Thermodynamics of strongly-coupled lattice QCD in the chiral limit

Hélvio Vairinhos
ETH zürich

in collaboration with:
Philippe de Forcrand,
Paul Romatschke,
Wolfgang Unger

University of Southampton
29 Jul 2016

Introduction

Let us consider $U(N_c)$ lattice QCD, with $N_f = 1$ massless staggered fermion, on a $N_s^3 \times N_t$ lattice, at finite temperature.

- The Goldstone boson, associated with the spontaneous breaking of the remnant $U(1)$ chiral symmetry, is interacting ($f_\pi^2 \neq 0$). But, in the $T \ll f_\pi$ regime, it is essentially a free particle. At low temperatures, it is expected to behave like a Stefan-Boltzmann gas.
- We study the thermal properties of $U(3)$ and $SU(3)$ lattice QCD, in the strong lattice coupling limit ($\beta = 0$) and chiral limit ($m_q = 0$), with a focus on the expected (near) ideal gas behavior.
- Possibility of using algorithms (of the worm type), which are very efficient in strong and chiral limits, and at low temperatures.
- Precursor to an extensive precision study of the equation of state of lattice QCD, in the strong coupling limit: finite quark mass, baryon density, etc.

Thermodynamics of a free massless boson on the lattice

The thermal properties of free massless bosons on a $N_s^3 \times N_t$ anisotropic lattice, with anisotropy $\gamma = \frac{a}{a_t}$, have been studied in detail. [Karsch-Engels-Satz '82]

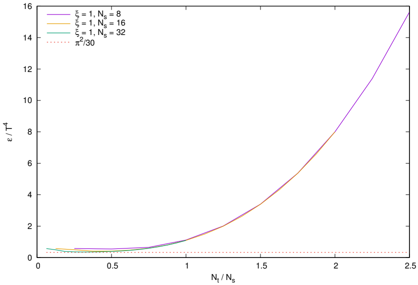
Energy density:

$$a^4(\varepsilon - \varepsilon_0) = -\frac{\gamma^3}{N_s^3 N_t} \sum_{\vec{j} \neq 0} \frac{\sin^2(\pi j_0/N_t)}{b^2 + \gamma^2 \sin^2(\pi j_0/N_t)}, \qquad b^2 = \sum_{i=1}^{3} \sin^2(\pi j_i/N_s)$$

$$a^4 \varepsilon_0 = \frac{\gamma^3}{N_s^3} \sum_{\vec{j}} \left(b^2 + \gamma^2 + b\sqrt{\gamma^2 + b^2}\right)^{-1}$$

Interaction energy:
(holds on any finite lattice)

$$\varepsilon - 3p = 0$$



Lessons:

- Finite-size effects are considerable
- For mild lattice corrections, use: $N_s \geq 2N_t$ and $\gamma \geq 2$
- Ideal gas behavior in the $\gamma \to \infty$ (continuous time) limit

$U(3)$ lattice QCD as a monomer-dimer system

Analytical integration over $U_{x\mu}$ and $\psi_x, \bar{\psi}_x$ in $U(3)$ lattice QCD yields the partition function of a **monomer-dimer system**: [Rossi-Wolff '84]

$$Z = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{2a_t m_q \sum_x \bar{\psi}_x \psi_x + \sum_{x,\mu} \gamma^{\delta_{\mu 0}} \eta_{x\mu}(\bar{\psi}_x U_{x\mu} \psi_{x+\hat{\mu}} - \mathrm{h.c.})}$$

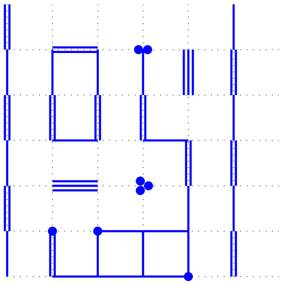
$$= \sum_{\{n,k\}} \left(\prod_{x,\mu} \frac{(3-k_{x\mu})!}{3!k_{x\mu}!} \right) (2a_t m_q)^{N_M} \gamma^{2N_{Dt}}$$

$$n_x, k_{x\mu} \in \{0,1,2,3\}, \qquad N_M = \textstyle\sum_x n_x, \qquad N_{Dt} = \textstyle\sum_x k_{x0}$$

- Admissible configurations satisfy **Grassmann constraints:**

$$n_x + \textstyle\sum_{\pm\mu} k_{x\mu} \stackrel{!}{=} 3$$

- Configurations are generated using a **directed path (worm) algorithm:** very efficient, especially in the chiral limit. [Adams-Chandrasekharan '03]



Thermodynamics of $U(3)$ lattice QCD in the chiral limit

We work directly in the **chiral limit**: $m_q = 0 \iff N_M = 0$ (**dimers only**).

