an accident. The vacuum states of the theory form the coset space G/H.

A coset is a collection of elements in G. Two elements g_1 and g_2 are in the same coset iff $g_1 = g_2 h$ for some $h \in H$.

In general G/H is not a group.

$$S^1 \sim U(1) \qquad S^3 \sim SU(2)$$

but S^2 and $S^{N \geq 4}$ are not group manifolds.

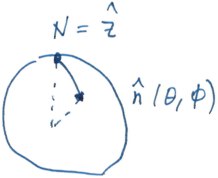
Why do cosets correspond to vacua? Because

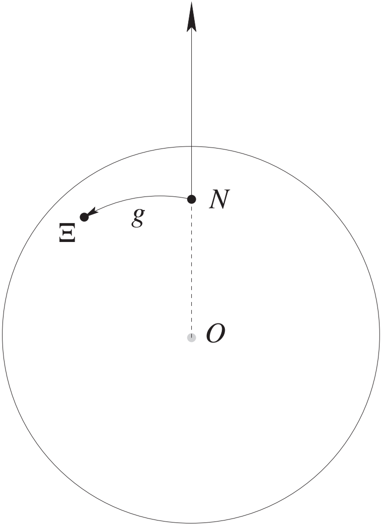
$$g_1\phi_0 = g_2 h\phi_0 = g_2\phi_0 \qquad \text{since} \qquad h\phi_0 = \phi_0$$

Potential energy V is unchanged for motion along an orbit — dim G/H massless modes which are the Goldstone bosons.

For $O(3)$:

$$g = e^{-i\phi J_z} e^{-i\theta J_y} e^{-i\psi J_z} \quad \text{rotates} \quad \hat{z} \to \hat{n}(\theta,\phi) \qquad \forall\psi$$

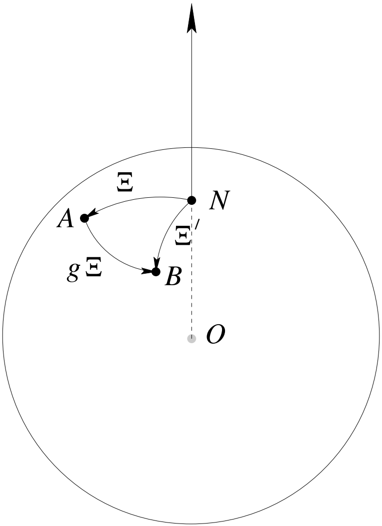




In the $O(3)$ model, a general rotation g takes $\hat{z}$ to some point on the sphere. One can get to this point by a rotation Ξ along a line of longitude. Ξ and g can differ by a rotation around the z axis.

Ξ has as many parameters as the number of Goldstone bosons (flat directions).

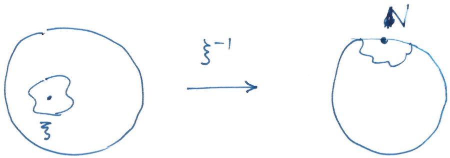
A constant Ξ can be removed, so interactions only depend on $\partial\Xi$.



In the $O(3)$ model, Ξ is a rotation along a line of longitude. Ξ' and $g\,\Xi$ can differ by a rotation around the z axis.

This happens because the manifold G/H is curved.

$$g\,\Xi(x) = \Xi'(x)\,h(x)$$



One can map any point on the vacuum manifold to the origin using Ξ^{-1}. Therefore, can construct a general *G* invariant theory by writing down an *H* invariant theory at $\pi = 0$.

One can construct rotationally invariant interactions on a sphere by rotating any point to the North pole, and making sure the interactions are invariant in a neighborhood of the North pole.

Note that

$$g\,\Xi(x) = \Xi'(x)\,h(x)$$

$h(x)$ depends on x through $\pi(x)$. This looks like a local symmetry, but it is not.

G/H sigma model is a fiber bundle, and has a lot of the mathematical structure of a gauge theory. However, we have $\Xi(x)$ which tells us where we are in group space, which has no analog for a pure gauge theory.

$h(x)$ has been misused — turned into "hidden local symmetry", and claimed to predict properties of vector mesons, etc.

