`Novel' unitarity problem in the UV (at ~100 TeV)

Perturbative growth of Electro-Weak high-multiplicity processes and probes of fundamental physics

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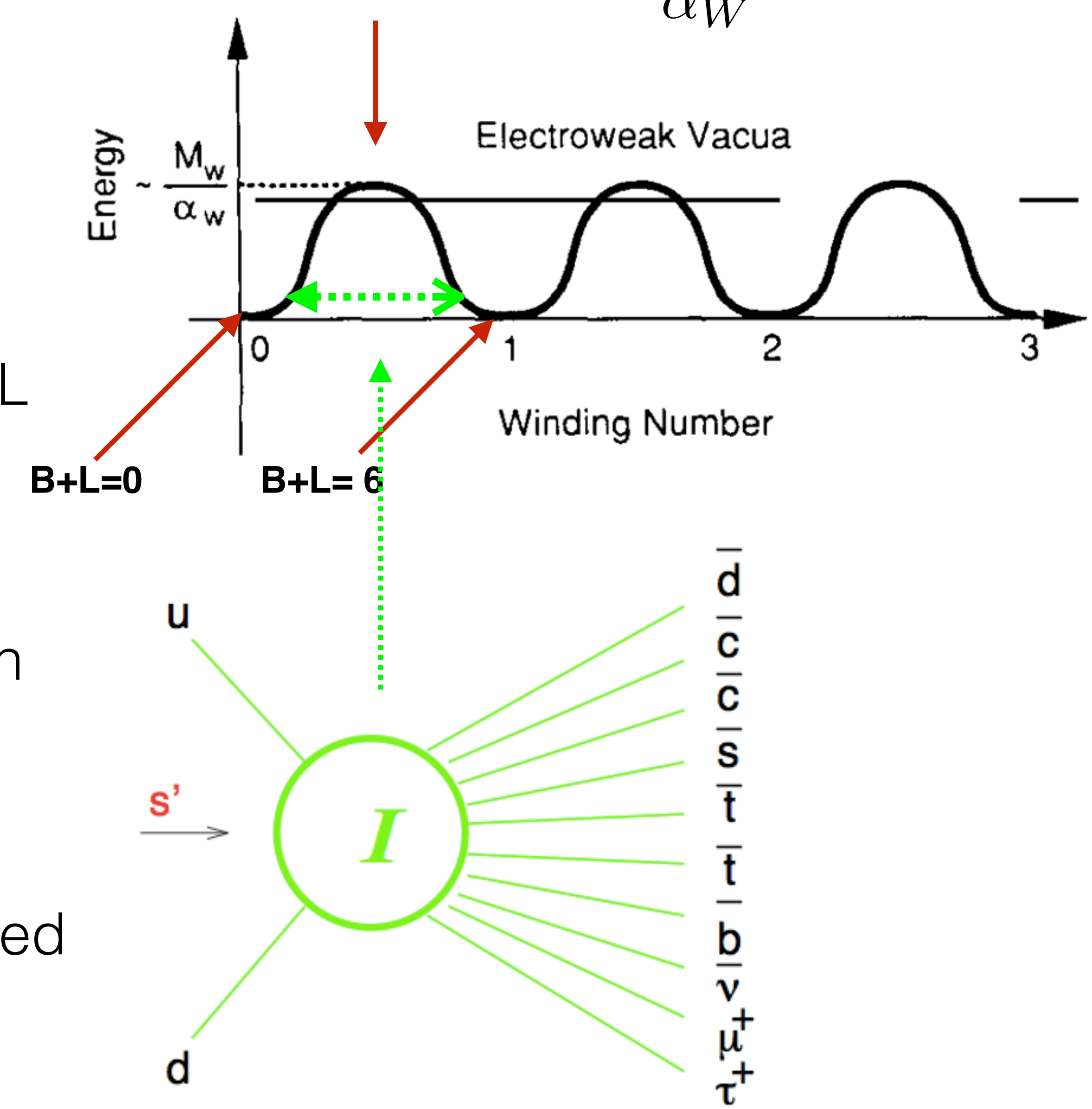
I. Motivation

- Start with a simple motivation (next 2 pages):
- Sphaleron mass is a new scale in the SM at ~10 TeV so that at >> 30 TeV a possibility of new phenomena in the SM.
- => Our trusted weakly coupled perturbation theory breaks down
- [Interesting but technically irrelevant for the rest of the talk]
- —> The main part of the talk will be based on perturbation theory; its high-E behaviour presents an easier unsolved problem to tackle.

Baryon + Lepton number violation (Recall)

$$E_{\rm sph} = c_{\rm sph}\,\frac{m_W}{\alpha_W} \approx 10\,{\rm TeV}$$

- Electroweak vacuum has a nontrivial structure (!) [SU(2)-sector]
- The saddle-point at the top of the barrier is the *sphaleron.* New EW scale ~ 10 TeV
- Transitions between the vacua change B+L (result of the ABJ anomaly):
  Delta (B+L)= 3 x (1+1) ; Delta (B-L)=0
- *Instantons* are tunnelling solutions between the vacua. They mediate B+L violation
- 3 x (1 lepton + 3 quarks) = 12 fermions
  12 left-handed fermion doublets are involved
- There are EW processes which are not described by perturbation theory!



$$q + q \to 7\bar{q} + 3\bar{l} + n_W W + n_Z Z + n_h H$$

- Electroweak sector of the SM is always seen as perturbative. If these instanton processes can be detected —> a truly remarkable breakthrough in realising & understanding non-perturbative EW dynamics

- **Numbers of W’s, Z’s and H’s** produced in the final state at 30-100 TeV energies is allowed to be large, ~ 1/alpha => a technical consequence of this fact is that the instanton crossection receives an **exponential enhancement with energy**

- The B+L processes are accompanied by ~50 EW vector & H bosons; charged Lepton number can also be measured —> unique experimental signature of the final state

- The rate of the B+L processes is still not known theoretically. There are optimistic phenomenological models with ~pb or ~fb crossections, and there are pessimistic models with unobservable rates even at infinite energy.

- Very hard theoretical problem, new computational methods are needed.

- Since the final state is essentially backgroundless, the obesrvability of the rate can be always settled experimentally.

- B+L processes provide **physics opportunities** which are **completely unique to the very high energy pp machine**. This cannot be done anywhere else

II. On Mass Threshold at Tree Level A(1* -> n)

- Recursion relations & classical solutions general technique - Brown 1992
- Results in factorial growth of amplitudes in:
- (a) unbroken phi^4 theory
- (b) scalar theory with the VEV
- (c) Gauge-Higgs theory (spontaneously broken gauge theory)
- generalise to more realistic 2 -> n

Tree-level n-point Amplitudes on mass threshold

The amplitude $\mathcal{A}_{1\to n}$ for the field ϕ to create n particles in the ϕ^4 theory,

$$\mathcal{L}_\rho(\phi) \,=\, \frac{1}{2}\,(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 - \frac{1}{4}\lambda\phi^4 + \rho\,\phi\,,$$

is derived by applying the LSZ reduction technique:

$$\langle n|\phi(x)|0\rangle \,=\, \lim_{\rho\to 0}\left[\prod_{j=1}^{n}\lim_{p_j^2\to M^2}\int d^4x_j e^{ip_j\cdot x_j}(M^2-p_j^2)\frac{\delta}{\delta\rho(x_j)}\right]\langle 0_{\rm out}|\phi(x)|0_{\rm in}\rangle_\rho\,.$$

Tree-level approximation is obtained via $\langle 0_{\rm out}|\phi(x)|0_{\rm in}\rangle_\rho \longrightarrow \phi_{\rm cl}(x)$ where $\phi_{\rm cl}(x)$ is a solution to the classical field equation.

On **mass threshold** limit all outgoing particles are produced at rest, $\vec{p}_j = 0$ and we set all $p_j^\mu = (\omega, \vec{0})$ and $\rho(x) = \rho(t) = \rho_0(\omega)\,e^{i\omega t}$. Hence,

$$(M^2-p_j^2)\frac{\delta}{\delta\rho(x_j)} \longrightarrow (M^2-\omega^2)\frac{\delta}{\delta\rho(t_j)} = \frac{\delta}{\delta z(t_j)}\,,$$

$$z(t) \,:=\, \frac{\rho_0(\omega)\,e^{i\omega t}}{M^2-\omega^2-i\epsilon} \,:=\, z_0\;e^{i\omega t}\,, \quad z_0 = \text{finite const}$$

Tree-level n-point Amplitudes on mass threshold

Take the required on-shell limit $\omega \to M$ simultaneously with sending the amplitude of the source to zero, $\rho_0(\omega) \to 0$ such that z_0 remains finite.

For each external leg operator acting on $\phi_{\rm cl}$

$$\int d^4x_j e^{ip_j\cdot x_j}(M^2 - p_j^2)\frac{\delta}{\delta\rho(x_j)} \cdot \phi_{\rm cl}(t) \;=\; e^{i\omega t}\,\frac{\partial\phi_{\rm cl}}{\partial z(t)} \;=\; \frac{\partial\phi_{\rm cl}}{\partial z_0}\,.$$

The tree-level amplitude $\mathcal{A}_{1\to n}$ at the n-particle threshold is thus given by

$$\mathcal{A}_{1\to n} \;=\; \langle n|\phi(0)|0\rangle \;=\; \left(\frac{\partial}{\partial z}\right)^n \phi_{\rm cl}\Big|_{z=0}\;.$$

To give the generating function of amplitudes at multiparticle thresholds, the solution must contain only the positive frequency components of the form e^{+inMt} where n is the number of final state particles in the amplitude $\mathcal{A}_{1\to n}$.