Thermodynamical quantities are derived from the partition function:

$$Z(\gamma) = \sum_{\{k\}} \left(\prod_{x,\mu} \frac{(3-k_{x\mu})!}{3!k_{x\mu}!} \right) \gamma^{2N_{Dt}}$$

which has the bare anisotropy coupling γ as the only free parameter; the physical anisotropy is parameterized by $\xi(\gamma) = \frac{a}{a_t}$.

Energy density:

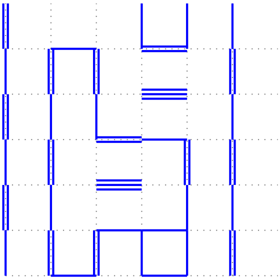
$$a^3 a_t\, \varepsilon = -\frac{a^3 a_t}{V} \left. \frac{\partial \log Z}{\partial T^{-1}} \right|_V = \frac{\xi}{\gamma} \frac{d\gamma}{d\xi} \langle 2n_{Dt} \rangle$$

Pressure:

$$a^3 a_t\, p = a^3 a_t T \left. \frac{\partial \log Z}{\partial V} \right|_T = \frac{\xi}{3\gamma} \frac{d\gamma}{d\xi} \langle 2n_{Dt} \rangle$$

Interaction energy: $\varepsilon - 3p = 0$

Entropy density: $s = \dfrac{4\varepsilon}{3T}$



Anisotropy calibration

Grassmann constraints imply locally **conserved currents**:
[Chandrasekharan-Jiang '03]

$$j_{x\mu} = \sigma_x \left(k_{x\mu} - \frac{3}{8} \right) \quad \Longrightarrow \quad \sum\nolimits_{\pm\mu} (j_{x\mu} - j_{x-\hat{\mu},\mu}) = 0$$

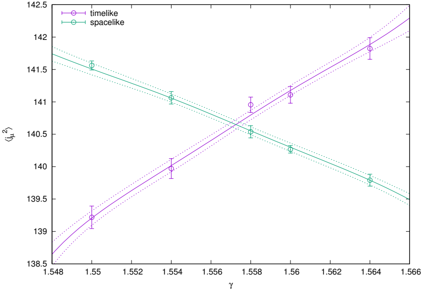
The variance of the associated **conserved charges**,

$$j_\mu = \sum\nolimits_{x \perp \hat{\mu}} j_{x\mu}$$

should coincide in a hypercubic volume, and this provides a very precise method to calibrate the anisotropy, $\xi(\gamma) = \frac{a}{a_t}$, with the help of multi-histogram reweighing:

Renormalization criterion:

$$a_t N_t = a N_s$$
$$\Downarrow$$
$$\left\langle j_t^2 \right\rangle (\gamma_c) \stackrel{!}{=} \left\langle j_s^2 \right\rangle (\gamma_c)$$



Anisotropy calibration

Grassmann constraints imply locally **conserved currents**:
[Chandrasekharan-Jiang '03]

$$j_{x\mu} = \sigma_x \left(k_{x\mu} - \frac{3}{8} \right) \implies \sum\nolimits_{\pm\mu} (j_{x\mu} - j_{x-\hat{\mu},\mu}) = 0$$

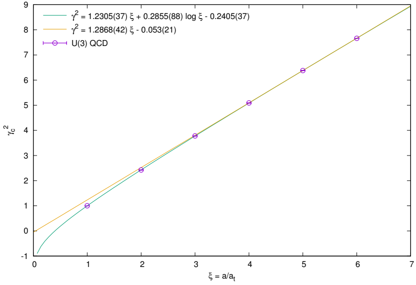
The variance of the associated **conserved charges**,

$$j_\mu = \sum\nolimits_{x\perp\hat{\mu}} j_{x\mu}$$

should coincide in a hypercubic volume, and this provides a very precise method to calibrate the anisotropy, $\xi(\gamma) = \frac{a}{a_t}$, with the help of multi-histogram reweighing:

$$\xi(\gamma) \sim \gamma^2$$

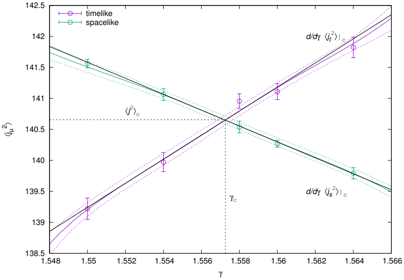
but the prefactor deviates from the mean field prediction ($\xi = \gamma^2$)