Back to the Linear Sigma Model

One choice is to pick

$$T^a = \left[\begin{array}{cccc|c} 0 & * \cdots * & & * & 0 \\ * & * \cdots * & & * & 0 \\ * & * \cdots * & & * & 0 \\ \hline 0 & & \cdots & 0 & 0 \end{array}\right]$$

$$iX^1 = \left[\begin{array}{cccc|c} 0 & 0 \cdots 0 & & * & 1 \\ * & 0 \cdots 0 & & * & 0 \\ * & 0 \cdots 0 & & * & 0 \\ \hline -1 & & \cdots & 0 & 0 \end{array}\right], \quad \text{etc.}$$

$$\phi(x) = \xi(x) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v + h(x) \end{pmatrix} = [v + h(x)]\, \xi(x) \phi_0$$

Decomposition into angular and radial coordinates.

$$\xi(x) = \exp iX \cdot \pi = \exp \frac{1}{v} \begin{bmatrix} 0 & \dots & 0 & \pi_1 \\ 0 & \dots & 0 & \pi_2 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & \pi_{N-1} \\ -\pi_1 & \dots & -\pi_{N-1} & 0 \end{bmatrix}$$

The Lagrangian is

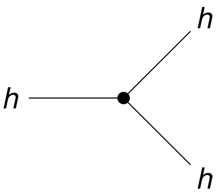
$$L = \frac{1}{2}\partial_\mu h\,\partial^\mu h + \frac{1}{2}(h+v)^2 \phi_{\mathbf{0}}^T \partial_\mu \xi^T \partial^\mu \xi \phi_0 - \frac{1}{4}\lambda\left(h^2 + 2hv\right)^2$$

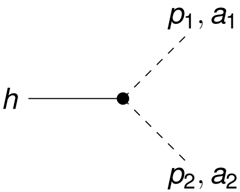
and the potential does not depend on $\boldsymbol{\xi}$.

$$L = \frac{1}{2}\partial_\mu h\,\partial^\mu h - \frac{1}{4}\lambda\left(h^2+2hv\right)^2 + \frac{1}{2}\left(1+\frac{h}{v}\right)^2\left[\partial_\mu \boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi}\right]$$
$$+\frac{1}{6v^2}\left(1+\frac{h}{v}\right)^2\left[(\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi})^2 - (\boldsymbol{\pi}\cdot\boldsymbol{\pi})(\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi})\right]$$
$$+\ldots$$

An infinite series of terms, and all π interactions depend on at least one $\partial\pi$.

Cubic Vertices



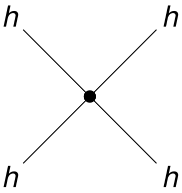


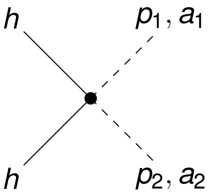
$$-6\lambda v$$

$$-\frac{2}{v} p_1 \cdot p_2 \delta_{a_1 a_2}$$

Momentum dependent $h\pi^2$ vertex.

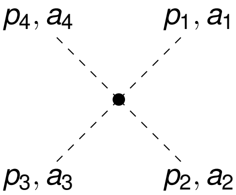
Quartic Vertices





-6λ

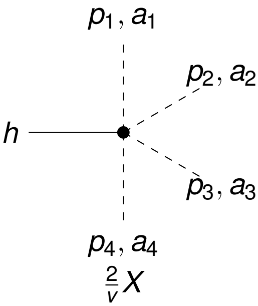
$-\frac{2}{v^2} p_1 \cdot p_2 \delta_{a_1 a_2}$

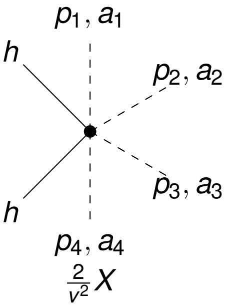


X

$$X = \frac{1}{3v^2}\Big\{ \delta_{a_1 a_2}\delta_{a_3 a_4}\left[2p_1 \cdot p_2 + 2p_3 \cdot p_4 - (p_1+p_2)\cdot(p_3+p_4)\right] + \delta_{a_1 a_3}\delta_{a_2 a_4}\left[2p_1 \cdot p_3 + 2p_2 \cdot p_4 - (p_1+p_3)\cdot(p_2+p_4)\right] + \delta_{a_1 a_4}\delta_{a_2 a_3}\left[2p_1 \cdot p_4 + 2p_2 \cdot p_3 - (p_1+p_4)\cdot(p_2+p_3)\right]\Big\}$$

More Fields


p_1, a_1
p_2, a_2
h
p_3, a_3
p_4, a_4
$\frac{2}{v}X$


p_1, a_1
h
p_2, a_2
p_3, a_3
h
p_4, a_4
$\frac{2}{v^2}X$

Both forms give the same S-matrix.

