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A Worm Algorithm for the Lattice $\mathbb{C}P^{N-1}$ Model

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Motivation

1. Why $\mathbb{C}P^{N-1}$?

$\rightarrow$ 2D $\mathbb{C}P^{N-1}$ is cheap toy model for 4D YM :

- non-trivial vacuum structure,
- confinement,
- asymptotic freedom.

2. Why consider (new) worm algorithm?

$\rightarrow$ Dual formulation of $\mathbb{C}P^{N-1}$ lattice model with N^2 d.o.f. per link [Chandrasekharan, PoS 2008]:

- worm algorithm found to be non-ergodic [Vetter, 2011 (semester thesis)],
- meantime found solution to ergodicity problem, $\leftarrow$ topic of this talk
- can easily be generalized to finite densities.

$\rightarrow$ Loop algorithm based on dual formulation with one d.o.f. per link [Wolff, arXiv:1001.2231]:

- no critical slowing down, but alg. can scale badly in some regions of config. space,
- (can it be generalized to finite density?)

$\rightarrow$ Most recent dual formulation in terms of $2N$ d.o.f. per link [Bruckmann et al., arXiv:1507.04253]:

- has not been tested so far,
- already allows for finite densities,
- is it more efficient than the N^2 d.o.f. per link version?

1. The $\mathbb{C}P^{N-1}$ Model: the two standard formulations (continuum)

- $\mathbb{C}P^{N-1}$ model as $\mathrm{U}(1)$ gauged non-linear $\mathrm{SU}(N)$ sigma model:
  - in d-dimensional continuous space:

$$S_A = -\frac{1}{g}\int \mathrm{d}^d x \, (\mathrm{D}_\mu z)^\dagger \cdot (\mathrm{D}_\mu z)$$

with:

- $z : \mathbb{R}^d \to \mathbb{C}^N$, a N-component complex scalar field with $z^\dagger \cdot z = 1$,
- $\mathrm{D}_\mu = \partial_\mu + \mathrm{i}\, A_\mu$: covariant derivative with respect to auxiliary $\mathrm{U}(1)$ gauge field A_μ,
- and coupling strength g.

- Using Euler-Lagrange equation for auxiliary gauge field $\rightarrow$ Quartic formulation:
  - in d-dimensional continuous space:

$$S_Q = -\frac{1}{g}\int \mathrm{d}^d x \left((\partial_\mu z^\dagger)\cdot(\partial_\mu z) + \frac{1}{4}\left(z^\dagger \cdot (\partial_\mu z) - (\partial_\mu z^\dagger)\cdot z\right)^2 \right)$$

with:

- $z : \mathbb{R}^d \to \mathbb{C}^N$, $z^\dagger \cdot z = 1$,
- coupling strength g.

1. The $\mathbb{C}P^{N-1}$ Model: the two standard formulations (lattice)

- $\mathbb{C}P^{N-1}$ model as $\mathrm{U}(1)$ gauged non-linear $\mathrm{SU}(N)$ sigma model:
  - on a d-dimensional lattice:

$$S_A = -\beta \sum_{x,\mu} \Big(z^\dagger(x) U_\mu(x) z(x+\widehat{\mu}) + z^\dagger(x) U_\mu^\dagger(x-\widehat{\mu}) z(x-\widehat{\mu}) - 2 \Big)$$

  with:
    - $z(x) \in \mathbb{C}^N$, a N-component complex vector with $z^\dagger(x) \cdot z(x) = 1$,
    - auxiliary link variable: $U_\mu(x) \in \mathrm{U}(1)$,
    - and dimensionless coupling β .

- Quartic formulation:
  - on a d-dimensional lattice:

$$S_Q = -\beta \sum_{x,\mu} \left| z^\dagger(x) \cdot z(x+\widehat{\mu}) \right|^2$$

  with:
    - $z(x) \in \mathbb{C}^N$, a N-component complex vector with $z^\dagger(x) \cdot z(x) = 1$,
    - and dimensionless coupling β .

- Quartic version: [cf. Chandrasekharan, PoS LATTICE 2008 (2008) 003]

  - Partition function

$$Z_Q = \int \mathcal{D}[z^\dagger, z] \exp\left(\beta \sum_{x,\mu} \left| z^\dagger(x) \cdot z(x+\widehat{\mu}) \right|^2 \right)$$

where $\mathcal{D}[z^\dagger, z] = \prod_x \delta\left(|z(x)|^2 - 1\right) \mathrm{d}^N \bar{z}(x)\, \mathrm{d}^N z(x)$.

1. The $\mathbb{C}P^{N-1}$ Model: flux-variable representation of partition function

- Quartic version: [cf. Chandrasekharan, PoS LATTICE 2008 (2008) 003]

  - Partition function

$$Z_Q = \int \mathcal{D}[z^{\dagger},z] \exp\left(\beta \sum_x \sum_{\mu=1}^{d} \sum_{a,b=1}^{N} \left(\bar{z}_a(x) z_b(x)\right)\left(\bar{z}_b(x+\widehat{\mu}) z_a(x+\widehat{\mu})\right)\right) ,$$

where $\mathcal{D}[z^{\dagger},z] = \prod_x \delta\left(|z(x)|^2 - 1\right) \mathrm{d}^N \bar{z}(x)\, \mathrm{d}^N z(x)$.

1. The $\mathbb{C}P^{N-1}$ Model: flux-variable representation of partition function

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$$Z_Q = \int \mathcal{D}[z^\dagger, z] \prod_x \prod_{\mu=1}^{d} \prod_{a,b=1}^{N} \exp\Big(\beta \big(\bar{z}_a(x) z_b(x)\big) \big(\bar{z}_b(x+\widehat{\mu}) z_a(x+\widehat{\mu})\big) \Big) ,$$

where $\mathcal{D}[z^\dagger, z] = \prod_x \delta\big(|z(x)|^2 - 1\big) \mathrm{d}^N \bar{z}(x) \, \mathrm{d}^N z(x)$.

- Quartic version: [cf. Chandrasekharan, PoS LATTICE 2008 (2008) 003]

  - Partition function

$$Z_Q = \int \mathcal{D}[z^\dagger, z] \prod_x \prod_{\mu=1}^{d} \prod_{a,b=1}^{N} \sum_{n_{x,\mu}^{ab}=0}^{\infty} \left\{ \frac{\beta^{n_{x,\mu}^{ab}}}{n_{x,\mu}^{ab}!} \left(\left(\bar{z}_a(x) z_b(x) \right) \left(\bar{z}_b(x+\widehat{\mu}) z_a(x+\widehat{\mu}) \right) \right)^{n_{x,\mu}^{ab}} \right\}$$

where $\mathcal{D}\left[z^\dagger, z\right] = \prod_x \delta\left(|z(x)|^2 - 1\right) \mathrm{d}^N \bar{z}(x) \, \mathrm{d}^N z(x)$.

$\cdots$

$\cdots$

1. The $\mathbb{C}P^{N-1}$ Model: flux-variable representation of partition function

- Quartic version:

$$Z_Q = \sum_{\{k,l\}} \prod_x \bigg\{ \bigg(\prod_{\mu=1}^{d} \prod_{a,b=1}^{N} \frac{\beta^{\frac{1}{2}(|k_{x,\mu}^{ab}|+k_{x,\mu}^{ab})+l_{x,\mu}^{ab}}}{\big(\frac{1}{2}(|k_{x,\mu}^{ab}|+k_{x,\mu}^{ab})+l_{x,\mu}^{ab}\big)!} \bigg)$$
$$\frac{\prod_a^N \Big(\delta\big(\sum_{\mu=1}^{d} \sum_{b=1}^{N} (k_{x,\mu}^{ab} - k_{x-\hat{\mu},\mu}^{ab}) \big) \big(\sum_{\mu=1}^{d} \sum_{b=1}^{N} \big(\frac{1}{2}(|k_{x,\mu}^{ab}| + |k_{x-\hat{\mu},\mu}^{ab}|) + l_{x,\mu}^{ab} + l_{x-\hat{\mu},\mu}^{ab} \big) \big)! \Big)}{\big(N - 1 + \sum_{\mu=1}^{d} \sum_{c,b=1}^{N} \big(\frac{1}{2}(|k_{x,\mu}^{cb}| + |k_{x-\hat{\mu},\mu}^{cb}|) + l_{x,\mu}^{cb} + l_{x-\hat{\mu},\mu}^{cb} \big) \big)!} \bigg\} ,$$

- with:

$$n_{x,\mu}^{ab} - n_{x,\mu}^{ba} = k_{x,\mu}^{ab} \in \mathbb{Z} \quad , \qquad \frac{1}{2}\big(n_{x,\mu}^{ab} + n_{x,\mu}^{ba} - |k_{x,\mu}^{ab}|\big) = l_{x,\mu}^{ab} \in \mathbb{N}_0 \,,$$

i.e.

$$n_{x,\mu}^{ab} = \frac{1}{2}\big(|k_{x,\mu}^{ab}| + k_{x,\mu}^{ab}\big) + l_{x,\mu}^{ab} \,.$$