Thus, the solution we are after is given by the Taylor expansion in $z(t)$,

$$\phi_{\rm cl}(t) \;=\; z(t) \,+\, \sum_{n=2}^{\infty} d_n\, z(t)^n$$

SUMMARY: Tree-level threshold amplitudes in phi^4

Brown 9209203

The generating function of tree amplitudes on multiparticle thresholds is a classical solution. It solves an ordinary differential equation with no source term,

$$d_t^2 \phi + M^2 \phi + \lambda \phi^3 \;=\; 0\,.$$

The solution contains only positive frequency harmonics, i.e. the Taylor expansion in $z(t)$,

$$\phi_{\rm cl}(t) \;=\; z(t) \,+\, \sum_{n=2}^{\infty} d_n \, z(t)^n$$

Coefficients d_n determine the actual amplitudes by differentiation w.r.t. z,

$$\mathcal{A}_{1\to n} \;=\; \left(\frac{\partial}{\partial z}\right)^{n} \phi_{\rm cl} \Big|_{z=0} \;=\; n!\, d_n \qquad \textbf{Factorial growth!!}$$

$$\phi_{\rm cl}(t) \;=\; \frac{z(t)}{1-\frac{\lambda}{8M^2}\, z(t)^2} \qquad \mathcal{A}_{1\to n} \;=\; n!\, \left(\frac{\lambda}{8M^2}\right)^{\frac{n-1}{2}}$$

SUMMARY: apply to phi^4 with SSB (Higgs-like)

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$$\mathcal{L}(h) \; = \; \frac{1}{2}\left(\partial h\right)^2 - \frac{\lambda}{4}\left(h^2 - v^2\right)^2 \; ,$$

The classical equation for the spatially uniform field $h(t)$,

$$d_t^2 h \; = \; -\lambda\, h^3 + \lambda v^2\, h \, ,$$

again has a closed-form solution with correct initial conditions $h_{\rm cl} = v + z + \ldots$

$$h_{\rm cl}(t) \; = \; v\, \frac{1 + \frac{z(t)}{2v}}{1 - \frac{z(t)}{2v}} \, , \quad \text{where} \quad z(t) \; = \; z_0\, e^{iM_h t} \; = \; z_0\, e^{i\sqrt{2\lambda}\, v\, t}$$

$$h_{\rm cl}(t) \; = \; 2v \sum_{n=0}^{\infty} \left(\frac{z(t)}{2v}\right)^n d_n \; = \; v \, + \, 2v \sum_{n=1}^{\infty} \left(\frac{z(t)}{2v}\right)^n \, ,$$

i.e. with $d_0 = 1/2$ and all $d_{n \geq 1} = 1$.

$$\mathcal{A}_{1\to n} \; = \; \left(\frac{\partial}{\partial z}\right)^n h_{\rm cl}\Big|_{z=0} \; = \; n!\,(2v)^{1-n} \qquad \textbf{Factorial growth!!}$$

Gauge-Higgs theory: Tree-level threshold amplitudes

VVK 1404.4876

$$\mathcal{L} = -\frac{1}{4} F^{a\,\mu\nu} F^{a}_{\mu\nu} + |D_\mu H|^2 - \lambda \left(|H|^2 - \frac{v^2}{2} \right)^2 , \qquad H = \frac{1}{\sqrt{2}}\,(0, h)$$

The particle content is given by the neutral Higgs h, and a triplet of massive vector bosons $W^\pm$ and Z^0

$$M_h = \sqrt{2\lambda}\, v \simeq 125.66\,\mathrm{GeV}\,, \quad M_V = \frac{gv}{2} \simeq 80.384\,\mathrm{GeV}\,, \quad \kappa := \frac{g}{2\sqrt{2\lambda}} = \frac{M_V}{M_h}$$

Multiparticle threshold limit: a single virtual state decays into n Higgs bosons and m vector bosons, all with vanishing spatial momenta:

$$p^\mu_{\rm in} = (nM_h + mM_V, \vec{0}) \to \sum_{j=1}^{n} p^\mu_j + \sum_{k=1}^{m} p^\mu_k \,, \quad \text{where} \quad p^\mu_j = (M_h, \vec{0})\,, \quad p^\mu_k = (M_V, \vec{0})\,.$$

The transversality condition, $p^\mu A^a_\mu = 0$, allows us to set $A^a_0 = 0$.

Degrees of freedom left: $\{h(t), A^a_m(t)\}$ with $m = 1, 2, 3 = T_1, T_2, L$ polarisations of the triplet $(a = 1, 2, 3)$ of massive vector bosons.

Gauge-Higgs theory: Tree-level threshold amplitudes

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The Lagrangian with spatially-independent fields,

$$\mathcal{L} = \frac{1}{2}(d_t A^a_m)^2 + \frac{1}{2}(d_t h)^2 - \frac{g^2}{8} h^2 (A^a_m)^2 - \frac{g^2}{4}\left((A^a_m)^2 (A^b_n)^2 - (A^a_m A^a_n)^2\right) - \frac{\lambda}{4}\left(h^2 - v^2\right)^2$$

results in the equations of motion for $h_{\rm cl}(t)$ and $A^a_{m\,{\rm cl}}(t)$:

$$\begin{aligned} d_t^2 h &= -\lambda\, h^3 + \lambda v^2\, h - \frac{g^2}{4}(A^a_m)^2 h\,, \\ d_t^2 A^a_m &= -\frac{g^2}{4} h^2 A^a_m - g^2\left((A^b_n)^2 A^a_m - (A^a_n A^b_n) A^b_m\right) \end{aligned}$$

Final simplification – drop transverse polarisations from the final state:

$$d_t^2 h = -\lambda\, h^3 + \lambda v^2\, h - \frac{g^2}{4}(A^a_L)^2 h\,, \quad d_t^2 A^a_L = -\frac{g^2}{4} h^2 A^a_L$$

This system of equations can now be solved iteratively in complex variables

$$z(t) = z_0\, e^{iM_h t}\,, \qquad w^a(t) = w^a_0\, e^{iM_V t}\,.$$

Gauge-Higgs theory: Tree-level threshold amplitudes

VVK 1404.4876

These equations are solved by iterations (numerically) with Mathematica. The double Taylor expansion of the generating functions takes the form:

$$
\begin{aligned}
h_{\rm cl}(z, w^a) &= 2v \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} d(n, 2k) \left(\frac{z}{2v}\right)^n \left(\frac{w^a w^a}{(2v)^2}\right)^k , \\
A^a_{L\,{\rm cl}}(z, w^a) &= w^a \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a(n, 2k) \left(\frac{z}{2v}\right)^n \left(\frac{w^a w^a}{(2v)^2}\right)^k ,
\end{aligned}
$$

where $d(n, 2k)$ and $a(n, 2k)$ are determined from the iterative solution of EOM.

By repeatedly differentiating these with respect to z and w^a for the Higgs to n Higgses and m longitudinal Z bosons threshold amplitude we get,

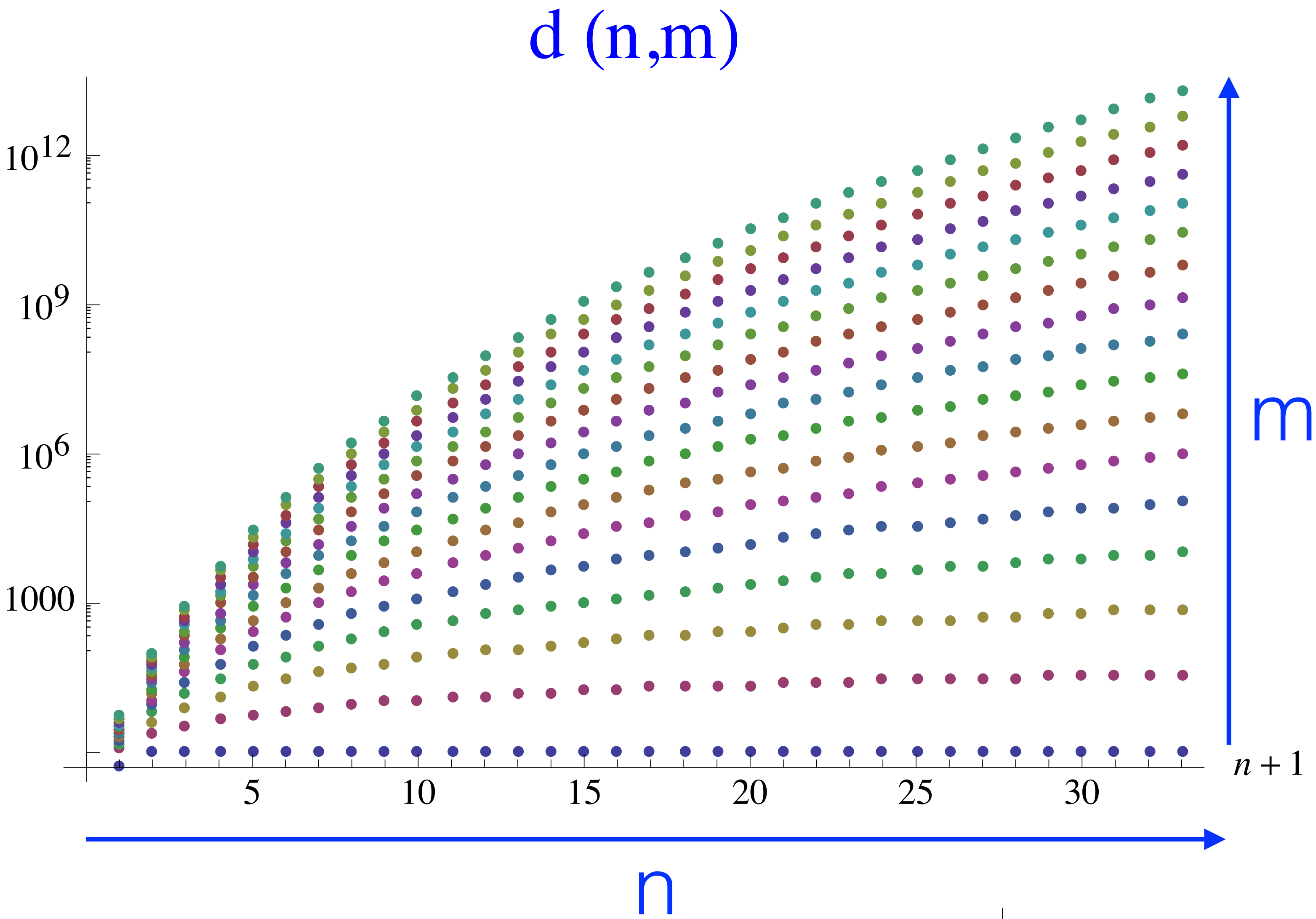
$$
\mathcal{A}(h \to n \times h + m \times Z_L) = (2v)^{1-n-m}\, n!\, m!\, d(n, m) ,
$$