The number and type of graphs are different in the two theories.

Graphs which look the same have different expressions in the two theories

There are many ways of writing the nonlinear version, and individual graphs depend on the parameterization, but the S-matrix does not.

Can make a field redefinition

$$\phi = \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_{N-1} \\ v + h \end{bmatrix} = [v + h(x)]\, \xi(x)\phi_0$$

so the two are completely equivalent.

Standard Model

$$\phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} \qquad \widetilde{\phi} = \begin{bmatrix} \left(\phi^0\right)^\dagger \\ -\left(\phi^+\right)^\dagger \end{bmatrix} = \epsilon_{ij}\phi_j^\dagger$$

$$\phi = \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi_2 + i\varphi_1 \\ \varphi_0 - i\varphi_3 \end{bmatrix}, \quad \Phi = \varphi_0 + i\boldsymbol{\tau}\cdot\boldsymbol{\varphi} \quad = \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi_0 + i\varphi_3 & i\varphi_1 + \varphi_2 \\ i\varphi_1 - \varphi_2 & \varphi_0 - i\varphi_3 \end{bmatrix}$$

$$\phi \to L\phi \qquad \Phi \to L\Phi \qquad SU(2)$$

$$\phi \to e^{i\alpha/2}\phi \qquad \Phi \to \Phi e^{-i\alpha\tau_3/2} \qquad U(1)$$

The SM has an $O(4)$ symmetry in the Higgs sector,

$$\phi^\dagger \phi = \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_0^2 \qquad\qquad V(\phi^\dagger\phi)$$

$$\Phi \to L \Phi R^\dagger \qquad\qquad SU(2)_L \times SU(2)_R \sim O(4)$$

with custodial $SU(2)$ symmetry unbroken

$$\langle \Phi \rangle = \frac{v}{\sqrt{2}} \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \qquad \langle \Phi \rangle \to U \Phi U^\dagger, \qquad L = R = U$$

even after EW symmetry breaking.

$$\phi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_0 \end{bmatrix} \to (v+h)\, e^{iX\cdot\pi} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

For spontaneously broken symmetry:

$$\phi \to e^{iX\cdot\pi} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad h$$

No longer a connection between π and h, or between v and h.

In the SM, the Higgs does two things

- It breaks the gauge symmetry and gives the W and Z a mass
- It has Yukawa couplings which give the fermion a mass $\leftarrow$ problematic for other alternatives

Technicolor: Spontaneously broken $SU(2)_L \times SU(2)_R$ chiral symmetry by scaling $f_\pi \to v$.

$$h \bar{l} e \to \overline{Q} Q \bar{l} e \qquad \text{extended techicolor, } \ldots$$

Composite Higgs: Some kind of strong interaction with $G \to H$ where $SU(2) \times U(1) \subset H$, and the SM doublet ϕ is a Goldstone boson. If ϕ couples to quarks and leptons, need them to be composite. This is ignored by just writing an effective operator

$$\bar{l}\, \Xi\, e \qquad \Xi \;\; \text{a composite of unspecified strong dynamics}$$

One can write and EFT using

1. ϕ, i.e. the SM + higher dimension operators
2. Ξ, h + higher dimension operators

In case (2),

$$L = -m_W^2 \left(1 + \frac{h}{v}\right)^2 W^+ W^- \quad \rightarrow -m_W^2 \left(1 + 2c_1 \frac{h}{v} + c_2 \frac{h^2}{v^2} + \ldots\right) W^+ W^-$$

Yukawa interactions with a function $y(h/v)$ of h/v.

$$y\left(\frac{h}{v}\right) \bar{l}\, \Xi\, \phi_0\, e$$

Case (1) is a special case of (2)