- Quartic version:

$$Z_Q = \sum_{\{k,l\}} \prod_x \Bigg\{ \Bigg(\prod_{\mu=1}^{d} \prod_{a,b=1}^{N} \frac{\beta^{\frac{1}{2}(|k_{x,\mu}^{ab}|+k_{x,\mu}^{ab})+l_{x,\mu}^{ab}}}{\left(\frac{1}{2}(|k_{x,\mu}^{ab}|+k_{x,\mu}^{ab})+l_{x,\mu}^{ab}\right)!} \Bigg) \frac{\prod_{a}^{N} \Big(\delta\big(\sum_{\mu=1}^{d} \sum_{b=1}^{N} (k_{x,\mu}^{ab} - k_{x-\widehat{\mu},\mu}^{ab}) \big) \big(\sum_{\mu=1}^{d} \sum_{b=1}^{N} \big(\frac{1}{2}(|k_{x,\mu}^{ab}| + |k_{x-\widehat{\mu},\mu}^{ab}|) + l_{x,\mu}^{ab} + l_{x-\widehat{\mu},\mu}^{ab} \big) \big)! \Big)}{\big(N - 1 + \sum_{\mu=1}^{d} \sum_{c,b=1}^{N} \big(\frac{1}{2}(|k_{x,\mu}^{cb}| + |k_{x-\widehat{\mu},\mu}^{cb}|) + l_{x,\mu}^{cb} + l_{x-\widehat{\mu},\mu}^{cb} \big) \big)!} \Bigg\} ,$$

- with:

$$n_{x,\mu}^{ab} - n_{x,\mu}^{ba} = k_{x,\mu}^{ab} \in \mathbb{Z} \quad , \qquad \frac{1}{2}\left(n_{x,\mu}^{ab} + n_{x,\mu}^{ba} - |k_{x,\mu}^{ab}|\right) = l_{x,\mu}^{ab} \in \mathbb{N}_0 \, ,$$

  i.e.

$$n_{x,\mu}^{ab} = \frac{1}{2}\left(|k_{x,\mu}^{ab}| + k_{x,\mu}^{ab}\right) + l_{x,\mu}^{ab} \, .$$

- $N(N-1)/2$ anti-symmetric $k_{x,\mu}^{ab}$, subject to constraints.

- Quartic version:

$$Z_Q = \sum_{\{k,l\}} \prod_x \Bigg\{ \bigg(\prod_{\mu=1}^{d} \prod_{a,b=1}^{N} \frac{\beta^{\frac{1}{2}(|k^{ab}_{x,\mu}|+k^{ab}_{x,\mu})+l^{ab}_{x,\mu}}}{\left(\frac{1}{2}(|k^{ab}_{x,\mu}|+k^{ab}_{x,\mu})+l^{ab}_{x,\mu}\right)!} \bigg) \frac{\prod_a^N \Big(\delta\big(\sum_{\mu=1}^{d}\sum_{b=1}^{N}(k^{ab}_{x,\mu}-k^{ab}_{x-\widehat{\mu},\mu})\big)\big(\sum_{\mu=1}^{d}\sum_{b=1}^{N}\big(\frac{1}{2}(|k^{ab}_{x,\mu}|+|k^{ab}_{x-\widehat{\mu},\mu}|)+l^{ab}_{x,\mu}+l^{ab}_{x-\widehat{\mu},\mu}\big)\big)! \Big)}{\big(N-1+\sum_{\mu=1}^{d}\sum_{c,b=1}^{N}\big(\frac{1}{2}(|k^{cb}_{x,\mu}|+|k^{cb}_{x-\widehat{\mu},\mu}|)+l^{cb}_{x,\mu}+l^{cb}_{x-\widehat{\mu},\mu}\big)\big)!} \Bigg\} ,$$

- with:

$$n^{ab}_{x,\mu} - n^{ba}_{x,\mu} = k^{ab}_{x,\mu} \in \mathbb{Z} \quad , \qquad \frac{1}{2}\left(n^{ab}_{x,\mu} + n^{ba}_{x,\mu} - \left|k^{ab}_{x,\mu}\right|\right) = l^{ab}_{x,\mu} \in \mathbb{N}_0 \, ,$$

i.e.

$$n^{ab}_{x,\mu} = \frac{1}{2}\left(|k^{ab}_{x,\mu}| + k^{ab}_{x,\mu}\right) + l^{ab}_{x,\mu} \, .$$

- $N(N-1)/2$ anti-symmetric $k^{ab}_{x,\mu}$, subject to constraints.
- $N(N+1)/2$ symmetric $l^{ab}_{x,\mu}$, free from constraints.

1. The $\mathbb{C}P^{N-1}$ Model: flux-variable representation of partition function

- Quartic version:

$$Z_Q = \sum_{\{k,l\}} \prod_x \bigg\{ \bigg(\prod_{\mu=1}^{d} \prod_{a,b=1}^{N} \frac{\beta^{\frac{1}{2}(|k^{ab}_{x,\mu}|+k^{ab}_{x,\mu})+l^{ab}_{x,\mu}}}{\big(\frac{1}{2}(|k^{ab}_{x,\mu}|+k^{ab}_{x,\mu})+l^{ab}_{x,\mu}\big)!} \bigg) \frac{\prod_{a}^{N}\bigg(\delta\big(\sum_{\mu=1}^{d}\sum_{b=1}^{N}(k^{ab}_{x,\mu}-k^{ab}_{x-\widehat{\mu},\mu})\big)\big(\sum_{\mu=1}^{d}\sum_{b=1}^{N}\big(\frac{1}{2}(|k^{ab}_{x,\mu}|+|k^{ab}_{x-\widehat{\mu},\mu}|)+l^{ab}_{x,\mu}+l^{ab}_{x-\widehat{\mu},\mu}\big)\big)!\bigg)}{\big(N-1+\sum_{\mu=1}^{d}\sum_{c,b=1}^{N}\big(\frac{1}{2}(|k^{cb}_{x,\mu}|+|k^{cb}_{x-\widehat{\mu},\mu}|)+l^{cb}_{x,\mu}+l^{cb}_{x-\widehat{\mu},\mu}\big)\big)!} \bigg\} ,$$

  - with:

$$n^{ab}_{x,\mu} - n^{ba}_{x,\mu} = k^{ab}_{x,\mu} \in \mathbb{Z} \quad , \qquad \frac{1}{2}\big(n^{ab}_{x,\mu} + n^{ba}_{x,\mu} - |k^{ab}_{x,\mu}|\big) = l^{ab}_{x,\mu} \in \mathbb{N}_0 \, ,$$

  i.e.

$$n^{ab}_{x,\mu} = \frac{1}{2}\big(|k^{ab}_{x,\mu}| + k^{ab}_{x,\mu}\big) + l^{ab}_{x,\mu} \, .$$

  - $N(N-1)/2$ anti-symmetric $k^{ab}_{x,\mu}$, subject to constraints.
  - $N(N+1)/2$ symmetric $l^{ab}_{x,\mu}$, free from constraints.

$\rightarrow$ in total N^2 degrees of freedom per link.

- Auxiliary $\mathrm{U}(1)$ version v1:

  - Partition function

$$Z_A = \int \mathcal{D}[z^\dagger, z, U] \exp\left(\beta \sum_{x,\mu}\left(z^\dagger(x) U_\mu(x) z(x+\widehat{\mu}) + z^\dagger(x) U_\mu^\dagger(x-\widehat{\mu}) z(x-\widehat{\mu})\right)\right),$$

  where

    - $\mathcal{D}\left[z^\dagger, z, U\right] = \prod_x \delta\left(|z(x)|^2 - 1\right) \mathrm{d}^N \bar{z}(x)\, \mathrm{d}^N z(x)\, \mathrm{d}U_\mu(x)$
    - $\mathrm{d}U_\mu(x)$: $\mathrm{U}(1)$ Haar measure for link variable $U_\mu(x)$.

- Auxiliary $\mathrm{U}(1)$ version v1:
  - Partition function

$$Z_{A,v1} = \int \mathcal{D}[z^{\dagger},z] \prod_{x,\mu} \{ I_0\big(2\beta\,\big|z^{\dagger}(x)\cdot z(x+\widehat{\mu})\big|\big)\,\mathrm{e}^{-2\beta} \}\,,$$

where

- $\mathcal{D}\big[z^{\dagger},z\big] = \prod_x \delta\big(|z(x)|^2 - 1\big)\,\mathrm{d}^N\bar{z}(x)\,\mathrm{d}^N z(x)\,,$
- have used $\mathrm{e}^{x(t+1/t)} = \sum\limits_{k=-\infty}^{\infty} I_k(2x)\,t^k$ and integrated out $U_\mu(x) = \mathrm{e}^{\mathrm{i}\theta_{x,\mu}}$.