and for the longitudinal Z decaying into n Higgses and $m + 1$ vector bosons,

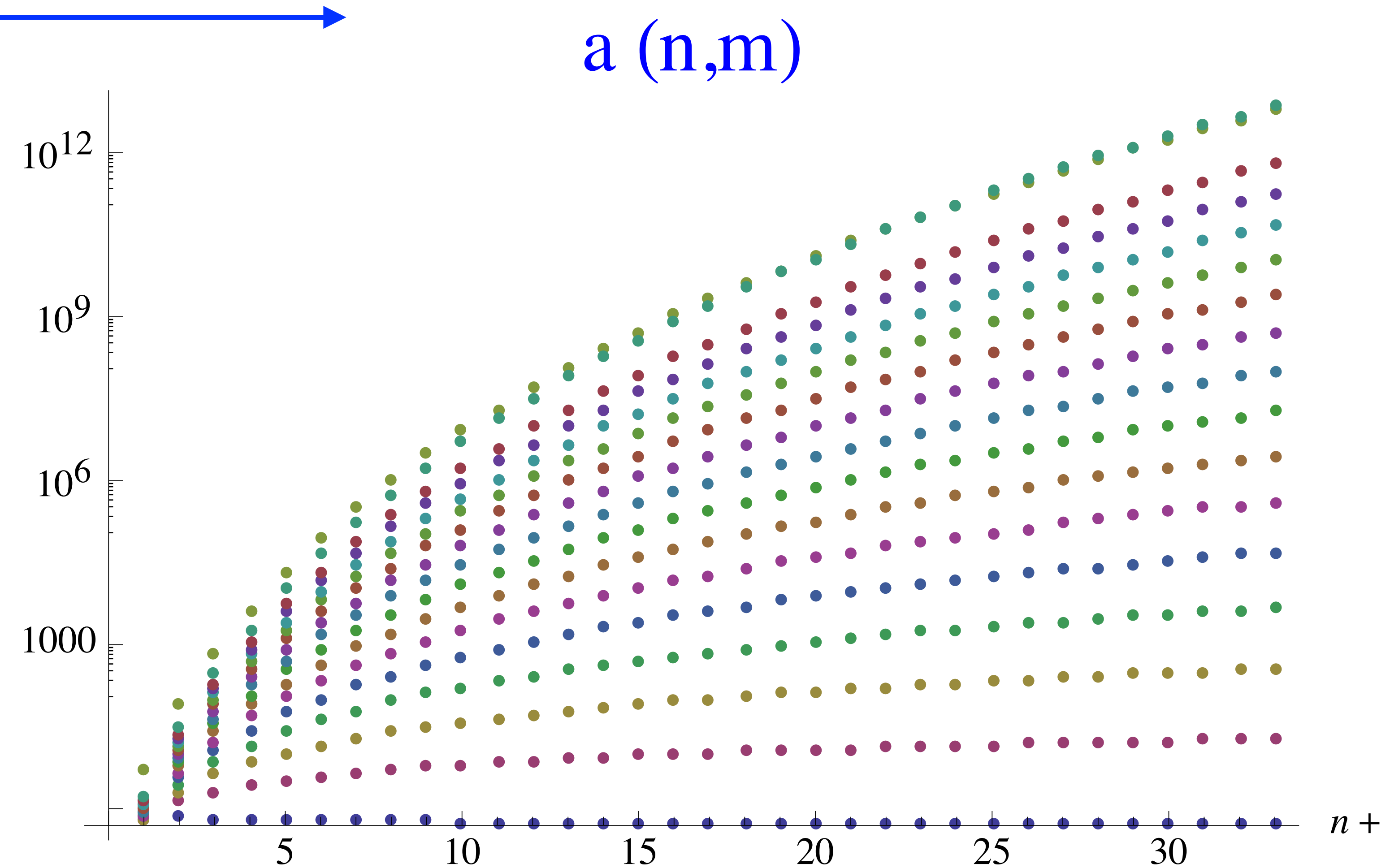
$$
\mathcal{A}(Z_L \to n \times h + (m+1) \times Z_L) = \frac{1}{(2v)^{n+m}}\, n!\, (m+1)!\, a(n, m) .
$$

Factorial growth reemains (in n and in m) !

d(n,m)	m = 0	2	4	6	8	10	12	14	16	m = 18	20	22	24	26	28	30	32
n=0	$\frac{1}{2}$	1.285	1.462	1.604	1.763	1.937	2.127	2.335	2.563	2.812	3.085	3.384	3.711	4.07	4.463	4.893	5.365
1	1	2.508	4.304	6.422	8.914	1.183×10^{1}	1.523×10^{1}	1.917×10^{1}	2.373×10^{1}	2.899×10^{1}	3.505×10^{1}	4.2×10^{1}	4.996×10^{1}	5.907×10^{1}	6.946×10^{1}	8.131×10^{1}	9.479×10^{1}
2	1	3.69	8.53	1.607×10^{1}	2.696×10^{1}	4.196×10^{1}	6.199×10^{1}	8.809×10^{1}	1.215×10^{2}	1.637×10^{2}	2.162×10^{2}	2.812×10^{2}	3.607×10^{2}	4.575×10^{2}	5.745×10^{2}	7.152×10^{2}	8.836×10^{2}
3	1	4.868	1.414×10^{1}	3.215×10^{1}	6.327×10^{1}	1.131×10^{2}	1.886×10^{2}	2.988×10^{2}	4.545×10^{2}	6.694×10^{2}	9.601×10^{2}	1.347×10^{3}	1.854×10^{3}	2.512×10^{3}	3.355×10^{3}	4.427×10^{3}	5.779×10^{3}
4	1	6.045	2.112×10^{1}	5.628×10^{1}	1.271×10^{2}	2.566×10^{2}	4.772×10^{2}	8.338×10^{2}	1.387×10^{3}	2.218×10^{3}	3.433×10^{3}	5.169×10^{3}	7.602×10^{3}	1.096×10^{4}	1.552×10^{4}	2.164×10^{4}	2.976×10^{4}
5	1	7.22	2.947×10^{1}	9.006×10^{1}	2.297×10^{2}	5.167×10^{2}	1.06×10^{3}	2.027×10^{3}	3.661×10^{3}	6.319×10^{3}	1.05×10^{4}	1.689×10^{4}	2.643×10^{4}	4.039×10^{4}	6.045×10^{4}	8.883×10^{4}	1.284×10^{5}
6	1	8.394	3.92×10^{1}	1.351×10^{2}	3.839×10^{2}	9.528×10^{2}	2.138×10^{3}	4.439×10^{3}	8.656×10^{3}	1.604×10^{4}	2.847×10^{4}	4.875×10^{4}	8.09×10^{4}	1.307×10^{5}	2.062×10^{5}	3.185×10^{5}	4.828×10^{5}
7	1	9.568	5.031×10^{1}	1.93×10^{2}	6.047×10^{2}	1.641×10^{3}	$4.\times10^{3}$	8.962×10^{3}	1.876×10^{4}	3.716×10^{4}	7.021×10^{4}	1.275×10^{5}	2.237×10^{5}	3.808×10^{5}	6.316×10^{5}	1.023×10^{6}	1.624×10^{6}
8	1	1.074×10^{1}	6.278×10^{1}	2.654×10^{2}	9.089×10^{2}	2.678×10^{3}	7.045×10^{3}	1.695×10^{4}	3.792×10^{4}	7.992×10^{4}	1.602×10^{5}	3.074×10^{5}	5.686×10^{5}	1.018×10^{6}	1.771×10^{6}	3.003×10^{6}	4.978×10^{6}
9	1	1.191×10^{1}	7.662×10^{1}	3.538×10^{2}	1.315×10^{3}	4.181×10^{3}	1.181×10^{4}	3.035×10^{4}	7.229×10^{4}	1.616×10^{5}	3.424×10^{5}	6.929×10^{5}	1.347×10^{6}	2.53×10^{6}	4.607×10^{6}	8.163×10^{6}	1.411×10^{7}
10	1	1.309×10^{1}	9.184×10^{1}	4.599×10^{2}	1.844×10^{3}	6.292×10^{3}	1.898×10^{4}	5.194×10^{4}	1.312×10^{5}	3.101×10^{5}	6.926×10^{5}	1.474×10^{6}	3.007×10^{6}	5.91×10^{6}	1.124×10^{7}	2.077×10^{7}	3.738×10^{7}
11	1	1.426×10^{1}	1.084×10^{2}	5.853×10^{2}	2.517×10^{3}	9.177×10^{3}	2.947×10^{4}	8.55×10^{4}	2.283×10^{5}	5.689×10^{5}	1.336×10^{6}	2.983×10^{6}	6.37×10^{6}	1.308×10^{7}	2.595×10^{7}	4.992×10^{7}	9.339×10^{7}
12	1	1.543×10^{1}	1.264×10^{2}	7.316×10^{2}	3.36×10^{3}	1.303×10^{4}	4.437×10^{4}	1.361×10^{5}	3.832×10^{5}	1.004×10^{6}	2.473×10^{6}	5.78×10^{6}	1.289×10^{7}	2.762×10^{7}	5.705×10^{7}	1.141×10^{8}	2.216×10^{8}
13	1	1.66×10^{1}	1.457×10^{2}	9.003×10^{2}	4.398×10^{3}	1.809×10^{4}	6.508×10^{4}	2.104×10^{5}	6.228×10^{5}	1.712×10^{6}	4.415×10^{6}	1.078×10^{7}	2.508×10^{7}	5.593×10^{7}	1.201×10^{8}	2.493×10^{8}	5.02×10^{8}
14	1	1.777×10^{1}	1.664×10^{2}	1.093×10^{3}	5.66×10^{3}	2.459×10^{4}	9.327×10^{4}	3.17×10^{5}	9.843×10^{5}	2.832×10^{6}	7.63×10^{6}	1.943×10^{7}	4.707×10^{7}	1.091×10^{8}	2.433×10^{8}	5.236×10^{8}	1.092×10^{9}
15	1	1.894×10^{1}	1.885×10^{2}	1.312×10^{3}	7.175×10^{3}	3.285×10^{4}	1.31×10^{5}	4.669×10^{5}	1.517×10^{6}	4.56×10^{6}	1.281×10^{7}	3.397×10^{7}	8.556×10^{7}	2.059×10^{8}	4.76×10^{8}	1.061×10^{9}	2.287×10^{9}
16	1	2.011×10^{1}	2.119×10^{2}	1.557×10^{3}	8.974×10^{3}	4.319×10^{4}	1.805×10^{5}	6.737×10^{5}	2.287×10^{6}	7.168×10^{6}	2.097×10^{7}	5.78×10^{7}	1.511×10^{8}	3.771×10^{8}	9.026×10^{8}	2.08×10^{9}	4.635×10^{9}
17	1	2.129×10^{1}	2.367×10^{2}	1.832×10^{3}	1.109×10^{4}	5.597×10^{4}	2.449×10^{5}	9.543×10^{5}	3.378×10^{6}	1.102×10^{7}	3.353×10^{7}	9.592×10^{7}	2.6×10^{8}	6.719×10^{8}	1.663×10^{9}	3.961×10^{9}	9.11×10^{9}
18	1	2.246×10^{1}	2.629×10^{2}	2.137×10^{3}	1.357×10^{4}	7.161×10^{4}	3.272×10^{5}	1.33×10^{6}	4.9×10^{6}	1.662×10^{7}	5.248×10^{7}	1.557×10^{8}	4.369×10^{8}	1.168×10^{9}	2.987×10^{9}	7.342×10^{9}	1.741×10^{10}
19	1	2.363×10^{1}	2.904×10^{2}	2.475×10^{3}	1.643×10^{4}	9.057×10^{4}	4.314×10^{5}	1.825×10^{6}	6.991×10^{6}	2.461×10^{7}	8.056×10^{7}	2.474×10^{8}	7.184×10^{8}	1.984×10^{9}	5.237×10^{9}	1.327×10^{10}	3.243×10^{10}
20	1	2.48×10^{1}	3.193×10^{2}	2.846×10^{3}	1.972×10^{4}	1.133×10^{5}	5.62×10^{5}	2.471×10^{6}	9.823×10^{6}	3.585×10^{7}	1.215×10^{8}	3.86×10^{8}	1.158×10^{9}	3.3×10^{9}	8.983×10^{9}	2.346×10^{10}	5.9×10^{10}
21	1	2.597×10^{1}	3.496×10^{2}	3.252×10^{3}	2.349×10^{4}	1.405×10^{5}	7.238×10^{5}	3.303×10^{6}	1.361×10^{7}	5.143×10^{7}	1.803×10^{8}	5.916×10^{8}	1.832×10^{9}	5.383×10^{9}	1.51×10^{10}	4.059×10^{10}	1.05×10^{11}
22	1	2.714×10^{1}	3.813×10^{2}	3.695×10^{3}	2.777×10^{4}	1.726×10^{5}	9.227×10^{5}	4.364×10^{6}	1.862×10^{7}	7.276×10^{7}	2.635×10^{8}	8.925×10^{8}	2.849×10^{9}	8.627×10^{9}	2.491×10^{10}	6.888×10^{10}	1.831×10^{11}
23	1	2.831×10^{1}	4.143×10^{2}	4.177×10^{3}	3.261×10^{4}	2.103×10^{5}	1.165×10^{6}	5.705×10^{6}	2.517×10^{7}	1.016×10^{8}	3.798×10^{8}	1.326×10^{9}	4.362×10^{9}	1.36×10^{10}	4.038×10^{10}	1.148×10^{11}	3.134×10^{11}
24	1	2.948×10^{1}	4.487×10^{2}	4.699×10^{3}	3.806×10^{4}	2.543×10^{5}	1.459×10^{6}	7.384×10^{6}	3.366×10^{7}	1.402×10^{8}	5.403×10^{8}	1.944×10^{9}	6.581×10^{9}	2.11×10^{10}	6.44×10^{10}	1.88×10^{11}	5.269×10^{11}
25	1	3.065×10^{1}	4.845×10^{2}	5.262×10^{3}	4.417×10^{4}	3.054×10^{5}	1.811×10^{6}	9.471×10^{6}	4.454×10^{7}	1.913×10^{8}	7.595×10^{8}	2.813×10^{9}	9.794×10^{9}	3.227×10^{10}	1.012×10^{11}	3.031×10^{11}	8.715×10^{11}
26	1	3.182×10^{1}	5.216×10^{2}	5.869×10^{3}	5.098×10^{4}	3.644×10^{5}	2.232×10^{6}	1.204×10^{7}	5.84×10^{7}	2.584×10^{8}	1.056×10^{9}	4.021×10^{9}	1.439×10^{10}	4.87×10^{10}	1.567×10^{11}	4.815×10^{11}	1.419×10^{12}
27	1	3.299×10^{1}	5.601×10^{2}	6.52×10^{3}	5.854×10^{4}	4.322×10^{5}	2.731×10^{6}	1.519×10^{7}	7.588×10^{7}	3.456×10^{8}	1.452×10^{9}	5.684×10^{9}	2.089×10^{10}	7.255×10^{10}	2.394×10^{11}	7.543×10^{11}	2.277×10^{12}
28	1	3.416×10^{1}	5.999×10^{2}	7.219×10^{3}	6.692×10^{4}	5.097×10^{5}	3.32×10^{6}	1.902×10^{7}	9.778×10^{7}	4.579×10^{8}	1.977×10^{9}	7.95×10^{9}	2.999×10^{10}	1.068×10^{11}	3.613×10^{11}	1.166×10^{12}	3.605×10^{12}
29	1	3.533×10^{1}	6.412×10^{2}	7.965×10^{3}	7.617×10^{4}	5.979×10^{5}	4.012×10^{6}	2.365×10^{7}	1.25×10^{8}	6.017×10^{8}	2.668×10^{9}	1.101×10^{10}	4.259×10^{10}	1.555×10^{11}	5.389×10^{11}	1.781×10^{12}	5.634×10^{12}
30	1	3.65×10^{1}	6.838×10^{2}	8.761×10^{3}	8.634×10^{4}	6.98×10^{5}	4.819×10^{6}	2.921×10^{7}	1.587×10^{8}	7.842×10^{8}	3.569×10^{9}	1.51×10^{10}	5.988×10^{10}	2.24×10^{11}	7.948×10^{11}	2.688×10^{12}	8.698×10^{12}
31	1	3.767×10^{1}	7.278×10^{2}	9.608×10^{3}	9.75×10^{4}	8.11×10^{5}	5.757×10^{6}	3.586×10^{7}	$2.\times10^{8}$	1.014×10^{9}	4.734×10^{9}	2.053×10^{10}	8.341×10^{10}	3.195×10^{11}	1.16×10^{12}	4.012×10^{12}	1.327×10^{13}
32	1	3.884×10^{1}	7.731×10^{2}	1.051×10^{4}	1.097×10^{5}	9.382×10^{5}	6.842×10^{6}	4.376×10^{7}	2.505×10^{8}	1.303×10^{9}	6.232×10^{9}	2.769×10^{10}	1.152×10^{11}	4.513×10^{11}	1.676×10^{12}	5.927×10^{12}	2.004×10^{13}