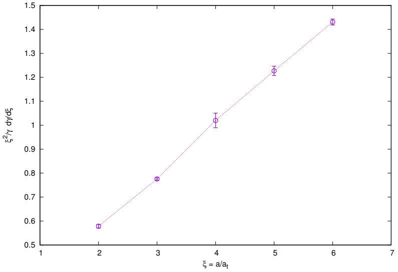


Running anisotropy

At the same time, the running anisotropy, $\frac{d\xi}{d\gamma}$, can be computed directly from the critical value of $\langle j_\mu^2 \rangle$, and the critical value of their slopes:

$$\xi \frac{d\gamma}{d\xi} = \frac{\langle j^2 \rangle_c}{\left(\frac{d}{d\gamma} \langle j_t^2 \rangle - \frac{d}{d\gamma} \langle j_s^2 \rangle \right)_{\gamma_c}}$$

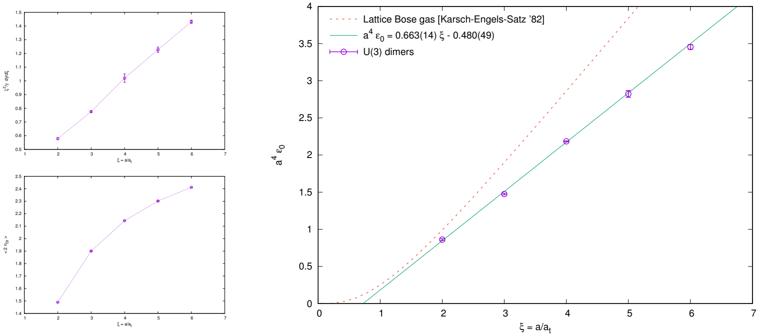




Energy density at $T = 0$

The $T = 0$ contribution must be carefully subtracted from the energy density:

$$a^4 \varepsilon_0(\xi) = \lim_{N_s \to \infty} \frac{\xi^2}{\gamma} \frac{d\gamma}{d\xi} \langle 2 n_{Dt} \rangle \Big|_{N_t = \xi N_s \ \text{(hypercubic)}}$$

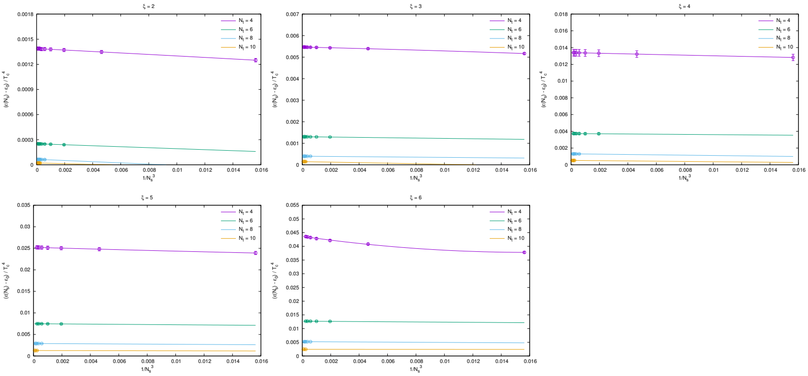


- Linear scaling, similar to the lattice gas of a free massless boson. [Karsch-Engels-Satz '82]

Energy density at finite T

We take thermodynamic extrapolations ($N_s \to \infty$) of the energy density, normalized by $aT_c = 1.8843(1)$ [Forcrand-Unger '11], at fixed ξ and N_t, assuming:

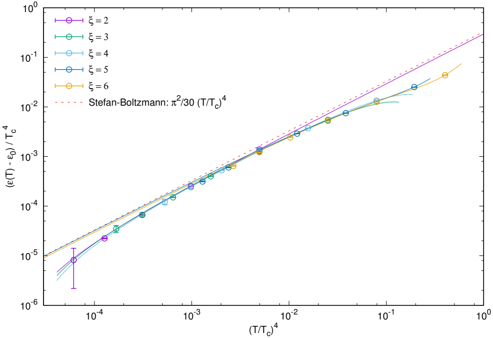
$$\frac{\epsilon(N_s, N_t, \xi) - \epsilon_0(\xi)}{T_c^4} = \frac{\epsilon(T) - \epsilon_0}{T_c^4} + \frac{c_1}{N_s^3} + \frac{c_2}{N_s^6} + \cdots$$



We use $N_s \gtrsim 2N_t$ and $\xi \geq 2$, for minimal lattice effects

Energy density at finite T

...and after all extrapolations:



Preliminary fits are added for illustration:

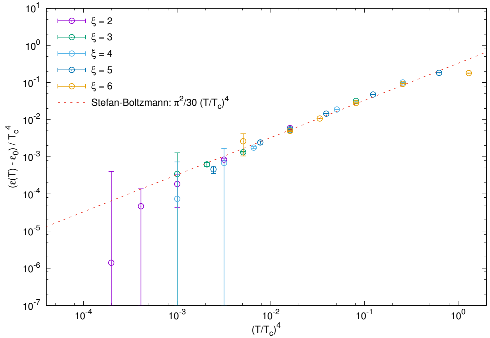
$$\frac{\epsilon(T) - \epsilon_0}{T_c^4} \approx c_2 \left(\frac{T}{T_c}\right)^2 + c_4 \left(\frac{T}{T_c}\right)^4 + c_6 \left(\frac{T}{T_c}\right)^6 + c_8 \left(\frac{T}{T_c}\right)^8 + \cdots$$

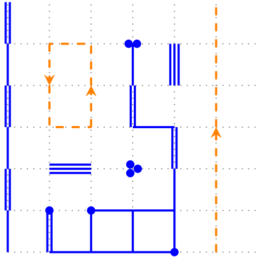
- Near T_c: Repulsion between pions $\Rightarrow$ energy decreases
- At low T: Near-ideal gas of a single massless boson
- At very low T: $1/\xi$ corrections?

Energy density at finite T, in $SU(3)$ QCD

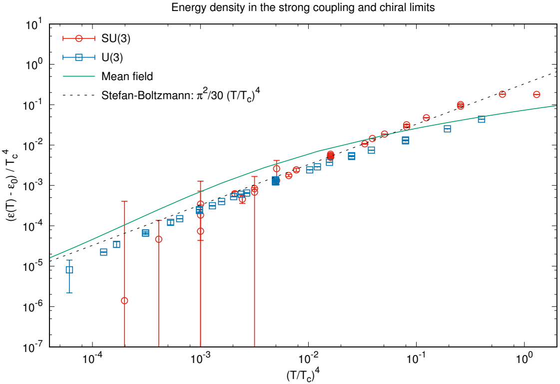
We perform the same study in $SU(3)$ QCD, for which there is a **monomer-dimer-loop** representation of the partition function, and for which the energy density receives **baryonic corrections**:

$$a^3 a_t\, \varepsilon = \mu_B \rho_B - \frac{a^3 a_t}{V} \left.\frac{\partial \log Z}{\partial T^{-1}}\right|_{V,\mu_B} = \frac{\xi}{\gamma}\frac{d\gamma}{d\xi}\,\langle 2n_{Dt} + 3n_{Bt}\rangle$$





Comparison between $U(3)$ and $SU(3)$



- Near T_c:
  - $U(3)$: Repulsion between pions $\Rightarrow$ energy decreases
  - $SU(3)$: Energy is higher than in $U(3)$ due to baryonic modes
- At low T: Pion gas is effectively free (up to $1/\xi$-corrections?)
- Qualitative consistency with mean field, at large-N_c

Summary and outlook

- In the strong coupling limit, $U(N_c)$ lattice QCD with a massless staggered quark describes an ideal gas of massless pions, at low temperatures.
- We study the thermodynamics of $N_f{=}1$ $U(3)$ and $SU(3)$ lattice QCD, in the chiral and strong coupling limits, by simulating the dimer representation of this system with a directed path algorithm.
- We propose a prescription for a very precise renormalization of the bare anisotropy coupling, and for the determination of its running.
- We determine, with high precision, the dependence of the energy density on the temperature, thanks to an accurate subtraction of the $T = 0$ contribution. In that regime, the system describes a near-ideal pion gas, just spoiled by massive modes near T_c.