The $\mathbb{C}P^{N-1}$ Model: flux-variable representation of partition function

- Auxiliary $\mathrm{U}(1)$ version v1:
  - Partition function

$$Z_{A,v1} = \sum_{\{k,l\}} \prod_x \Bigg\{ \Bigg(\prod_{\mu=1}^{d} \frac{\mathrm{e}^{-2\beta}}{\big(\sum_{a,b=1}^{N} \big(\frac{1}{2} |k_{x,\mu}^{ab}| + l_{x,\mu}^{ab} \big) \big)!} \Bigg(\prod_{a,b=1}^{N} \frac{\beta^{(|k_{x,\mu}^{ab}| + k_{x,\mu}^{ab}) + 2 l_{x,\mu}^{ab}}}{\big(\frac{1}{2} \big(|k_{x,\mu}^{ab}| + k_{x,\mu}^{ab} \big) + l_{x,\mu}^{ab} \big)!} \Bigg) \Bigg)$$
$$\frac{\prod_a^N \delta\big(\sum_{\mu=1}^{d} \sum_{b=1}^{N} \big(k_{x,\mu}^{ab} - k_{x-\widehat{\mu},\mu}^{ab} \big) \big) \big(\sum_{\mu=1}^{d} \sum_{b=1}^{N} \big(\frac{1}{2} \big(|k_{x,\mu}^{ab}| + |k_{x-\widehat{\mu},\mu}^{ab}| \big) + l_{x,\mu}^{ab} + l_{x-\widehat{\mu},\mu}^{ab} \big) \big)!}{\big(N - 1 + \sum_{\mu=1}^{d} \sum_{c,b=1}^{N} \big(\frac{1}{2} \big(|k_{x,\mu}^{cb}| + |k_{x-\widehat{\mu},\mu}^{cb}| \big) + l_{x,\mu}^{cb} + l_{x-\widehat{\mu},\mu}^{cb} \big) \big)!} \Bigg\} .$$

  - same as Z_Q except for additional link weight .
  - In total also N^2 degrees of freedom per link.

- Auxiliary $\mathrm{U}(1)$ version v2: [Bruckmann et al., arXiv:1507.04253]
  - Partition function

$$Z_{A,v2} = \sum_{\{k,l\}} \Bigg\{ \prod_x \Bigg(\prod_\nu \mathrm{e}^{-2\beta}\, \delta\big(\sum_a k^a_{x,\nu}\big) \prod_a \mathrm{e}^{\tilde{\mu}_a k^a_{x,\nu} \delta_{\nu,d}} \frac{\beta^{|k^a_{x,\nu}|+2 l^a_{x,\nu}}}{\big(|k^a_{x,\nu}| + l^a_{x,\nu}\big)!\, l^a_{x,\nu}!} \Bigg) \cdot \frac{\prod_a \delta\big(\sum_\nu (k^a_{x,\nu} - k^a_{x-\hat{\nu}})\big) \big(\sum_\nu \big(\frac{1}{2}\big(|k^a_{x,\nu}| + |k^a_{x-\hat{\nu},\nu}|\big) + l^a_{x,\nu} + l^a_{x-\hat{\nu},\nu}\big)\big)!}{\big(N - 1 + \sum_a \sum_\nu \big(\frac{1}{2}\big(|k^a_{x,\nu}| + |k^a_{x-\hat{\nu},\nu}|\big) + l^a_{x,\nu} + l^a_{x-\hat{\nu},\nu}\big)\big)!} \Bigg\} .$$

- Auxiliary $\mathrm{U}(1)$ version v2: [Bruckmann et al., arXiv:1507.04253]
  - Partition function

$$Z_{A,v2} = \sum_{\{k,l\}} \Big\{ \prod_x \Big(\prod_\nu \mathrm{e}^{-2\beta} \, \delta\big(\sum_a k^a_{x,\nu}\big) \prod_a \mathrm{e}^{\tilde{\mu}_a k^a_{x,\nu} \delta_{\nu,d}} \, \frac{\beta^{|k^a_{x,\nu}|+2l^a_{x,\nu}}}{\big(|k^a_{x,\nu}| + l^a_{x,\nu}\big)!\, l^a_{x,\nu}!} \Big) \cdot \frac{\prod_a \delta\big(\sum_\nu (k^a_{x,\nu} - k^a_{x-\widehat{\nu}})\big) \big(\sum_\nu \big(\frac{1}{2}\big(|k^a_{x,\nu}| + |k^a_{x-\widehat{\nu},\nu}|\big) + l^a_{x,\nu} + l^a_{x-\widehat{\nu},\nu}\big)\big)!}{\big(N - 1 + \sum_a \sum_\nu \big(\frac{1}{2}\big(|k^a_{x,\nu}| + |k^a_{x-\widehat{\nu},\nu}|\big) + l^a_{x,\nu} + l^a_{x-\widehat{\nu},\nu}\big)\big)!} \Big\} \, .$$

  - In total only $2N$ degrees of freedom per link: N constrained $k^a_{x,\nu}$ and N unconstrained $l^a_{x,\nu}$ variables.

- Auxiliary $\mathrm{U}(1)$ version v2: [Bruckmann et al., arXiv:1507.04253]
  - Partition function

$$Z_{A,v2} = \sum_{\{k,l\}} \Bigg\{ \prod_x \bigg(\prod_\nu \mathrm{e}^{-2\beta} \, \delta\Big(\sum_a k^a_{x,\nu}\Big) \prod_a \mathrm{e}^{\tilde{\mu}_a k^a_{x,\nu} \delta_{\nu,d}} \frac{\beta^{|k^a_{x,\nu}|+2 l^a_{x,\nu}}}{\big(|k^a_{x,\nu}| + l^a_{x,\nu}\big)!\, l^a_{x,\nu}!} \bigg) \cdot \frac{\prod_a \delta\Big(\sum_\nu \big(k^a_{x,\nu} - k^a_{x-\widehat{\nu}}\big)\Big) \Big(\sum_\nu \big(\frac{1}{2}\big(|k^a_{x,\nu}| + |k^a_{x-\widehat{\nu},\nu}|\big) + l^a_{x,\nu} + l^a_{x-\widehat{\nu},\nu}\big)\Big)!}{\Big(N-1+\sum_a\sum_\nu \big(\frac{1}{2}\big(|k^a_{x,\nu}| + |k^a_{x-\widehat{\nu},\nu}|\big) + l^a_{x,\nu} + l^a_{x-\widehat{\nu},\nu}\big)\Big)!} \Bigg\} .$$

  - In total only $2\,N$ degrees of freedom per link: N constrained $k^a_{x,\nu}$ and N unconstrained $l^a_{x,\nu}$ variables.
  - Two types of constraints: new *on-link constraint* and usual *on-site constraint*.

- Auxiliary $\mathrm{U}(1)$ version v2: [Bruckmann et al., arXiv:1507.04253]
  - Partition function

$$Z_{A,v2} = \sum_{\{k,l\}}\Big\{\prod_x\Big(\prod_\nu \mathrm{e}^{-2\beta}\,\delta\big(\sum_a k^a_{x,\nu}\big)\prod_a \mathrm{e}^{\tilde{\mu}_a k^a_{x,\nu}\delta_{\nu,d}}\,\frac{\beta^{|k^a_{x,\nu}|+2l^a_{x,\nu}}}{\big(|k^a_{x,\nu}|+l^a_{x,\nu}\big)!\,l^a_{x,\nu}!}\Big)$$
$$\cdot\frac{\prod_a \delta\Big(\sum_\nu\big(k^a_{x,\nu}-k^a_{x-\hat{\nu}}\big)\Big)\Big(\sum_\nu\big(\tfrac{1}{2}\big(|k^a_{x,\nu}|+|k^a_{x-\hat{\nu},\nu}|\big)+l^a_{x,\nu}+l^a_{x-\hat{\nu},\nu}\big)\Big)!}{\Big(N-1+\sum_a\sum_\nu\big(\tfrac{1}{2}\big(|k^a_{x,\nu}|+|k^a_{x-\hat{\nu},\nu}|\big)+l^a_{x,\nu}+l^a_{x-\hat{\nu},\nu}\big)\Big)!}\Big\}\,.$$

  - In total only $2N$ degrees of freedom per link: N constrained $k^a_{x,\nu}$ and N unconstrained $l^a_{x,\nu}$ variables.
  - Two types of constraints: new *on-link constraint* and usual *on-site constraint*.
  - Chemical potentials $\tilde{\mu}_a$ couple only to $k^a_{x,d}$ variables $\Rightarrow$ on-site constraint represents charge conservation.

1. The $\mathbb{C}P^{N-1}$ Model: flux-variable representation of partition function

- Auxiliary $\mathrm{U}(1)$ version v2: [Bruckmann et al., arXiv:1507.04253]

  - Partition function

$$Z_{A,v2} = \sum_{\{k,l\}} \Bigg\{ \prod_x \Bigg(\prod_\nu \mathrm{e}^{-2\beta}\, \delta\big(\sum_a k^a_{x,\nu}\big) \prod_a \mathrm{e}^{\tilde{\mu}_a k^a_{x,\nu} \delta_{\nu,d}}\, \frac{\beta^{|k^a_{x,\nu}|+2 l^a_{x,\nu}}}{\big(|k^a_{x,\nu}| + l^a_{x,\nu}\big)!\, l^a_{x,\nu}!} \Bigg)$$

$$\cdot \frac{\prod_a \delta\Big(\sum_\nu \big(k^a_{x,\nu} - k^a_{x-\widehat{\nu}}\big)\Big)\Big(\sum_\nu \big(\tfrac{1}{2}\big(|k^a_{x,\nu}| + |k^a_{x-\widehat{\nu},\nu}|\big) + l^a_{x,\nu} + l^a_{x-\widehat{\nu},\nu}\big)\Big)!}{\Big(N - 1 + \sum_a \sum_\nu \big(\tfrac{1}{2}\big(|k^a_{x,\nu}| + |k^a_{x-\widehat{\nu},\nu}|\big) + l^a_{x,\nu} + l^a_{x-\widehat{\nu},\nu}\big)\Big)!} \Bigg\} .$$