In addition to **n! m!** the coefficients themselves grow with n, m



(More) physical 2 -> n processes

- Our discussion so far had a slight deficiency. We were considering matrix elements for an initial state of a single highly virtual boson $1^* \to n$.

- In reality one should look at physical scattering processes which are $2 \to n$ with two on-shell initial particles.

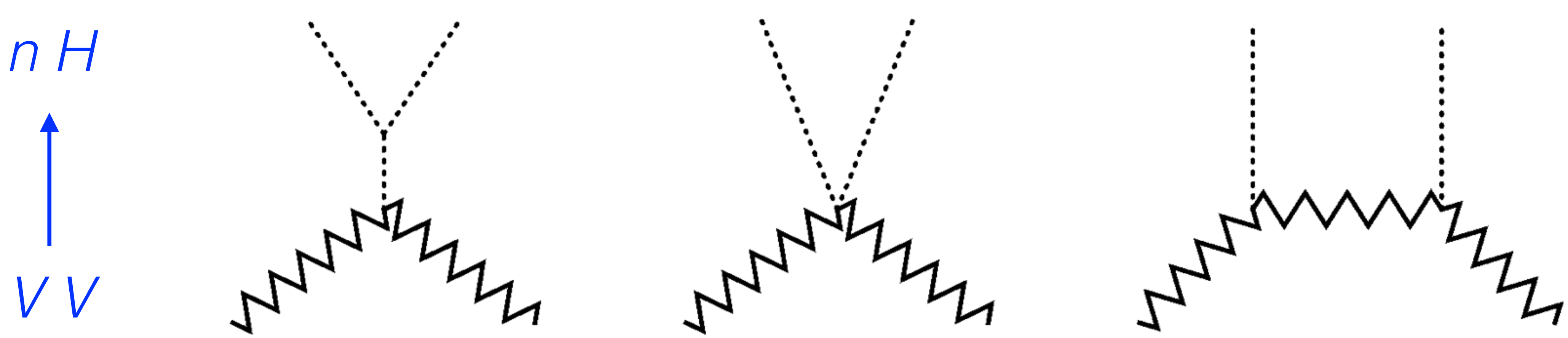
- In the pure ϕ^4 scalar field theory, both in the unbroken and in the broken phase, it is actually known that tree-level amplitudes on the multiparticle threshold for $2 \to n$ processes are **exactly vanishing**.

- But the pure scalar ϕ^4 theory (with a single self-coupling constant) is known to be a special case; this vanishing was expected to hold in the SM only for special fine-tuned values of the couplings (Vector and Higgs masses).

Voloshin; Smith; Argyres, Kleiss, Papadopoulous 1992-94

- Let us investigate the process $VV \to nH$ following

Jaeckel and VVK 1411.5633



(More) physical 2 -> n processes

The recursion relation for the $VV \to nH$ process can be obtained from the $V^* \to V + nH$ equation:

$$d_t^2 A = -\frac{g^2}{4} h^2 A$$

in the background of the Higgs solution $h^2 = v^2 \left(\frac{1+\frac{z}{2v}}{1-\frac{z}{2v}}\right)^2$ with $z = z_0 \exp(iM_h t)$.

Include a non-vanishing 3-momentum flowing through the vector bosons and the kinetic energy of the second V (made incoming),

$$-(nm_h - E_V)^2 A = -\left[\vec{p}^{\,2} + \frac{g^2}{4} h^2\right] A\,, \qquad E_V := \sqrt{\vec{p}^{\,2} + m_V^2} = \frac{n}{2} m_h$$

The Taylor expansion and the amplitude are

$$A \;=\; \sum_k a_n \left(\frac{z}{2v}\right)^n\,, \qquad \mathcal{A}_{2\to n} \;=\; n!\,(2v)^{-n}\, a_n.$$

Factorial growth persists, the amplitude is non-vanishing for $M_V/M_h \neq 1/\sqrt{2}$ and the analytic solution for Taylor coefficients a_n is known.

(some more detail for) *VV* -> nH process

Comparing coefficients of z^n on both sides of the defining equation:

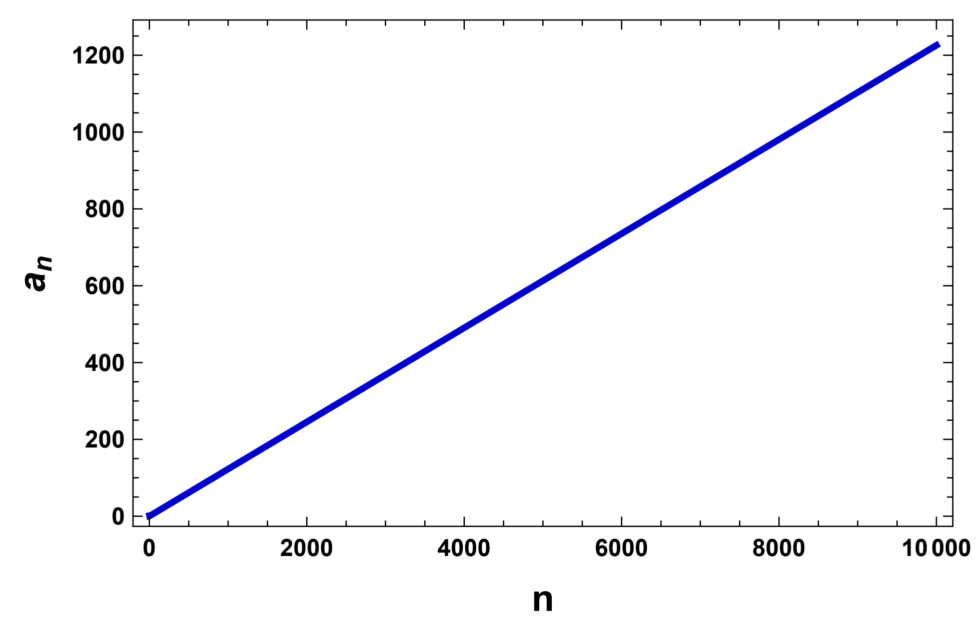
$$-(nm_h - E_V)^2 A = -\left[\vec{p}^2 + \frac{g^2}{4}h^2\right] A$$

we find the relation,

$$a_l = \frac{4\kappa^2}{l^2 - 2E_V l} \sum_{k=1}^{l-1} k\, a_k$$

This can be rewritten in terms of the recursion relations,

$$\begin{aligned} b_k &= b_{k-1} + c_{k-1} + a_{k-1} \\ c_k &= c_{k-1} + a_{k-1} \\ a_k &= \frac{4\kappa^2}{k^2 - 2E_V k} b_k, \end{aligned}$$



with the initial values,

$$b_0 = 0, \quad c_0 = 0, \quad a_0 = 1.$$

Matrix element coefficients a_n for $VV \to nH$ process grow linearly at high n.

