Progress on the lattice QCD calculation of rare kaon decays: $K^+ \to \pi^+ \ell^+ \ell^-$

A. Lawson
RBC & UKQCD Collaborations

LATTICE 2016

University of Southampton

July 28, 2016

RBC & UKQCD Collaborations

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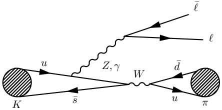
Papers

- "Prospects for a lattice computation of rare kaon decay amplitudes: I $K \to \pi\ell^+\ell^-$ decays", N. H. Christ, X. Feng., A. Portelli and C. T. Sachrajda, PRD 2015. arXiv:1507.03094.
- "Prospects for a lattice computation of rare kaon decay amplitudes: II $K \to \pi\nu\bar{\nu}$ decays", N. H. Christ, X. Feng., A. Portelli and C. T. Sachrajda, PRD 2016. arXiv:1605.04442
- "First exploratory calculation of the long distance contributions to the rare kaon decays $K \to \pi\ell^+\ell^-$" N. H. Christ, X. Feng., A. Juttner, A. Lawson, A. Portelli and C. T. Sachrajda. In preparation.
- $K \to \pi\nu\bar{\nu}$ numerical paper in preparation.

Outline

Rare Kaon Decays

- The processes $K \to \pi\ell\bar{\ell}$ proceed via a flavour changing neutral current (FCNC).
  - $s \to d\ell\bar{\ell}$ transition.
- FCNCs are forbidden at tree level in the SM.
  - Ideal probes for new physics!
- NA62 and KOTO experiments dedicated to studying $K \to \pi\nu\bar{\nu}$.
- This talk concerns long-distance contributions to $K^+ \to \pi^+\ell^+\ell^-$ decays
  - Techniques applicable also to $K_S \to \pi^0\ell^+\ell^-$ decays.



Example long-distance contribution to $K \to \pi\ell\bar{\ell}$ (QCD effects not shown).

[1] CERN-SPSC-2005-013, SPSC-P-326 (2005).
[2] KOTO Collaboration, T. Yamanaka, PTEP 2012 (2012) 02B006.
[3] KOTO Collaboration, E. Iwai, Nucl. Phys. B Proc. Suppl. 233 (2012) 279.

Lattice Methodology

$K \to \pi\ell\bar{\ell}$ on the Lattice

- We compute the amplitude of $K \to \pi\gamma^*$,

$$A_\mu\left(q^2\right) = \int d^4x \left\langle \pi(p) \right| T\left[J_\mu(0) H_W(x)\right] \left| K(k) \right\rangle$$

- The effective Weak Hamiltonian for a $|\Delta S| = 1$ transition is:

$$H_W = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \left(\sum_{i=1}^{2} C_i \left(Q_i^u - Q_i^c\right) + \sum_{j=3}^{8} C_j Q_j + \mathscr{O}\left(\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}}\right)\right).$$

- The Wilson coefficients C_1 and C_2 are substantially larger than the others, hence we require only

$$Q_1^q = \left(\bar{s}_i \gamma_\mu^L d_i\right)\left(\bar{q}_j \gamma_\mu^L q_j\right), \qquad Q_2^q = \left(\bar{s}_i \gamma_\mu^L q_i\right)\left(\bar{q}_j \gamma_\mu^L d_j\right).$$

- We can choose the current to be either the local or conserved lattice vector current.

[4] G. Buchalla et al., *Rev. Mod. Phys.* 68 (1996) 1125-1144. hep-ph/9512380
[5] N. H. Christ et al., *Phys. Rev.* D92 (2015) 094512. arXiv:1507.03094
[6] G. Isidori et al., *Phys. Lett.* B633 (2005) 75-83. hep-lat/0506026

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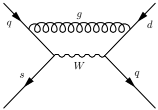

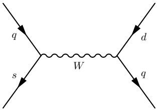

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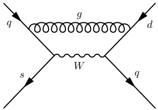

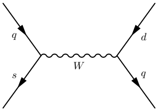

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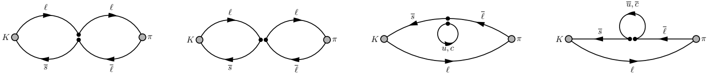
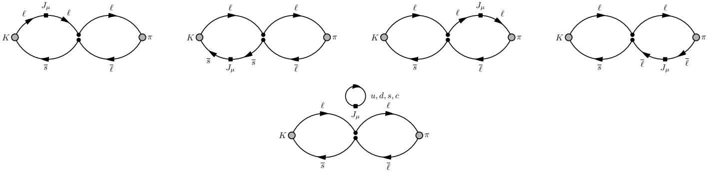
Wick Contractions

- Performing the Wick contractions for just the H_W diagrams, we obtain 4 different classes of diagrams:

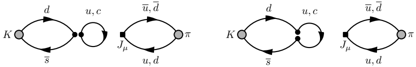


Connected **W**ing **E**ye **S**aucer

- Adding in the current, we obtain 5 diagrams per H_W class:

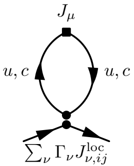


- For the $K_S \to \pi^0 \ell^+ \ell^-$ decay we additionally have:



Renormalisation

- When H_W approaches J_μ the S & E diagram classes are superficially quadratically divergent (resembles HVP).



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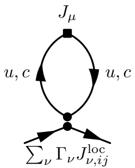

- J^V_μ, conserved current:
  - Electromagnetic gauge invariance implies the divergence can at most be logarithmic.
    - Cancelled by GIM mechanism.
    - One of the first lattice calculations to employ GIM!
  - Also possible to calculate in 3-flavour theory, using NPR to remove logarithmic divergence.