  - In total only $2N$ degrees of freedom per link: N constrained $k^a_{x,\nu}$ and N unconstrained $l^a_{x,\nu}$ variables.
  - Two types of constraints: new *on-link constraint* and usual *on-site constraint*.
  - Chemical potentials $\tilde{\mu}_a$ couple only to $k^a_{x,d}$ variables $\Rightarrow$ on-site constraint represents charge conservation.
  - Shortcut derivation for how to couple Z_Q and $Z_{A,v1}$ to chemical potentials $\tilde{\mu}_a$: identify $k^a_{x,\nu} = \sum_b k^{ab}_{x,\nu}$

    $\Rightarrow$ additional weight for temporal links:

$$\prod_a \mathrm{e}^{\tilde{\mu}_a \sum_b k^{ab}_{x,d}} .$$

- Auxiliary $\mathrm{U}(1)$ version v2: [Bruckmann et al., arXiv:1507.04253]

  - Partition function

$$Z_{A,v2} = \sum_{\{k,l\}} \Bigg\{ \prod_x \bigg(\prod_\nu \mathrm{e}^{-2\beta\, \delta\left(\sum_a k^a_{x,\nu}\right)} \prod_a \mathrm{e}^{\tilde{\mu}_a k^a_{x,\nu} \delta_{\nu,d}} \frac{\beta^{|k^a_{x,\nu}|+2l^a_{x,\nu}}}{\left(|k^a_{x,\nu}| + l^a_{x,\nu}\right)!\, l^a_{x,\nu}!} \bigg) \cdot \frac{\prod_a \delta\left(\sum_\nu \left(k^a_{x,\nu} - k^a_{x-\hat{\nu}}\right)\right) \left(\sum_\nu \left(\frac{1}{2}\left(|k^a_{x,\nu}| + |k^a_{x-\hat{\nu},\nu}|\right) + l^a_{x,\nu} + l^a_{x-\hat{\nu},\nu}\right)\right)!}{\left(N - 1 + \sum_a \sum_\nu \left(\frac{1}{2}\left(|k^a_{x,\nu}| + |k^a_{x-\hat{\nu},\nu}|\right) + l^a_{x,\nu} + l^a_{x-\hat{\nu},\nu}\right)\right)!} \Bigg\} .$$

  - In total only $2N$ degrees of freedom per link: N constrained $k^a_{x,\nu}$ and N unconstrained $l^a_{x,\nu}$ variables.

  - Two types of constraints: new *on-link constraint* and usual *on-site constraint*.

  - Chemical potentials $\tilde{\mu}_a$ couple only to $k^a_{x,d}$ variables $\Rightarrow$ on-site constraint represents charge conservation.

  - Shortcut derivation for how to couple Z_Q and $Z_{A,v1}$ to chemical potentials $\tilde{\mu}_a$: identify $k^a_{x,\nu} = \sum_b k^{ab}_{x,\nu}$

    $\Rightarrow$ additional weight for temporal links:

$$\prod_a \mathrm{e}^{\tilde{\mu}_a \sum_b k^{ab}_{x,d}} .$$

  - Note: due to anti-symmetry of $k^{ab}_{x,\nu}$ in indices (a,b), on-link constraint which is present in $Z_{A,v2}$ is automatically satisfied in $Z_{A,v1}$.

$$Z_{A,v2} = \sum_{\{k,l\}} \Big\{ \prod_z \Big(\prod_\nu e^{-2\beta} \delta\big(\sum_c k^c_{z,\nu}\big) \prod_c e^{\tilde{\mu}_c k^c_{z,\nu} \delta_{\nu,d}} \frac{\beta^{|k^c_{z,\nu}|+2l^c_{z,\nu}}}{(|k^c_{z,\nu}|+l^c_{z,\nu})!\, l^c_{z,\nu}!} \Big)$$

$$\cdot \Big(\prod_c \delta\big(\sum_\nu (k^c_{z,\nu} - k^c_{z-\widehat{\nu}})\big) \Big)$$

$$\cdot \frac{\prod_c \big(\sum_\nu \big(\frac{1}{2} \big(|k^c_{z,\nu}| + |k^c_{z-\widehat{\nu},\nu}| \big) + l^c_{z,\nu} + l^c_{z-\widehat{\nu},\nu} \big) \big)!}{\big(N - 1 + \sum_c \sum_\nu \big(\frac{1}{2} \big(|k^c_{z,\nu}| + |k^c_{z-\widehat{\nu},\nu}| \big) + l^c_{z,\nu} + l^c_{z-\widehat{\nu},\nu} \big) \big)!} \Big\}$$

- can be simulated by ordinary worm.

$$Z_{A,v2} = \sum_{\{k,l\}} \Big\{ \prod_z \Big(\prod_\nu \mathrm{e}^{-2\beta} \delta\big(\sum_c k^c_{z,\nu}\big) \prod_c \mathrm{e}^{\tilde{\mu}_c k^c_{z,\nu} \delta_{\nu,d}} \frac{\beta^{|k^c_{z,\nu}|+2 l^c_{z,\nu}}}{\big(|k^c_{z,\nu}| + l^c_{z,\nu}\big)!\, l^c_{z,\nu}!} \Big)$$

$$\cdot \Big(\prod_c \delta\big(\sum_\nu (k^c_{z,\nu} - k^c_{z-\widehat{\nu}})\big) \Big)$$

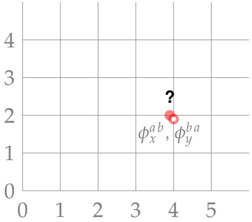
$$\cdot \frac{\prod_c \big(\sum_\nu \big(\tfrac{1}{2} \big(|k^c_{z,\nu}| + |k^c_{z-\widehat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\widehat{\nu},\nu} \big)\big)!}{\big(N - 1 + \sum_c \sum_\nu \big(\tfrac{1}{2} \big(|k^c_{z,\nu}| + |k^c_{z-\widehat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\widehat{\nu},\nu} \big)\big)!} \Big\}$$

- can be simulated by ordinary worm.
- Due to on-link constraint: always two k variables of the same link have to be updated simultaneously in opposite directions, e.g.

$$k^a_{x,\nu} \to k^a_{x,\nu} + 1 \quad , \quad k^b_{x,\nu} \to k^b_{x,\nu} - 1 \quad , \quad a \neq b.$$

$$
\begin{aligned}
Z^{ab}_{A,v2,2}(x,y) = \sum_{\{k,l\}} \Big\{ & \prod_z \Big(\prod_\nu \mathrm{e}^{-2\beta}\,\delta\big(\sum_c k^c_{z,\nu}\big) \prod_c \mathrm{e}^{\tilde{\mu}_c k^c_{z,\nu} \delta_{\nu,d}} \frac{\beta^{|k^c_{z,\nu}|+2 l^c_{z,\nu}}}{\big(|k^c_{z,\nu}| + l^c_{z,\nu}\big)!\, l^c_{z,\nu}!} \Big) \\
& \cdot \Big(\prod_c \delta\Big(\big(\delta^{c,b} - \delta^{c,a}\big)\big(\delta_{x,z} - \delta_{y,z}\big) + \sum_\nu \big(k^c_{z,\nu} - k^c_{z-\hat{\nu}}\big)\Big)\Big) \\
& \cdot \frac{\prod_c \Big(\sum_\nu \Big(\frac{1}{2}\big(\delta^{c,a} + \delta^{c,b}\big)\big(\delta_{x,z} + \delta_{y,z}\big) + \frac{1}{2}\big(|k^c_{z,\nu}| + |k^c_{z-\hat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\hat{\nu},\nu}\Big)\Big)!}{\Big(N - 1 + \delta_{x,z} + \delta_{y,z} + \sum_c \sum_\nu \Big(\frac{1}{2}\big(|k^c_{z,\nu}| + |k^c_{z-\hat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\hat{\nu},\nu}\Big)\Big)!} \Big\}
\end{aligned}
$$

- can be simulated by ordinary worm.
- Due to on-link constraint: always two k variables of the same link have to be updated simultaneously in opposite directions.



where $\phi^{ab}_x = z_a(x)\,\bar{z}_b(x)$

2. Simulation Methods: ordinary worm algorithm for simulating $Z_{A,v2}$

$$Z^{ab}_{A,v2,2}(x,y) = \sum_{\{k,l\}} \Big\{ \prod_z \Big(\prod_\nu e^{-2\beta} \delta\big(\sum_c k^c_{z,\nu}\big) \prod_c e^{\tilde{\mu}_c k^c_{z,\nu} \delta_{\nu,d}} \frac{\beta^{|k^c_{z,\nu}|+2l^c_{z,\nu}}}{\big(|k^c_{z,\nu}|+l^c_{z,\nu}\big)!\, l^c_{z,\nu}!} \Big)$$

$$\cdot \Big(\prod_c \delta\Big(\big(\delta^{c,b}-\delta^{c,a}\big)\big(\delta_{x,z}-\delta_{y,z}\big) + \sum_\nu \big(k^c_{z,\nu} - k^c_{z-\hat{\nu}}\big) \Big) \Big)$$

$$\cdot \frac{\prod_c \Big(\sum_\nu \Big(\frac{1}{2}\big(\delta^{c,a}+\delta^{c,b}\big)\big(\delta_{x,z}+\delta_{y,z}\big) + \frac{1}{2}\big(|k^c_{z,\nu}| + |k^c_{z-\hat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\hat{\nu},\nu} \Big) \Big)!}{\Big(N - 1 + \delta_{x,z} + \delta_{y,z} + \sum_c \sum_\nu \Big(\frac{1}{2}\big(|k^c_{z,\nu}| + |k^c_{z-\hat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\hat{\nu},\nu} \Big) \Big)!} \Big\}$$