Factorial growth remains

Jaeckel & VVK 1411.5633

III. Off the multi-particle threshold

- Tree level recursion relations & classical equations
- Non-relativistic kinematics in the multi-particle final state
- Integration over the n-particle phase space
- The holy grail

- VVK 1411.2925 following the approach of
  Libanov, Rubakov, Son & Troitsky 9407381 and Son 9505338 in unbroken phi^4

Off-threshold in phi^4 with SSB (Higgs-like)

$$-\,(\partial^\mu \partial_\mu + M_h^2)\,\varphi \;=\; 3\lambda v\,\varphi^2 \;+\; \lambda\,\varphi^3$$

This classical equation for $\varphi(x) = h(x) - v$ determines directly the structure of the recursion relation for tree-level scattering amplitudes:

$$(P_{\rm in}^2 - M_h^2)\,\mathcal{A}_n(p_1 \ldots p_n) \;=\; 3\lambda v \sum_{n_1,n_2}^{n} \delta^n_{n_1+n_2} \sum_{\mathcal{P}} \mathcal{A}_{n_1}(p_1^{(1)},\ldots,p_{n_1}^{(1)})\,\mathcal{A}_{n_2}(p_1^{(2)}\ldots p_{n_2}^{(2)})$$

$$+\lambda \sum_{n_1,n_2,n_3}^{n} \delta^n_{n_1+n_2+n_3} \sum_{\mathcal{P}} \mathcal{A}_{n_1}(p_1^{(1)}\ldots p_{n_1}^{(1)})\,\mathcal{A}_{n_2}(p_1^{(2)}\ldots p_{n_2}^{(2)})\,\mathcal{A}_{n_3}(p_1^{(3)}\ldots p_{n_2}^{(3)})$$

Away from the multi-particle threshold, the external particles 3-momenta $\vec{p}_i$ are non-vanishing. In the non-relativistic limit, the leading momentum-dependent contribution to the amplitudes is proportional to $E_n^{\rm kin}$ (Galilean Symmetry),

$$\mathcal{A}_n(p_1 \ldots p_n) \;=\; \mathcal{A}_n + \mathcal{M}_n\,E_n^{\rm kin} \;:=\; \mathcal{A}_n + \mathcal{M}_n\,n\,\varepsilon\,,$$

$$\varepsilon = \frac{1}{n\,M_h}\,E_n^{\rm kin} \;=\; \frac{1}{n}\,\frac{1}{2M_h^2}\,\sum_{i=1}^{n} \vec{p}_i^{\,2}\,.$$

In the non-relativistic limit we have $\varepsilon \ll 1$.

Off-threshold in phi^4 with SSB (Higgs-like)

In the CoM frame the incoming momentum is $P^\mu_{\rm in} = (nM_h(1+\epsilon),\vec{0})$ and the l.h.s. of the recursion rel-n is $M_h^2\left(n^2(1+\varepsilon)^2-1\right)\mathcal{A}_n(p_1\ldots p_n)$. Expand in ε:

$$M_h^2(n^2-1)\,\mathcal{A}_n \;=\; 3\lambda v \sum_{n_1,n_2}^{n} \delta^n_{n_1+n_2} \frac{n!}{n_1!n_2!}\mathcal{A}_{n_1}\,\mathcal{A}_{n_2} + \lambda \sum_{n_1,n_2,n_3}^{n} \delta^n_{n_1+n_2+n_3}\frac{n!}{n_1!n_2!n_3!}\mathcal{A}_{n_1}\,\mathcal{A}_{n_2}\,\mathcal{A}_{n_3}\,.$$

$$\begin{aligned}\left(n^2-1\right) n\,\varepsilon\,\mathcal{M}_n \,+\, 2n^2\,\varepsilon\,\mathcal{A}_n \;\;&=\;\; 6\,\frac{\lambda v}{M_h^2}\sum_{n_1,n_2}^{n}\delta^n_{n_1+n_2}\sum_{\mathcal{P}} E^{\rm kin}_{n_1}\,\mathcal{M}_{n_1}\,\mathcal{A}_{n_2}\\ &\quad + 3\,\frac{\lambda}{M_h^2}\sum_{n_1,n_2,n_3}^{n}\delta^n_{n_1+n_2+n_3}\sum_{\mathcal{P}} E^{\rm kin}_{n_1}\,\mathcal{M}_{n_1}\,\mathcal{A}_{n_2}\,\mathcal{A}_{n_3}\,.\end{aligned}$$

1st equation determines the known threshold $\mathcal{A}_n$, the 2nd equation is for $\mathcal{M}_n$.
Solution for the full amplitude in the non-rel limit is:

$$\mathcal{A}_n(p_1\ldots p_n) \;=\; n!\,(2v)^{1-n}\,\left(1-\frac{7}{6}\,n\,\varepsilon\;-\;\frac{1}{6}\,\frac{n}{n-1}\,\varepsilon\;+\;\ldots\right).$$

There are corrections to this expression at higher orders in ε, but it holds to the order ε^1 for any value of n.

- VVK 1411.2925

Off-threshold in phi^4 with SSB (Higgs-like)

$$\mathcal{A}_n(p_1 \ldots p_n) \;=\; n!\,(2v)^{1-n}\,\left(1-\frac{7}{6}\,n\,\varepsilon\;-\;\frac{1}{6}\,\frac{n}{n-1}\,\varepsilon+\mathcal{O}(\varepsilon^2)\right).$$

An important observation is that by exponentiating the order-$n\varepsilon$ contribution, one obtains the expression for the amplitude which solves the original recursion relation to all orders in $(n\varepsilon)^m$ in the large-n non-relativistic limit,

$$\mathcal{A}_n(p_1 \ldots p_n) \;=\; n!\,(2v)^{1-n}\,\exp\left[-\frac{7}{6}\,n\,\varepsilon\right], \quad n\to\infty\,, \quad \varepsilon\to 0\,, \quad n\varepsilon=\text{fixed}\,.$$

Simple corrections of order ε, with coefficients that are not-enhanced by n are expected, but the expression is correct to all orders $n\varepsilon$ in the double scaling large-n limit. The exponential factor can be absorbed into the z variable so that

$$\varphi(z) = \sum_{n=1}^{\infty} d_n\,\left(z\,e^{-\frac{7}{6}\,\varepsilon}\right)^n\,,$$

remains a solution to the classical equation and the original recursion relations.

- VVK 1411.2925

The Gauge-Higgs theory above the threshold

These results can now be generalised to the Gauge-Higgs theory by solving a system of coupled recursion relations for the gauge and the higgs-field components with non-vanishing average kinetic energies $\varepsilon_h \ll 1$ and $\varepsilon_V \ll 1$ in the fianl state.

Our result for the tree-level multi-vector-boson multi-Higgs production amplitudes in the high-multiplicity double-scaling limit $n\varepsilon_h$ fixed and $m\varepsilon_V$ fixed, and with $\kappa = M_W/M_h = 80/125 \simeq 0.64$ is:

$$\mathcal{A}_{h^* \to n\times h + m\times Z_L} = (2v)^{1-n-m}\, n!\, m!\, d(n,m)\, \exp\left[-\frac{7}{6}\, n\, \varepsilon_h - 1.7\, m\, \varepsilon_V\right],$$

$$\mathcal{A}_{Z_L^* \to n\times h + (m+1)\times Z_L} = \frac{1}{(2v)^{n+m}}\, n!\, (m+1)!\, a(n,m)\, \exp\left[-\frac{7}{6}\, n\, \varepsilon_h - 1.7\, m\, \varepsilon_V\right]$$

- Note that multiplicity- and energy-dependent exponential form-factors now appear alongside the already-established factorials.
- We can now accomplish integrations over the multi-particle phase-space.

• VVK 1411.2925

Phase-space integration

$$\sigma_{n,m} = \int d\Phi_{n,m} \, \frac{1}{n!\, m!} \, \left|\mathcal{A}_{h^* \to n\times h + m \times Z_L}\right|^2 ,$$

The n-particle Lorentz-invariant phase space volume element

$$\int d\Phi_n = (2\pi)^4 \delta^{(4)}(P_{\rm in} - \sum_{j=1}^{n} p_j) \prod_{j=1}^{n} \int \frac{d^3 p_j}{(2\pi)^3 \, 2p_j^0} ,$$

in the large-n non-relativistic limit with $n\varepsilon_h$ fixed becomes,

$$\Phi_n \simeq \frac{1}{\sqrt{n}} \left(\frac{M_h^2}{2}\right)^n \exp\left[\frac{3n}{2}\left(\log\frac{\varepsilon_h}{3\pi} + 1\right) + \frac{n\varepsilon_h}{4} + \mathcal{O}(n\varepsilon_h^2)\right] .$$

Repeating the same steps now including vector boson emissions,

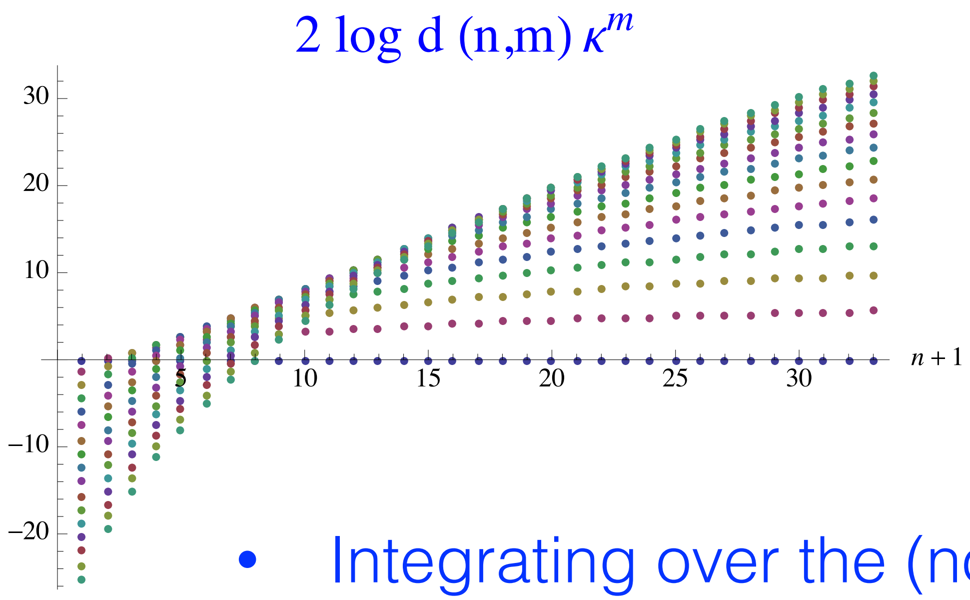
$$\sigma_{n,m} \sim \exp\Bigg[2\log d(n,m) + n\left(\log\frac{\lambda n}{4} - 1\right) + m\left(\log\left(\frac{g^2 m}{32}\right) - 1\right) + \frac{3n}{2}\left(\log\frac{\varepsilon_h}{3\pi} + 1\right) + \frac{3m}{2}\left(\log\frac{\varepsilon_V}{3\pi} + 1\right) - \frac{25}{12} n\varepsilon_h - 3.15\, m\varepsilon_V + \mathcal{O}(n\varepsilon_h^2 + m\varepsilon_V^2)\Bigg]$$