Next:

- Measure f_π^2, and compare with ChPT predictions.
- Extend the study of the equation of state of $U(3)$ and $SU(3)$ QCD to finite quark mass, finite baryon density, $N_f > 1$, etc.

Backup slides

$SU(3)$ lattice QCD as a monomer-dimer-loop system

Analytical integration over $U_{x\mu}$ and $\psi_x, \bar{\psi}_x$ in $SU(3)$ lattice QCD yields the partition function of a **monomer-dimer-loop system:** [Rossi-Wolff '84]

$$Z = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{2a_t m_q \sum_x \bar{\psi}_x\psi_x + \sum_{x,\mu} \gamma^{\delta_{\mu 0}} \eta_{x\mu}\left(e^{a_t\mu_q}\bar{\psi}_x U_{x\mu}\psi_{x+\hat{\mu}} - e^{-a_t\mu_q}\bar{\psi}_x U_{x\mu}\psi_{x+\hat{\mu}}\right)}$$

$$= \sum_{\{n,k,C\}} \frac{\sigma(C)}{N!^{|C|}} \left(\prod_x \frac{3!}{n_x!}\right)\left(\prod_{x,\mu}\frac{(3-k_{x\mu})!}{3!k_{x\mu}!}\right)(2a_t m_q)^{N_M}\gamma^{2N_{Dt}+3N_{Bt}} e^{3N_t a_t\mu_q\Omega(C)}$$

$$n_x, k_{x\mu} \in \{0,1,2,3\}, \qquad N_{Dt} = \sum_x k_{x0}, \qquad N_M = \sum_x n_x$$

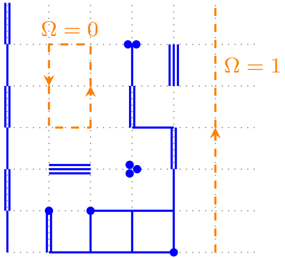
$$b_{x\mu} \in \{0,\pm 1\}, \qquad N_{Bt} = \sum_x |b_{x0}|$$

- Admissible configurations satisfy **Grassmann constraints:**

$$n_x + \sum\nolimits_{\pm\mu} k_{x\mu} \stackrel{!}{=} 3$$

$$\sum\nolimits_{x,\mu} |b_{x\mu}| \stackrel{!}{=} 0$$

- **Baryonic sign problem:** $\sigma(C) = \pm 1$



Thermodynamics of $SU(3)$ lattice QCD

Thermodynamical quantities are derived from the partition function:

$$Z = \sum_{\{n,k,C\}} \sigma(C) \left(\prod_x \frac{3!}{n_x!} \right) \left(\prod_{x,\mu} \frac{(3-k_{x\mu})!}{3!k_{x\mu}!} \right) (2a_t m_q)^{N_M} \gamma^{2N_{Dt}+3N_{Bt}} e^{3N_t a_t \mu_q \Omega(C)}$$

Baryon number density: $(\mu_B = 3\mu_q)$

$$a^3 \rho_B = a^3 \; \frac{T}{V} \frac{\partial \log Z}{\partial \mu_B}\bigg|_{V,T} = \frac{\langle \Omega \rangle}{N_s^3} = \langle \omega \rangle$$

Energy density:

$$a^3 a_t \, \varepsilon = \mu_B \rho_B - \frac{a^3 a_t}{V} \; \frac{\partial \log Z}{\partial T^{-1}}\bigg|_{V,\mu_B} = \frac{\xi}{\gamma} \frac{d\gamma}{d\xi} \langle 2n_{Dt} + 3n_{Bt} \rangle - \langle n_M \rangle$$

Pressure:

$$a^3 a_t \, p = a^3 a_t T \; \frac{\partial \log Z}{\partial V}\bigg|_{T,\mu_B} = \frac{\xi}{3\gamma} \frac{d\gamma}{d\xi} \langle 2n_{Dt} + 3n_{Bt} \rangle$$