- can be simulated by ordinary worm.
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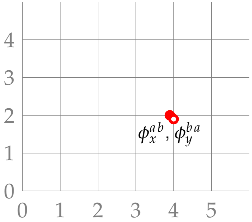

$$Z^{ab}_{A,v2,2}(x,y) = \sum_{\{k,l\}} \Big\{ \prod_{z} \Big(\prod_{\nu} \mathrm{e}^{-2\beta} \delta\big(\sum_{c} k^{c}_{z,\nu}\big) \prod_{c} \mathrm{e}^{\tilde{\mu}_c k^{c}_{z,\nu} \delta_{\nu,d}} \frac{\beta^{|k^{c}_{z,\nu}|+2l^{c}_{z,\nu}}}{(|k^{c}_{z,\nu}|+l^{c}_{z,\nu})!\, l^{c}_{z,\nu}!} \Big)$$
$$\cdot \Big(\prod_{c} \delta\big((\delta^{c,b}-\delta^{c,a})(\delta_{x,z}-\delta_{y,z}) + \sum_{\nu} (k^{c}_{z,\nu} - k^{c}_{z-\widehat{\nu}}) \big) \Big)$$
$$\cdot \frac{\prod_{c} \big(\sum_{\nu} \big(\tfrac{1}{2}(\delta^{c,a}+\delta^{c,b})(\delta_{x,z}+\delta_{y,z}) + \tfrac{1}{2}(|k^{c}_{z,\nu}| + |k^{c}_{z-\widehat{\nu},\nu}|) + l^{c}_{z,\nu} + l^{c}_{z-\widehat{\nu},\nu} \big) \big)!}{\big(N - 1 + \delta_{x,z} + \delta_{y,z} + \sum_{c}\sum_{\nu} \big(\tfrac{1}{2}(|k^{c}_{z,\nu}| + |k^{c}_{z-\widehat{\nu},\nu}|) + l^{c}_{z,\nu} + l^{c}_{z-\widehat{\nu},\nu} \big) \big)!} \Big\}$$

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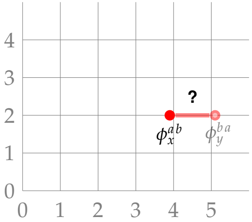

$$k^{a}_{y,1} \to k^{a}_{y,1}+1 \quad , \quad k^{b}_{y,1} \to k^{b}_{y,1}-1 \quad , \quad y \to y+\widehat{1} \ ?$$

$$Z^{ab}_{A,v2,2}(x,y) = \sum_{\{k,l\}} \Big\{ \prod_z \Big(\prod_\nu \mathrm{e}^{-2\beta} \delta\big(\sum_c k^c_{z,\nu}\big) \prod_c \mathrm{e}^{\tilde{\mu}_c k^c_{z,\nu} \delta_{\nu,d}} \frac{\beta^{|k^c_{z,\nu}|+2l^c_{z,\nu}}}{\big(|k^c_{z,\nu}|+l^c_{z,\nu}\big)!\, l^c_{z,\nu}!} \Big)$$

$$\cdot \Big(\prod_c \delta\big((\delta^{c,b}-\delta^{c,a})(\delta_{x,z}-\delta_{y,z}) + \sum_\nu (k^c_{z,\nu} - k^c_{z-\hat{\nu}}) \big) \Big)$$

$$\cdot \frac{\prod_c \big(\sum_\nu \big(\frac{1}{2}(\delta^{c,a}+\delta^{c,b})(\delta_{x,z}+\delta_{y,z}) + \frac{1}{2}\big(|k^c_{z,\nu}| + |k^c_{z-\hat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\hat{\nu},\nu} \big)\big)!}{\big(N-1+\delta_{x,z}+\delta_{y,z} + \sum_c \sum_\nu \big(\frac{1}{2}\big(|k^c_{z,\nu}| + |k^c_{z-\hat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\hat{\nu},\nu} \big)\big)!} \Big\}$$

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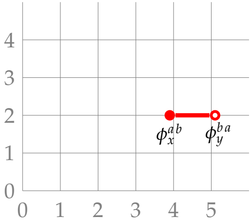

$$Z^{ab}_{A,v2,2}(x,y) = \sum_{\{k,l\}} \Big\{ \prod_z \Big(\prod_\nu \mathrm{e}^{-2\beta} \delta\big(\sum_c k^c_{z,\nu}\big) \prod_c \mathrm{e}^{\tilde{\mu}_c k^c_{z,\nu} \delta_{\nu,d}} \frac{\beta^{|k^c_{z,\nu}|+2l^c_{z,\nu}}}{\big(|k^c_{z,\nu}|+l^c_{z,\nu}\big)!\, l^c_{z,\nu}!} \Big)$$
$$\cdot \Big(\prod_c \delta\Big(\big(\delta^{c,b}-\delta^{c,a}\big)\big(\delta_{x,z}-\delta_{y,z}\big) + \sum_\nu \big(k^c_{z,\nu} - k^c_{z-\hat{\nu}}\big) \Big) \Big)$$
$$\cdot \frac{\prod_c \Big(\sum_\nu \Big(\frac{1}{2}\big(\delta^{c,a}+\delta^{c,b}\big)\big(\delta_{x,z}+\delta_{y,z}\big) + \frac{1}{2}\big(|k^c_{z,\nu}|+|k^c_{z-\hat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\hat{\nu},\nu}\Big)\Big)!}{\Big(N-1+\delta_{x,z}+\delta_{y,z}+\sum_c\sum_\nu \Big(\frac{1}{2}\big(|k^c_{z,\nu}|+|k^c_{z-\hat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\hat{\nu},\nu}\Big)\Big)!} \Big\}$$

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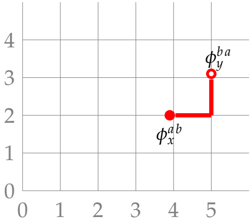

$$Z^{ab}_{A,v2,2}(x,y) = \sum_{\{k,l\}} \Big\{ \prod_{z} \Big(\prod_{\nu} \mathrm{e}^{-2\beta}\, \delta\big(\sum_{c} k^{c}_{z,\nu}\big) \prod_{c} \mathrm{e}^{\tilde{\mu}_c k^{c}_{z,\nu}\delta_{\nu,d}} \frac{\beta^{|k^{c}_{z,\nu}|+2l^{c}_{z,\nu}}}{\big(|k^{c}_{z,\nu}|+l^{c}_{z,\nu}\big)!\, l^{c}_{z,\nu}!} \Big)$$
$$\cdot \Big(\prod_{c} \delta\big((\delta^{c,b}-\delta^{c,a})(\delta_{x,z}-\delta_{y,z}) + \sum_{\nu} (k^{c}_{z,\nu} - k^{c}_{z-\hat{\nu}}) \big) \Big)$$
$$\cdot \frac{\prod_{c} \big(\sum_{\nu} \big(\frac{1}{2}(\delta^{c,a}+\delta^{c,b})(\delta_{x,z}+\delta_{y,z}) + \frac{1}{2}\big(|k^{c}_{z,\nu}| + |k^{c}_{z-\hat{\nu},\nu}|\big) + l^{c}_{z,\nu} + l^{c}_{z-\hat{\nu},\nu} \big) \big)!}{\big(N - 1 + \delta_{x,z} + \delta_{y,z} + \sum_{c}\sum_{\nu} \big(\frac{1}{2}\big(|k^{c}_{z,\nu}| + |k^{c}_{z-\hat{\nu},\nu}|\big) + l^{c}_{z,\nu} + l^{c}_{z-\hat{\nu},\nu} \big) \big)!} \Big\}$$

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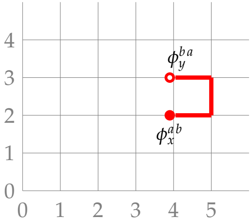

2. Simulation Methods: ordinary worm algorithm for simulating $Z_{A,v2}$

$$Z^{ab}_{A,v2,2}(x,y) = \sum_{\{k,l\}} \Big\{ \prod_z \Big(\prod_\nu \mathrm{e}^{-2\beta} \delta\big(\sum_c k^c_{z,\nu}\big) \prod_c \mathrm{e}^{\tilde{\mu}_c k^c_{z,\nu} \delta_{\nu,d}} \frac{\beta^{|k^c_{z,\nu}|+2l^c_{z,\nu}}}{\big(|k^c_{z,\nu}|+l^c_{z,\nu}\big)!\,l^c_{z,\nu}!} \Big)$$

$$\cdot \Big(\prod_c \delta\Big(\big(\delta^{c,b}-\delta^{c,a}\big)\big(\delta_{x,z}-\delta_{y,z}\big) + \sum_\nu \big(k^c_{z,\nu} - k^c_{z-\hat{\nu}}\big)\Big)\Big)$$

$$\cdot \frac{\prod_c \Big(\sum_\nu \Big(\tfrac{1}{2}\big(\delta^{c,a}+\delta^{c,b}\big)\big(\delta_{x,z}+\delta_{y,z}\big) + \tfrac{1}{2}\big(|k^c_{z,\nu}| + |k^c_{z-\hat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\hat{\nu},\nu}\Big)\Big)!}{\Big(N-1+\delta_{x,z}+\delta_{y,z}+\sum_c\sum_\nu\Big(\tfrac{1}{2}\big(|k^c_{z,\nu}| + |k^c_{z-\hat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\hat{\nu},\nu}\Big)\Big)!} \Big\}$$