- VVK 1411.2925

Summary: *Rising cross sections for multi-H,W,Z production*

- At high energies it becomes kinematically possible to produce multiple Higgs and massive vector W, Z bosons. These amplitudes grow factorially with the number of particles — overcome the suppression by powers of perturbative couplings.

$$\mathcal{A}_{h^*\to n\times h+m\times Z_L} \;=\; (2v)^{1-n-m}\, n!\, m!\, d(n,m)\, \exp\left[-\frac{7}{6}\, n\, \varepsilon_h - 1.7\, m\, \varepsilon_V\right]$$



$$E = n(1+\varepsilon_h)m_h + m(1+\varepsilon_V)m_V$$

$$\kappa := \frac{g}{2\sqrt{2\lambda}} = \frac{m_V}{m_h} \sim 80/125 \sim 0.65$$

- Integrating over the (non-relativistic) phase-space get the cross section

$$\sigma_{n,m} \sim \exp\left[2\log d(n,m) + n\left(\log\frac{\lambda n}{4} - 1\right) + m\left(\log\left(\frac{g^2 m}{32}\right) - 1\right) + \frac{3n}{2}\left(\log\frac{\varepsilon_h}{3\pi} + 1\right) + \frac{3m}{2}\left(\log\frac{\varepsilon_V}{3\pi} + 1\right) - \frac{25}{12}\, n\varepsilon_h - 3.15\, m\varepsilon_V + \mathcal{O}(n\varepsilon_h^2 + m\varepsilon_V^2)\right]$$

IV. The Holy Grail

In the non-rel. limit for perturbative Higgs bosons only production we obtained:

$$\sigma_n \propto \exp\left[n\left(\log\frac{\lambda n}{4} - 1\right) + \frac{3n}{2}\left(\log\frac{\varepsilon}{3\pi} + 1\right) - \frac{25}{12}\, n\,\varepsilon \right]$$

More generally, in the large-n limit with $\lambda n =$ fixed and $\varepsilon =$ fixed, one expects

$$\sigma_n \propto \exp\left[\frac{1}{\lambda}\, F_{\text{h.g.}}(\lambda n, \varepsilon)\right]$$

[e.g. Libanov, Rubakov, Troitsky review 1997]

where the *holy grail* function $F_{\text{h.g.}}$ is of the form,

$$\frac{1}{\lambda}\, F_{\text{h.g.}}(\lambda n, \varepsilon) = \frac{\lambda n}{\lambda}\,(f_0(\lambda n) + f(\varepsilon))$$

In our higgs model, i.e. the scalar theory with SSB,

$$f_0(\lambda n) = \log\frac{\lambda n}{4} - 1 \qquad \text{at tree level}$$

$$f(\varepsilon) \to \frac{3}{2}\left(\log\frac{\varepsilon}{3\pi} + 1\right) - \frac{25}{12}\,\varepsilon \qquad \text{for } \varepsilon \ll 1$$

Large-n limit with λn = fixed and ε = fixed,

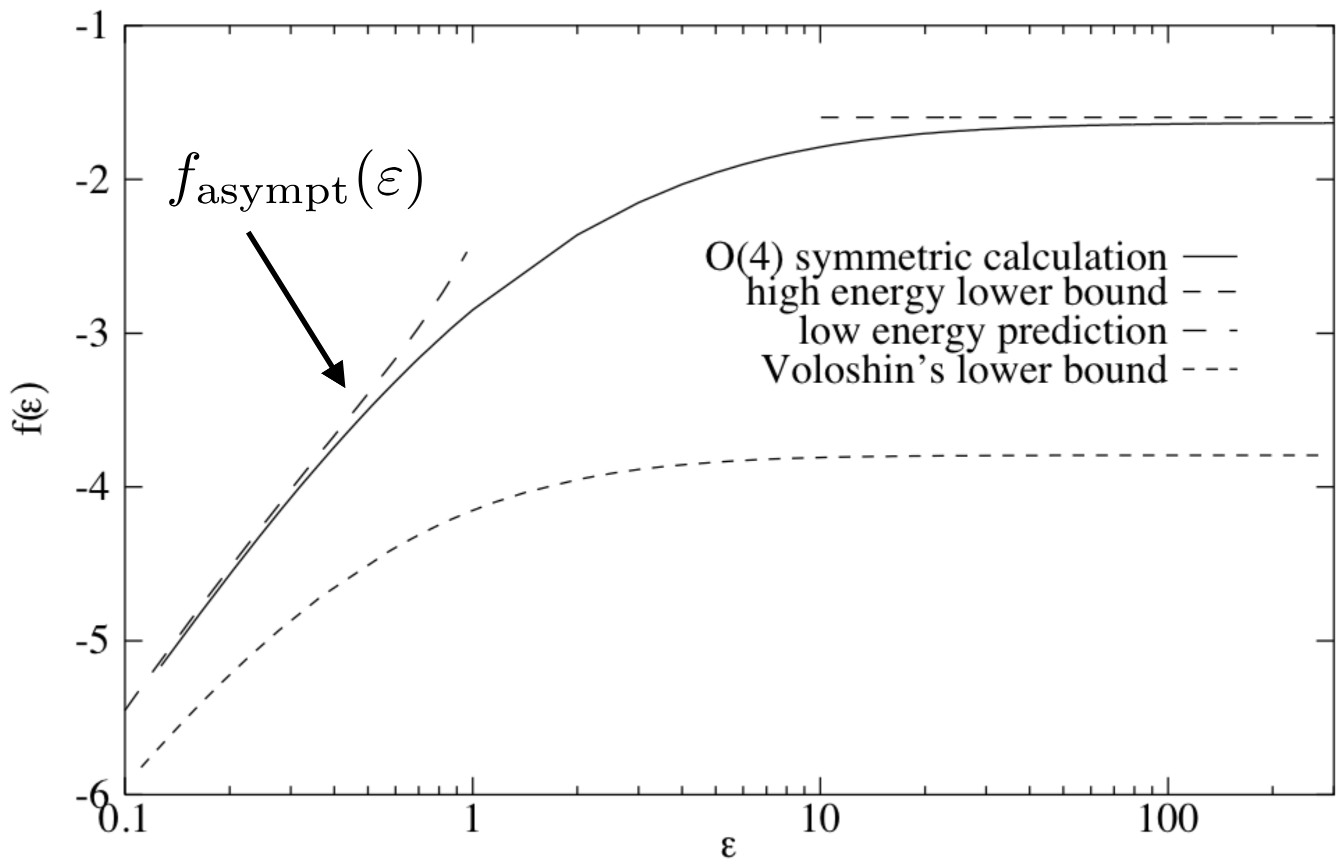
$$\frac{1}{n} \log \sigma_n \;=\; \frac{1}{\lambda n} F_{\rm h.g.}(\lambda n, \varepsilon) \;=\; f_0(\lambda n) + f(\varepsilon)$$

In the pure unbroken ϕ^4 theory

Unbroken phi^4

$$f_0(\lambda n) \;=\; \log \frac{\lambda n}{16} - 1 \qquad \text{at tree level}$$

$$f_{\rm asympt}(\varepsilon) \;=\; \frac{3}{2}\left(\log \frac{\varepsilon}{3\pi} + 1\right) - \frac{17}{12}\,\varepsilon \qquad \text{for } \varepsilon \ll 1$$



saturates at high epsilon (does not decrease)

negative f at eps->infinity on its own does not imply exponential suppression — common misinterpretation

f(epsilon) from Bezrukov, Libanov, Son, Troitsky 9512342

- VVK 1504.05023

Large-n limit with λn = fixed and ε = fixed,

$$\frac{1}{n}\log\sigma_n \;=\; \frac{1}{\lambda n}F_{\rm h.g.}(\lambda n,\varepsilon) \;=\; f_0(\lambda n)+f(\varepsilon)$$