Interaction energy: $\varepsilon - 3p = - \langle n_M \rangle$

Entropy density: $s = \dfrac{1}{T} \left(\dfrac{4\varepsilon}{3} - \mu_B \rho_B \right)$

Anisotropy calibration

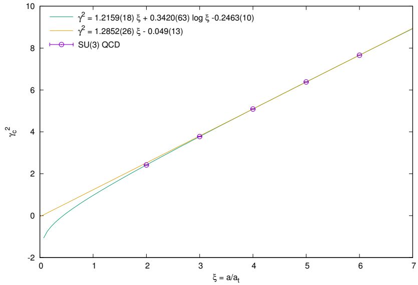
In the chiral limit, the Grassmann constraints imply locally conserved currents:

$$j_{x\mu} = \sigma_x \left(k_{x\mu} - \frac{3}{2}|b_{x\mu}| - \frac{3}{8} \right) \implies \sum_{\pm\mu} (j_{x\mu} - j_{x-\hat{\mu},\mu}) = 0$$

The variances of the associated conserved charges, $j_\mu = \sum_{x \perp \hat{\mu}} j_{x\mu}$ are used to calibrate the anisotropy, $\xi(\gamma) = \frac{a}{a_t}$, just like in the $U(3)$ case.

$$\xi(\gamma) \sim \gamma^2$$

but the prefactor again deviates from the mean field prediction ($\xi = \gamma^2$)

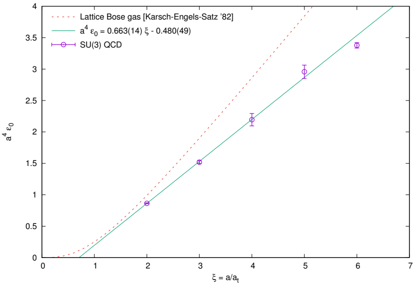


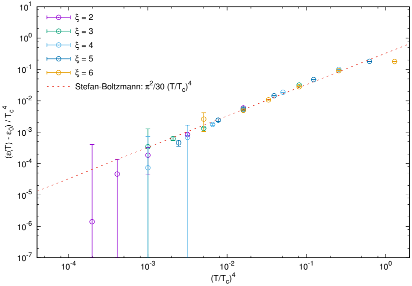
Energy density at $T = 0$ and $T > 0$

After subtracting the $T = 0$ contributions:

$$a^4 \varepsilon_0(\xi) = \lim_{N_s \to \infty} \frac{\xi^2}{\gamma} \frac{d\gamma}{d\xi} \left\langle 2n_{Dt} + 3n_{Bt} \right\rangle \Big|_{N_t = \xi N_s} \quad \text{(hypercubic)}$$

we obtain a similar plot for the energy density at finite T, in units of the $SU(3)$ critical temperature: $aT_c = 1.402(2)$ [Forcrand-Langelage-Philipsen-Unger '14]





Measuring of the running anisotropy

The variances of the currents j_μ scale with the volume of lattice slices $\perp \hat{\mu}$:

$$\begin{cases} \langle j_t^2 \rangle \propto a^3 \\ \langle j_s^2 \rangle \propto a^2 a_t \end{cases} \qquad \Rightarrow \qquad \frac{\langle j_t^2 \rangle}{\langle j_s^2 \rangle} = \frac{N_s}{N_t} \xi$$

The derivative of this ratio wrt the bare anisotropy coupling, at the critical value γ_c, is related to the running of the anisotropy coupling:

$$\frac{d}{d\gamma} \frac{\langle j_t^2 \rangle}{\langle j_s^2 \rangle} \bigg|_{\gamma_c} = \frac{1}{\langle j^2 \rangle_c} \left(\frac{d}{d\gamma} \langle j_t^2 \rangle - \frac{d}{d\gamma} \langle j_s^2 \rangle \right)_{\gamma_c} = \frac{N_s}{N_t} \frac{d\xi}{d\gamma} \bigg|_{\gamma_c} = \frac{1}{\xi} \frac{d\xi}{d\gamma} \bigg|_{\gamma_c}$$

Inverting the relation above, we finally obtain:

$$\xi \frac{d\gamma}{d\xi} = \frac{\langle j^2 \rangle_c}{\left(\frac{d}{d\gamma} \langle j_t^2 \rangle - \frac{d}{d\gamma} \langle j_s^2 \rangle \right)_{\gamma_c}}$$