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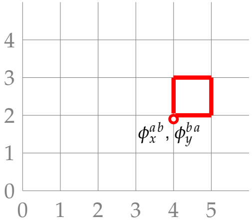

$$Z_{A,v2} = \sum_{\{k,l\}} \Big\{ \prod_z \Big(\prod_\nu \mathrm{e}^{-2\beta} \delta\big(\sum_c k^c_{z,\nu}\big) \prod_c \mathrm{e}^{\tilde{\mu}_c k^c_{z,\nu} \delta_{\nu,d}} \frac{\beta^{|k^c_{z,\nu}|+2 l^c_{z,\nu}}}{(|k^c_{z,\nu}| + l^c_{z,\nu})!\, l^c_{z,\nu}!} \Big)$$

$$\cdot \Big(\prod_c \delta\big(\sum_\nu (k^c_{z,\nu} - k^c_{z-\hat{\nu}})\big) \Big)$$

$$\cdot \frac{\prod_c \big(\sum_\nu \big(\tfrac{1}{2}\big(|k^c_{z,\nu}| + |k^c_{z-\hat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\hat{\nu},\nu}\big)\big)!}{\big(N - 1 + \sum_c \sum_\nu \big(\tfrac{1}{2}\big(|k^c_{z,\nu}| + |k^c_{z-\hat{\nu},\nu}|\big) + l^c_{z,\nu} + l^c_{z-\hat{\nu},\nu}\big)\big)!} \Big\}$$

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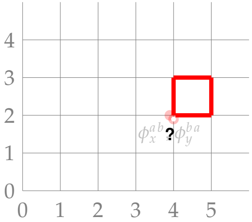

2. Simulation Methods: internal space sub-worm algorithm for simulating $Z_{Q/A,v1}$

$$Z_Q = \sum_{\{k,l\}} \prod_z \Bigg\{ \Bigg(\prod_{\mu=1}^{d} \prod_{c,b=1}^{N} \frac{\beta^{\frac{1}{2}(|k_{z,\mu}^{cb}|+k_{z,\mu}^{cb})+l_{z,\mu}^{cb}}}{\left(\frac{1}{2}(|k_{z,\mu}^{cb}|+k_{z,\mu}^{cb})+l_{z,\mu}^{cb}\right)!} \Bigg) \cdot \Bigg(\prod_{c}^{N} \delta\Big(\sum_{\mu=1}^{d} \sum_{b=1}^{N} (k_{z,\mu}^{cb} - k_{z-\widehat{\mu},\mu}^{cb}) \Big) \Bigg) \cdot \frac{\prod_{c}^{N} \Bigg(\Big(\sum_{\mu=1}^{d} \sum_{b=1}^{N} \big(\frac{1}{2}(|k_{z,\mu}^{cb}| + |k_{z-\widehat{\mu},\mu}^{cb}| + l_{z,\mu}^{cb} + l_{z-\widehat{\mu},\mu}^{cb}) \big) \Big)! \Bigg)}{\Big(N - 1 + \sum_{\mu=1}^{d} \sum_{c,b=1}^{N} \big(\frac{1}{2}(|k_{z,\mu}^{cb}| + |k_{z-\widehat{\mu},\mu}^{cb}|) + l_{z,\mu}^{cb} + l_{z-\widehat{\mu},\mu}^{cb} \big) \Big)!} \Bigg\} .$$

- could in principle be simulated by same algorithm as $Z_{A,v2}$ $\Rightarrow$ ergodicity problem! [R. Vetter, 2011]

2. Simulation Methods: internal space sub-worm algorithm for simulating $Z_{Q/A,v1}$

$$Z^{ab}_{Q,2}(x,y) = \sum_{\{k,l\}} \prod_{z} \Bigg\{ \Bigg(\prod_{\mu=1}^{d} \prod_{c,b=1}^{N} \frac{\beta^{\frac{1}{2}(|k^{cb}_{z,\mu}|+k^{cb}_{z,\mu})+l^{cb}_{z,\mu}}}{\left(\frac{1}{2}(|k^{cb}_{z,\mu}|+k^{cb}_{z,\mu})+l^{cb}_{z,\mu}\right)!} \Bigg)$$

$$\cdot \Bigg(\prod_{c}^{N} \delta\Big((\delta^{c,b} - \delta^{c,a})(\delta_{x,z} - \delta_{y,z}) + \sum_{\mu=1}^{d} \sum_{b=1}^{N} (k^{cb}_{z,\mu} - k^{cb}_{z-\widehat{\mu},\mu}) \Big) \Bigg)$$

$$\cdot \frac{\prod_{c}^{N} \Big(\big(\sum_{\mu=1}^{d} \sum_{b=1}^{N} \big(\frac{1}{2}(\delta^{c,a} + \delta^{c,b})(\delta_{x,z} + \delta_{y,z}) + \frac{1}{2}(|k^{cb}_{z,\mu}| + |k^{cb}_{z-\widehat{\mu},\mu}| + l^{cb}_{z,\mu} + l^{cb}_{z-\widehat{\mu},\mu}) \big) \big)! \Big)}{\big(N - 1 + \delta_{x,z} + \delta_{y,z} + \sum_{\mu=1}^{d} \sum_{c,b=1}^{N} \big(\frac{1}{2}(|k^{cb}_{z,\mu}| + |k^{cb}_{z-\widehat{\mu},\mu}|) + l^{cb}_{z,\mu} + l^{cb}_{z-\widehat{\mu},\mu} \big) \big)!} \Bigg\} .$$

- could in principle be simulated by same algorithm as $Z_{A,v2}$ $\Rightarrow$ ergodicity problem! [R. Vetter, 2011]

Internal space sub-worm algorithm:

- updates not just k^{ab} but all components of the k-matrices while sampling $Z^{ab}_{Q,2}(x,y)$

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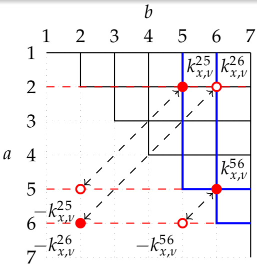

- could in principle be simulated by same algorithm as $Z_{A,v2}$ $\Rightarrow$ ergodicity problem! [R. Vetter, 2011]

Internal space sub-worm algorithm:

- updates not just k^{ab} but all components of the k-matrices while sampling $Z^{ab}_{Q,2}(x,y)$
- before moving head of worm from site x to site $x+\widehat{\nu}$, determine by sub-worm in **internal space** the best combination of $k^{ab}_{x,\nu}$ variables to propagate defect in delta-functions for $c = a_0, b_0$.

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2. Simulation Methods: internal space sub-worm algorithm for simulating $Z_{Q/A,v1}$

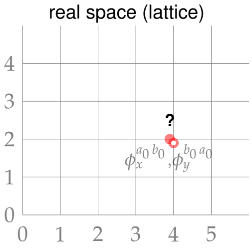
$$Z^{ab}_{Q,2}(x,y) = \sum_{\{k,l\}} \prod_z \Bigg\{ \Bigg(\prod_{\mu=1}^{d} \prod_{c,b=1}^{N} \frac{\beta^{\frac{1}{2}(|k^{cb}_{z,\mu}|+k^{cb}_{z,\mu})+l^{cb}_{z,\mu}}}{\big(\frac{1}{2}(|k^{cb}_{z,\mu}|+k^{cb}_{z,\mu})+l^{cb}_{z,\mu}\big)!} \Bigg)$$

$$\cdot \Bigg(\prod_{c}^{N} \delta\Big(\big(\delta^{c,b}-\delta^{c,a}\big)\big(\delta_{x,z}-\delta_{y,z}\big) + \sum_{\mu=1}^{d}\sum_{b=1}^{N} \big(k^{cb}_{z,\mu} - k^{cb}_{z-\widehat{\mu},\mu}\big)\Big)\Bigg)$$

$$\cdot \frac{\prod_{c}^{N} \Big(\big(\sum_{\mu=1}^{d} \sum_{b=1}^{N} \big(\frac{1}{2}\big(\delta^{c,a}+\delta^{c,b}\big)\big(\delta_{x,z}+\delta_{y,z}\big) + \frac{1}{2}\big(|k^{cb}_{z,\mu}| + |k^{cb}_{z-\widehat{\mu},\mu}| + l^{cb}_{z,\mu} + l^{cb}_{z-\widehat{\mu},\mu}\big)\big)\big)! \Big)}{\Big(N-1+\delta_{x,z}+\delta_{y,z} + \sum_{\mu=1}^{d}\sum_{c,b=1}^{N} \big(\frac{1}{2}\big(|k^{cb}_{z,\mu}| + |k^{cb}_{z-\widehat{\mu},\mu}|\big) + l^{cb}_{z,\mu} + l^{cb}_{z-\widehat{\mu},\mu}\big)\Big)!} \Bigg\} .$$