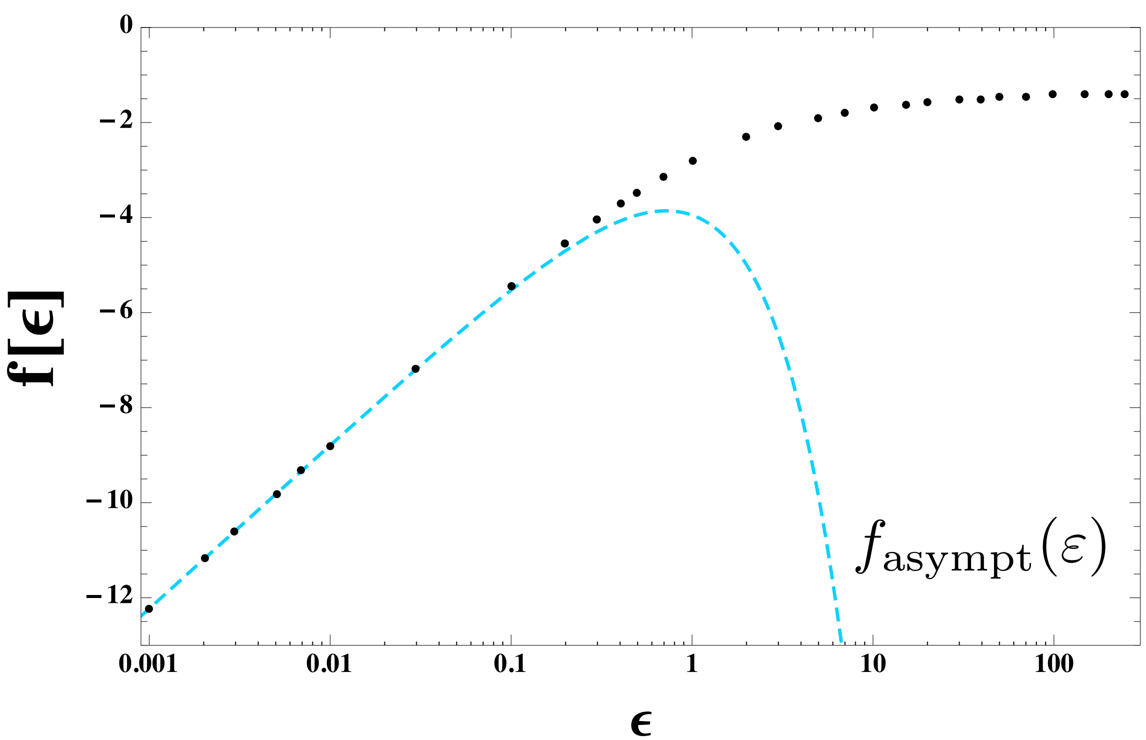
In the pure unbroken ϕ^4 theory

$$f_0(\lambda n) \;=\; \log\frac{\lambda n}{16}-1 \qquad \text{at tree level}$$

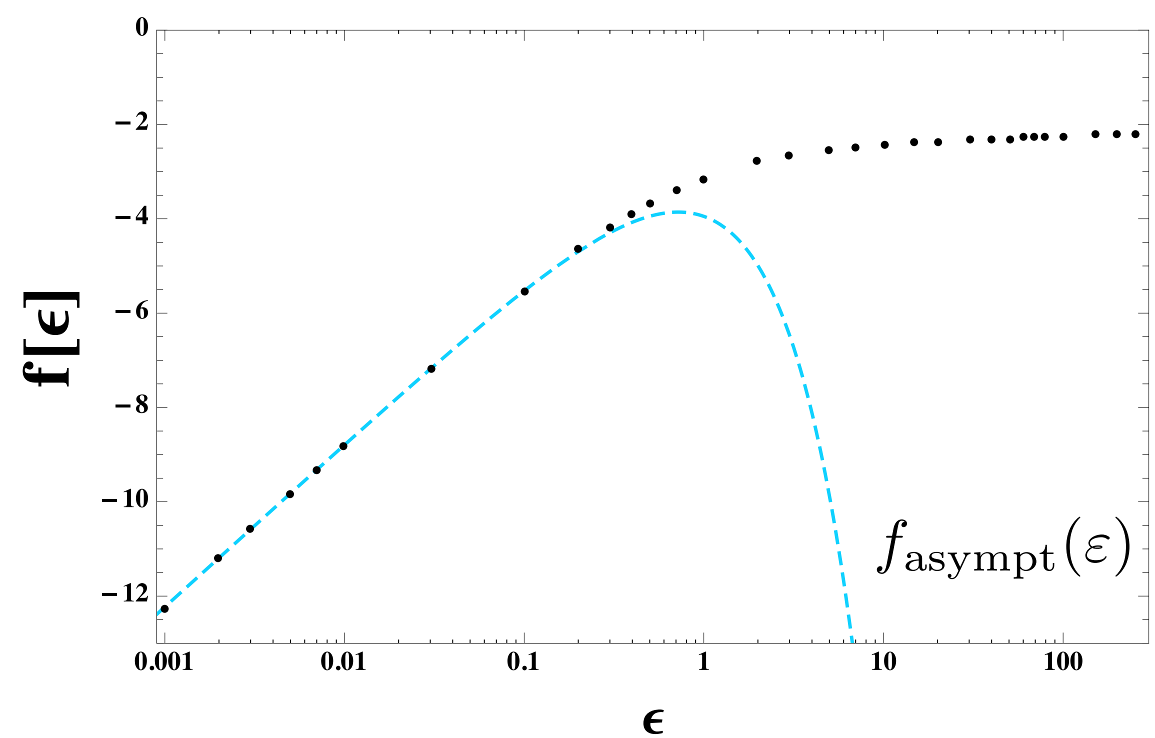
$$f_{\rm asympt}(\varepsilon) \;=\; \frac{3}{2}\left(\log\frac{\varepsilon}{3\pi}+1\right)-\frac{17}{12}\,\varepsilon \qquad \text{for } \varepsilon\ll 1$$

saturates at high epsilon

f[ϵ] no SSB theory

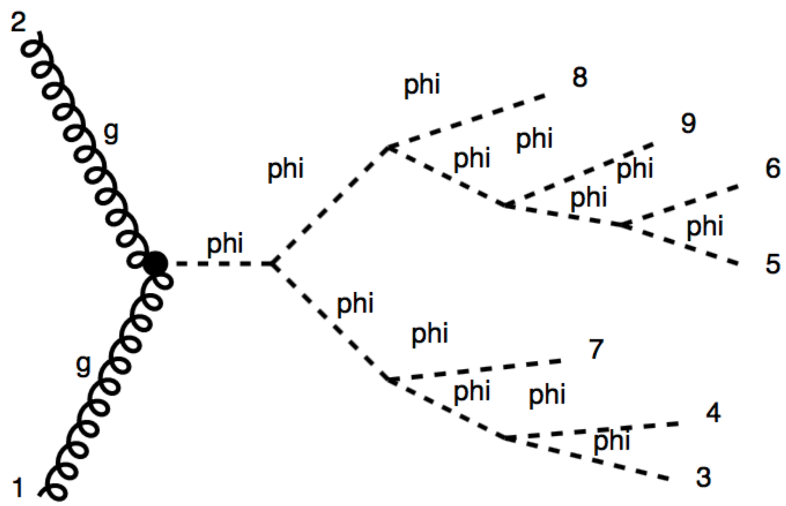


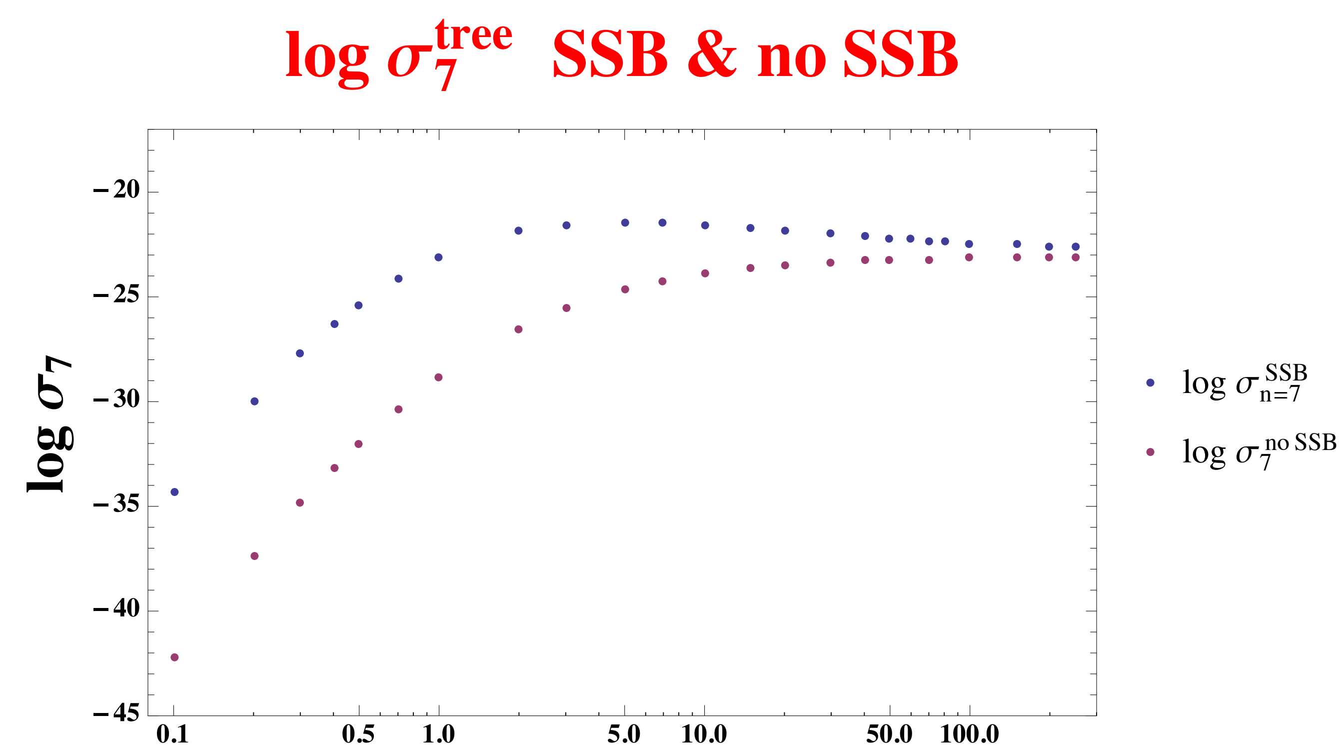
f[ϵ] SSB theory



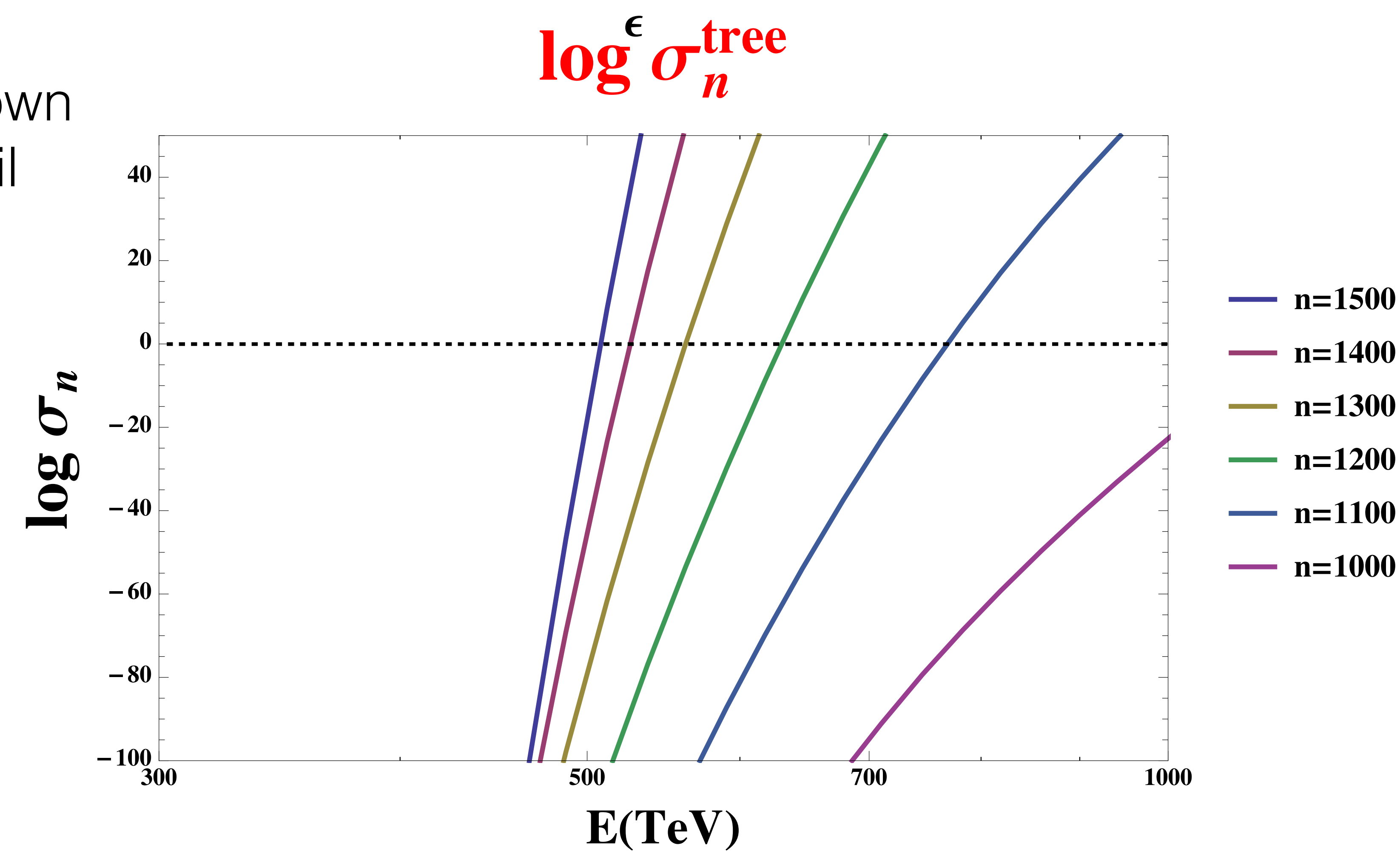
computed w MadGraph 2-> 7 for any epsilon and scaled to large n