- could in principle be simulated by same algorithm as $Z_{A,v2}$ $\Rightarrow$ ergodicity problem! [R. Vetter, 2011]

$\Rightarrow$ Internal space sub-worm algorithm:



where $\phi^{ab}_x = \bar{z}_a(x) z_b(x)$

2. Simulation Methods: internal space sub-worm algorithm for simulating $Z_{Q/A,v1}$

$$Z^{ab}_{Q,2}(x,y) = \sum_{\{k,l\}} \prod_z \Bigg\{ \Bigg(\prod_{\mu=1}^{d} \prod_{c,b=1}^{N} \frac{\beta^{\frac{1}{2}(|k^{cb}_{z,\mu}|+k^{cb}_{z,\mu})+l^{cb}_{z,\mu}}}{\big(\frac{1}{2}(|k^{cb}_{z,\mu}|+k^{cb}_{z,\mu})+l^{cb}_{z,\mu}\big)!} \Bigg)$$

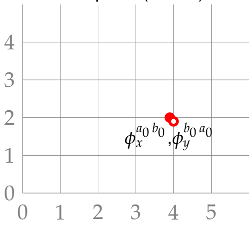
$$\cdot \Bigg(\prod_{c}^{N} \delta\Big(\big(\delta^{c,b}-\delta^{c,a}\big)\big(\delta_{x,z}-\delta_{y,z}\big) + \sum_{\mu=1}^{d} \sum_{b=1}^{N} \big(k^{cb}_{z,\mu} - k^{cb}_{z-\widehat{\mu},\mu}\big)\Big)\Bigg)$$

$$\cdot \frac{\prod_{c}^{N}\Big(\big(\sum_{\mu=1}^{d}\sum_{b=1}^{N}\big(\frac{1}{2}\big(\delta^{c,a}+\delta^{c,b}\big)\big(\delta_{x,z}+\delta_{y,z}\big) + \frac{1}{2}\big(|k^{cb}_{z,\mu}|+|k^{cb}_{z-\widehat{\mu},\mu}|+l^{cb}_{z,\mu}+l^{cb}_{z-\widehat{\mu},\mu}\big)\big)\big)\Big)!}{\Big(N-1+\delta_{x,z}+\delta_{y,z}+\sum_{\mu=1}^{d}\sum_{c,b=1}^{N}\big(\frac{1}{2}\big(|k^{cb}_{z,\mu}|+|k^{cb}_{z-\widehat{\mu},\mu}|\big)+l^{cb}_{z,\mu}+l^{cb}_{z-\widehat{\mu},\mu}\big)\Big)!}\Bigg\}.$$

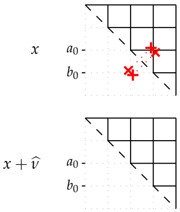
- could in principle be simulated by same algorithm as $Z_{A,v2}$ $\Rightarrow$ ergodicity problem! [R. Vetter, 2011]

$\Rightarrow$ Internal space sub-worm algorithm:

real space (lattice)



internal space (on-site constraints)



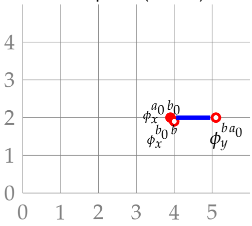
2. Simulation Methods: internal space sub-worm algorithm for simulating $Z_{Q/A,v1}$

$$Z^{ab}_{Q,2}(x,y) = \sum_{\{k,l\}} \prod_z \Bigg\{ \Bigg(\prod_{\mu=1}^{d} \prod_{c,b=1}^{N} \frac{\beta^{\frac{1}{2}(|k^{cb}_{z,\mu}|+k^{cb}_{z,\mu})+l^{cb}_{z,\mu}}}{\left(\frac{1}{2}(|k^{cb}_{z,\mu}|+k^{cb}_{z,\mu})+l^{cb}_{z,\mu}\right)!} \Bigg) \cdot \Bigg(\prod_c^N \delta\Big((\delta^{c,b}-\delta^{c,a})(\delta_{x,z}-\delta_{y,z}) + \sum_{\mu=1}^d \sum_{b=1}^N (k^{cb}_{z,\mu} - k^{cb}_{z-\hat{\mu},\mu})\Big)\Bigg) \cdot \frac{\prod_c^N \Big(\Big(\sum_{\mu=1}^d\sum_{b=1}^N \big(\tfrac12 (\delta^{c,a}+\delta^{c,b})(\delta_{x,z}+\delta_{y,z}) + \tfrac12(|k^{cb}_{z,\mu}|+|k^{cb}_{z-\hat{\mu},\mu}|+l^{cb}_{z,\mu}+l^{cb}_{z-\hat{\mu},\mu})\big)\Big)!\Big)}{\Big(N-1+\delta_{x,z}+\delta_{y,z}+\sum_{\mu=1}^d\sum_{c,b=1}^N\big(\tfrac12(|k^{cb}_{z,\mu}|+|k^{cb}_{z-\hat{\mu},\mu}|)+l^{cb}_{z,\mu}+l^{cb}_{z-\hat{\mu},\mu}\big)\Big)!} \Bigg\}.$$

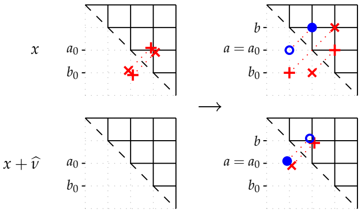
- could in principle be simulated by same algorithm as $Z_{A,v2}$ $\Rightarrow$ ergodicity problem! [R. Vetter, 2011]

$\Rightarrow$ Internal space sub-worm algorithm:

real space (lattice)



internal space (on-site constraints)



$$Z^{ab}_{Q,2}(x,y) = \sum_{\{k,l\}} \prod_z \Big\{ \Big(\prod_{\mu=1}^{d} \prod_{c,b=1}^{N} \frac{\beta^{\frac{1}{2}(|k^{cb}_{z,\mu}|+k^{cb}_{z,\mu})+l^{cb}_{z,\mu}}}{\big(\frac{1}{2}(|k^{cb}_{z,\mu}|+k^{cb}_{z,\mu})+l^{cb}_{z,\mu}\big)!} \Big)$$

$$\cdot \Big(\prod_{c}^{N} \delta\Big((\delta^{c,b}-\delta^{c,a})(\delta_{x,z}-\delta_{y,z}) + \sum_{\mu=1}^{d}\sum_{b=1}^{N} (k^{cb}_{z,\mu} - k^{cb}_{z-\hat{\mu},\mu}) \Big) \Big)$$

$$\cdot \frac{\prod_{c}^{N} \Big(\big(\sum_{\mu=1}^{d}\sum_{b=1}^{N} \big(\frac{1}{2}(\delta^{c,a}+\delta^{c,b})(\delta_{x,z}+\delta_{y,z}) + \frac{1}{2}(|k^{cb}_{z,\mu}| + |k^{cb}_{z-\hat{\mu},\mu}| + l^{cb}_{z,\mu} + l^{cb}_{z-\hat{\mu},\mu}) \big) \big)! \Big)}{\Big(N - 1 + \delta_{x,z} + \delta_{y,z} + \sum_{\mu=1}^{d}\sum_{c,b=1}^{N} \big(\frac{1}{2}(|k^{cb}_{z,\mu}| + |k^{cb}_{z-\hat{\mu},\mu}|) + l^{cb}_{z,\mu} + l^{cb}_{z-\hat{\mu},\mu} \big) \Big)!} \Big\} .$$

- could in principle be simulated by same algorithm as $Z_{A,v2}$ $\Rightarrow$ ergodicity problem! [R. Vetter, 2011]

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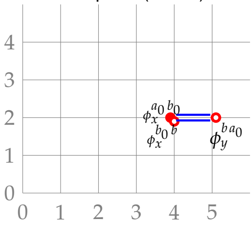
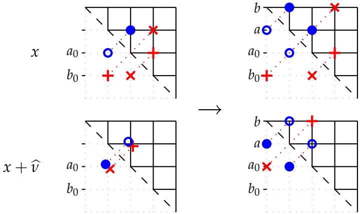


internal space (on-site constraints)



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$$\cdot \Bigg(\prod_{c}^{N} \delta\Big(\big(\delta^{c,b} - \delta^{c,a}\big)\big(\delta_{x,z} - \delta_{y,z}\big) + \sum_{\mu=1}^{d} \sum_{b=1}^{N} \big(k^{cb}_{z,\mu} - k^{cb}_{z-\widehat{\mu},\mu}\big)\Big) \Bigg)$$

$$\cdot \frac{\prod_{c}^{N} \Big(\big(\sum_{\mu=1}^{d} \sum_{b=1}^{N} \big(\frac{1}{2}\big(\delta^{c,a}+\delta^{c,b}\big)\big(\delta_{x,z}+\delta_{y,z}\big) + \frac{1}{2}\big(|k^{cb}_{z,\mu}| + |k^{cb}_{z-\widehat{\mu},\mu}| + l^{cb}_{z,\mu} + l^{cb}_{z-\widehat{\mu},\mu}\big)\big)\big)! \Big)}{\Big(N-1+\delta_{x,z}+\delta_{y,z} + \sum_{\mu=1}^{d} \sum_{c,b=1}^{N} \big(\frac{1}{2}\big(|k^{cb}_{z,\mu}| + |k^{cb}_{z-\widehat{\mu},\mu}|\big) + l^{cb}_{z,\mu} + l^{cb}_{z-\widehat{\mu},\mu}\big)\Big)!} \Bigg\} .$$

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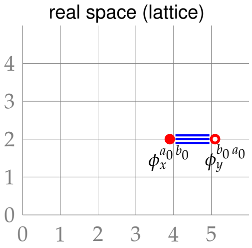
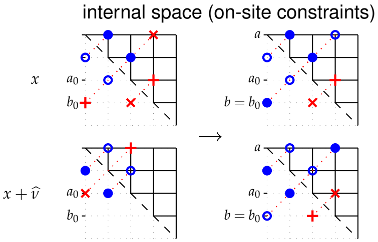
$\Rightarrow$ Internal space sub-worm algorithm:

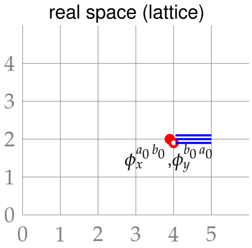


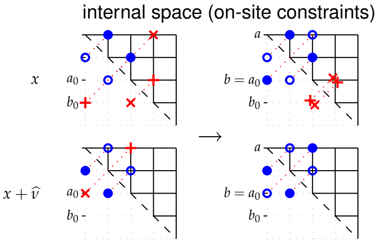


$$Z^{ab}_{Q,2}(x,y) = \sum_{\{k,l\}} \prod_z \Bigg\{ \Bigg(\prod_{\mu=1}^{d} \prod_{c,b=1}^{N} \frac{\beta^{\frac{1}{2}(|k^{cb}_{z,\mu}|+k^{cb}_{z,\mu})+l^{cb}_{z,\mu}}}{\big(\frac{1}{2}(|k^{cb}_{z,\mu}|+k^{cb}_{z,\mu})+l^{cb}_{z,\mu}\big)!} \Bigg) \cdot \Bigg(\prod_c^N \delta\Big((\delta^{c,b}-\delta^{c,a})(\delta_{x,z}-\delta_{y,z}) + \sum_{\mu=1}^d \sum_{b=1}^N \big(k^{cb}_{z,\mu} - k^{cb}_{z-\hat{\mu},\mu}\big) \Big) \Bigg) \cdot \frac{\prod_c^N \Big(\big(\sum_{\mu=1}^d \sum_{b=1}^N \big(\frac{1}{2}(\delta^{c,a}+\delta^{c,b})(\delta_{x,z}+\delta_{y,z}) + \frac{1}{2}(|k^{cb}_{z,\mu}|+|k^{cb}_{z-\hat{\mu},\mu}|+l^{cb}_{z,\mu}+l^{cb}_{z-\hat{\mu},\mu})\big)\big)\Big)!}{\Big(N-1+\delta_{x,z}+\delta_{y,z}+\sum_{\mu=1}^d\sum_{c,b=1}^N\big(\frac{1}{2}(|k^{cb}_{z,\mu}|+|k^{cb}_{z-\hat{\mu},\mu}|)+l^{cb}_{z,\mu}+l^{cb}_{z-\hat{\mu},\mu}\big)\Big)!} \Bigg\} .$$

- could in principle be simulated by same algorithm as $Z_{A,v2}$ $\Rightarrow$ ergodicity problem! [R. Vetter, 2011]

$\Rightarrow$ Internal space sub-worm algorithm:





2. Simulation Methods: cross-check of code

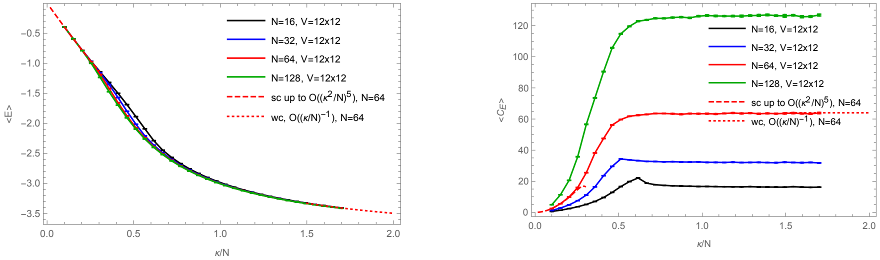
- Both algorithms for auxiliary $\mathrm{U}(1)$ version yield identical results for physical observables!

$\longrightarrow$ comparison with some high-precision numerical results from [J. Flynn et al., arXiv:1504.06292] :

L	β		sweeps	E	ξ_G	χ_m
72	0.8	[Flynn et al.]	80M	0.6670232(7)	4.5992(12)	28.0595(18)
		N^2 aux $\mathrm{U}(1)$	1M	0.6670233(143)	4.5771(232)	28.0528(531)
		$2N$ aux. $\mathrm{U}(1)$	10M	0.6670265(37)	4.5958(41)	28.0593(72)
96	0.85	[Flynn et al.]	80M	0.6222715(5)	6.3926(20)	46.863(4)
		N^2 aux $\mathrm{U}(1)$	1M	0.6222872(101)	6.3865(294)	46.863(56)
		$2N$ aux. $\mathrm{U}(1)$	10M	0.6222721(31)	6.3926(110)	46.845(26)
136	0.9	[Flynn et al.]	80M	0.5838365(3)	8.815(4)	78.202(8)
		N^2 aux $\mathrm{U}(1)$	1M	0.5838202(63)	8.855(81)	78.174(232)
		$2N$ aux. $\mathrm{U}(1)$	3M	0.5838326(29)	8.794(28)	78.191(72)
184	0.95	[Flynn et al.]	100M	0.55026689(20)	12.095(4)	130.707(15)
		N^2 aux $\mathrm{U}(1)$	1M	0.5502642(72)	12.279(149)	131.388(452)
		$2N$ aux. $\mathrm{U}(1)$	1M	0.5502692(36)	12.089(61)	130.824(239)

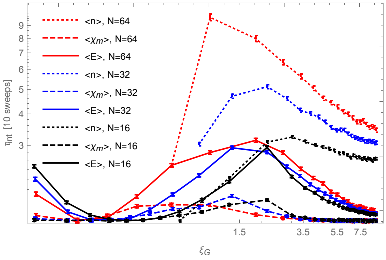
2. Simulation Methods: cross-check of code

- Both algorithms for auxiliary $\mathrm{U}(1)$ version yield identical results for physical observables!
- Comparison with analytic strong and weak coupling results: [Di Vecchia et al.,NPB 190 [FS3] (1981) 719-733]

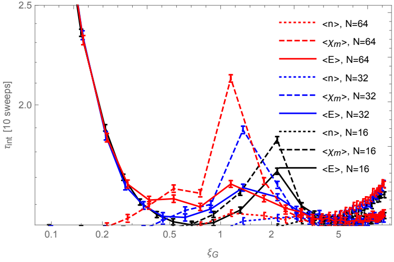


- Integrated auto-correlation time vs. correlation length:

$Z_{A,v1}$ (N^2 d.o.f. per link)



$Z_{A,v2}$ ($2N$ d.o.f. per link)



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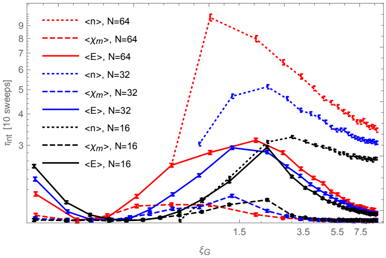
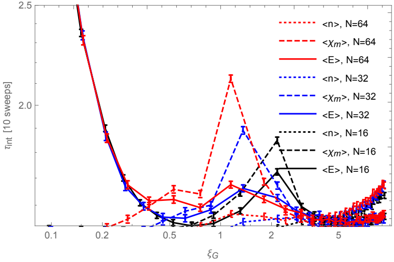
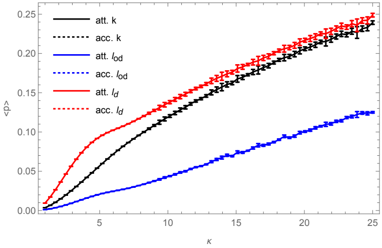


$Z_{A,v2}$ ($2N$ d.o.f. per link)

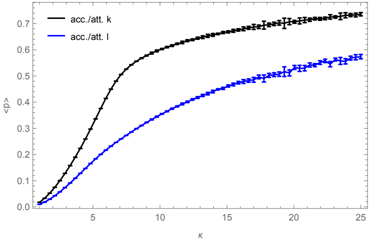


- Acceptance rates (per attempted update):

$Z_{A,v1}$ (N^2 d.o.f. per link)



$Z_{A,v2}$ ($2N$ d.o.f. per link)



4. Conclusion & Outlook

1. Conclusion:
   - Implemented and tested worm algorithms for two ($2N$ vs. N^2 d.o.f. per link) dual formulations of $\mathbb{C}\mathrm{P}^{N-1}$ model which work at finite densities.
   - internal space sub-worm algorithm to solve ergodicity problem in version with N^2 d.o.f. per link.
   - Both algorithms show so far essentially no critical slowing down for physical observables ($\langle n \rangle$, $\langle E \rangle$, χ_m).

2. Outlook:
   - Need results for larger systems to extract dynamical critical exponents.
   - Finite density results.
   - Topological charge and susceptibility (so far only for $\kappa < 0.5$).
   - Finite θ?

Thank you!