1. Compute cross-sections with MadGraph 2 -> 5,6,7 at all energies (i.e. arbitrary epsilon)





2. Scale to large n using the known n-dependence in the holy grail



- VVK 1504.05023

Loop corrections to tree-level amplitudes@threshold

The 1-loop corrected threshold amplitude for the pure n Higgs production:

$$\phi^4 \text{ with SSB}: \quad \mathcal{A}_{1\to n}^{\text{tree+1loop}} = n!\,(2v)^{1-n}\left(1 + n(n-1)\frac{\sqrt{3}\lambda}{8\pi}\right)$$

There are strong indications, based on the analysis of leading singularities of the multi-loop expansion around singular generating functions in scalar field theory, that the 1-loop correction exponentiates,

Libanov, Rubakov, Son, Troitsky 1994

$$\mathcal{A}_{1\to n} = \mathcal{A}_{1\to n}^{\text{tree}} \times \exp\left[B\,\lambda n^2 + \mathcal{O}(\lambda n)\right]$$

in the limit $\lambda \to 0$, $n \to \infty$ with λn^2 fixed. Here B is determined from the 1-loop calculation (as above) – *Smith; Voloshin 1992*):

$$\phi^4 \text{ with SSB}: \quad B = +\frac{\sqrt{3}}{8\pi},$$

$$\phi^4 \text{ w. no SSB}: \quad B = -\frac{1}{64\pi^2}\left(\log(7+4\sqrt{3}) - i\pi\right),$$

In the Higgs model, 1st equation leads to the exponential enhancement of the tree-level threshold amplitude at least in the leading order in $n^2\lambda$.

RETURN to the beginning of the section:

In the non-rel. limit for perturbative Higgs bosons only production we obtained:

$$\sigma_n \propto \exp\left[n\left(\log\frac{\lambda n}{4} - 1\right) + \frac{3n}{2}\left(\log\frac{\varepsilon}{3\pi} + 1\right) - \frac{25}{12}\,n\,\varepsilon\right] + 2\lambda n^2 B$$

More generally, in the large-n limit with $\lambda n =$ fixed and $\varepsilon =$ fixed, one expects

$$\sigma_n \propto \exp\left[\frac{1}{\lambda}\,F_{\rm h.g.}(\lambda n, \varepsilon)\right]$$

where the *holy grail* function $F_{\rm h.g.}$ is of the form,

$$\frac{1}{\lambda}\,F_{\rm h.g.}(\lambda n, \varepsilon) = \frac{\lambda n}{\lambda}\left(f_0(\lambda n) + f(\varepsilon)\right)$$

In our higgs model, i.e. the scalar theory with SSB,

$$f_0(\lambda n) = \log\frac{\lambda n}{4} - 1 + \lambda n\,\frac{\sqrt{3}}{4\pi}$$

a very significant enhancement though higher orders unknown!

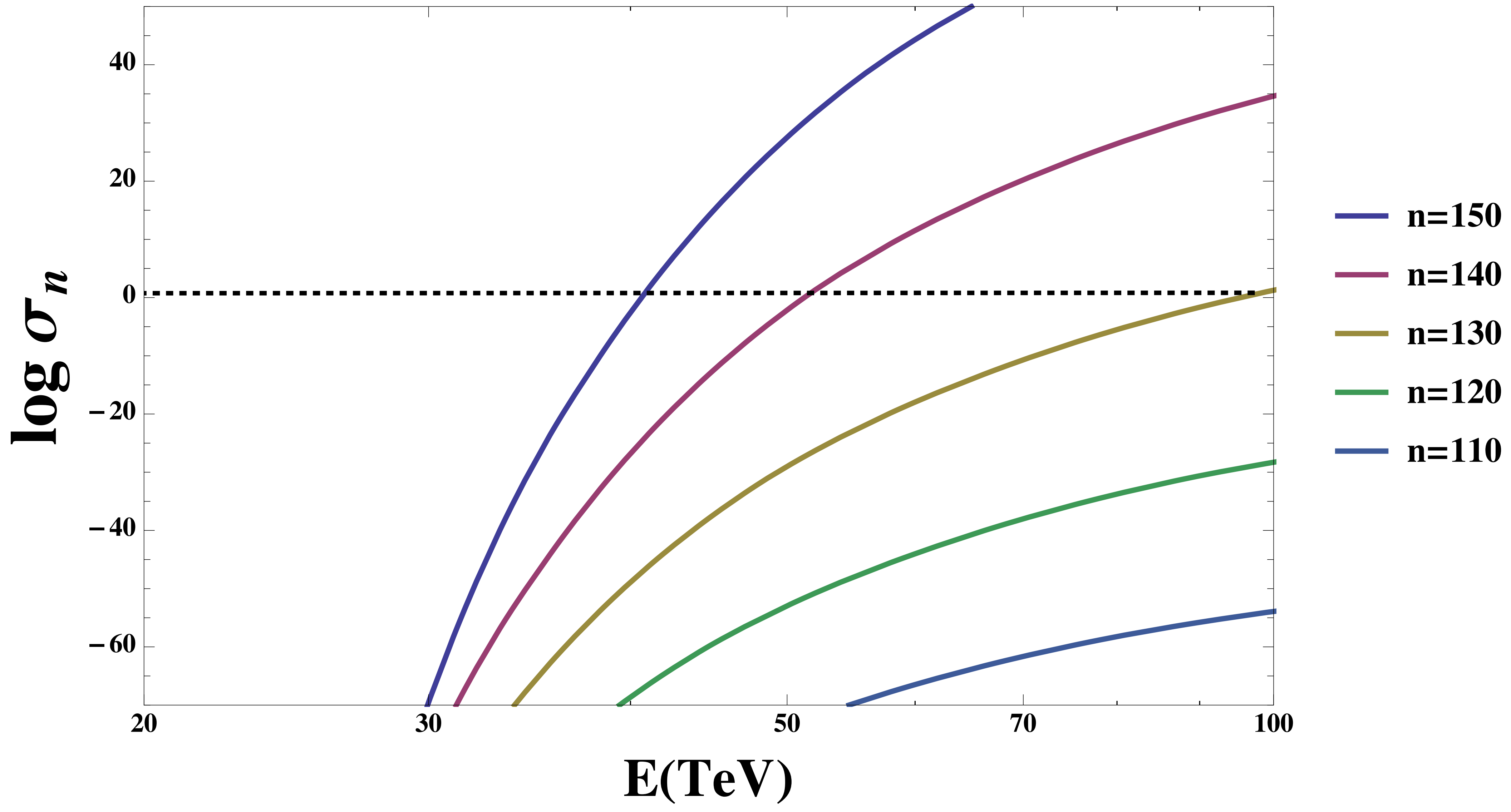
$$f(\varepsilon) \to \frac{3}{2}\left(\log\frac{\varepsilon}{3\pi} + 1\right) - \frac{25}{12}\,\varepsilon \qquad \text{for } \varepsilon \ll 1$$

1. Compute cross-sections with MadGraph 2 -> 5,6,7 at all energies (i.e. arbitrary epsilon)

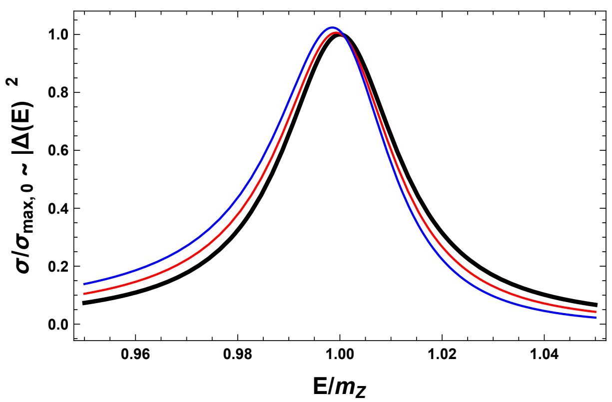
2. Scale to large n using the known n-dependence in the holy grail including the leading-loop factor to the exponent $+\lambda n \frac{\sqrt{3}}{4\pi}$

- VVK 1504.05023

$\log \sigma_n^{\text{loop}}$



- The perturbative cross section grows with energy, ultimately violating:
  - naive perturbative unitarity: $\sum_{n,\text{inelastic}} \int d\Phi_{n,m} |\mathcal{A}_{n,m}|^2 \leq 8\pi (l_{\max}+1)^2 \overset{?}{\sim} 8\pi$
  - the observed form of the Z-peak via the Kallen-Lehmann spectral representation:



$$\Delta(p) = \frac{Z}{p^2 - m^2} + \sum_{n \geq 2} \int_{(nm)^2}^{\infty} ds \, \frac{\int d\Phi_n |\mathcal{A}(1 \to n)|^2(s)}{p^2 - s}$$

$$\int_{(nm)^2}^{\infty} ds \, \frac{\int d\Phi_n |\mathcal{A}(1 \to n)|^2(s)}{p^2 - s} = \frac{1}{m^2}\mathcal{C}_1 + \frac{p^2}{m^4}\mathcal{C}_2 + \ldots \quad , \quad |\mathcal{C}_1| < 1 \quad , \quad |\mathcal{C}_2| < 1$$

  - the cosmic ray limit: upper bound comes from assuming that the effective cross section for inelastic scattering of cosmic rays is of the size of the universe. In this case high energy cosmic rays will be severely attenuated in conflict w observation.

- pert. SM cross sections exceed these bounds at energies:

E	$\lesssim$	810 TeV	naive unitarity limit
E	$\lesssim$	830 TeV	cosmic limit
E	$\lesssim$	300 TeV	asymptotic series truncation heuristic
E	$\lesssim$	100 TeV	adding $\simeq \varepsilon^2$ term in $f_{\text{our pert.}}(\varepsilon) \sim (1/n) \log \sigma_n$
E	$\lesssim$	35 TeV	include naive loop factor $\simeq \lambda n \frac{\sqrt{3}}{4\pi}$ in $f_0(\lambda n)$

- Jaeckel, VVK 1411.5633

V. Conclusions

- At (not too high) high energies perturbative Standard Model exhibits not only a formal breakdown, but will also be in conflict with observation Two options:

- At high energies (multiplicities) the Standard Model is fundamentally non-perturbative

- New physics beyond the Standard Model has to set in before the cross-sections become large

- New theoretical approaches & computational techniques have to be developed to determine the relevant energy scale - almost as exciting as probing this at the FCC -