[6] G. Isidori et al., *Phys. Lett.* B633 (2005) 75-83. hep-lat/0506026
[7] RBC-UKQCD Collaboration, *Phys. Rev.* D88 (2013) 014508. arXiv:1212.5931
[8] RBC-UKQCD Collaboration, *Phys. Rev.* D93 (2016) 114517. arXiv:1605.04442

Analysis

Rare Kaon Decay Correlators

- We measure the 4pt correlator:

$$\Gamma_\mu^{(4)}(t_H, t_J, \mathbf{k}, \mathbf{p}) = \sum_{\mathbf{x},\mathbf{y}} e^{-i\mathbf{q}\cdot\mathbf{x}} \langle \mathscr{O}(t_\pi, \mathbf{p}) | T \left[J_\mu(t_J, \mathbf{x}) H_W(t_H, \mathbf{y}) \right] | \mathscr{O}_K(0, \mathbf{k}) \rangle .$$

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- We extract the amplitude from the integrated correlator in the limit $T_A, T_B \to \infty$:

$$I_\mu(T_a,T_b,\mathbf{k},\mathbf{p}) = e^{-(E_\pi(\mathbf{p})-E_K(\mathbf{k}))t_J} \int_{t_J-T_a}^{t_J+T_b} dt_H \tilde{\Gamma}^{(4)}_\mu(t_H,t_J,\mathbf{k},\mathbf{p}),$$

where $$\tilde{\Gamma}^{(4)}_\mu = \frac{\Gamma^{(4)}_\mu}{Z_{\pi K}},\; Z_{\pi K} = \frac{Z_\pi Z_K^\dagger L^3}{4E_\pi(\mathbf{p})E_K(\mathbf{k})} e^{-t_\pi E_\pi(\mathbf{p})}.$$

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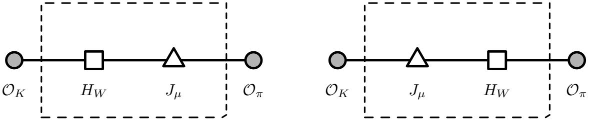

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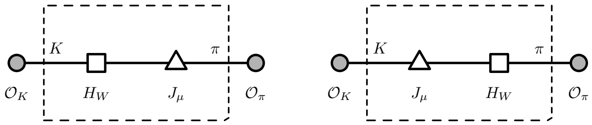

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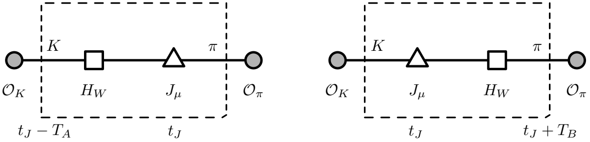

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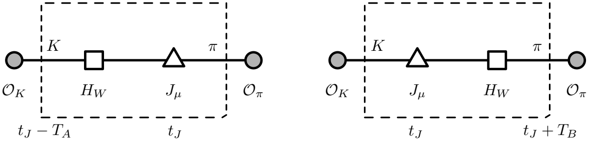

- The spectral representation of the integrated correlator is given by:

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\begin{aligned}
I_\mu\left(T_a, T_b, \mathbf{k}, \mathbf{p}\right) \quad = \quad & -\sum_n \frac{1}{2E_n} \frac{\langle \pi(\mathbf{p}) | J_\mu | n, \mathbf{k} \rangle \langle n, \mathbf{k} | H_W | K(\mathbf{k}) \rangle}{E_K(\mathbf{k}) - E_n} \left(1 - e^{\left(E_K(\mathbf{k}) - E_n\right) T_a}\right) \\
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\end{aligned}
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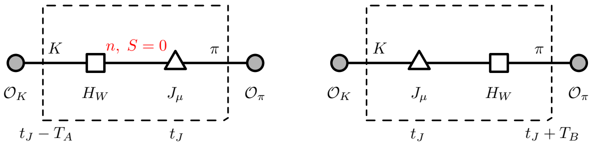

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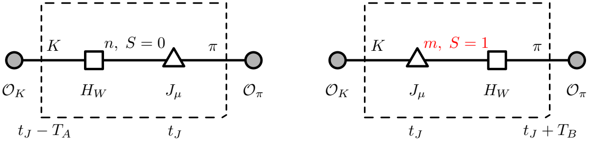

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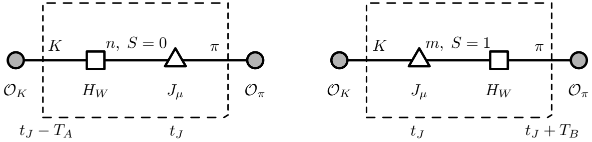

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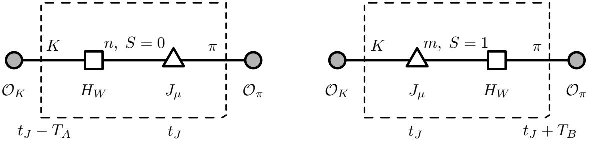
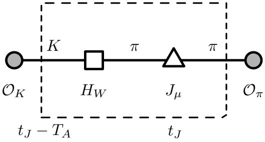


- The spectral representation of the integrated correlator is given by:

$$I_\mu(T_a,T_b,\mathbf{k},\mathbf{p}) = -\sum_n \frac{1}{2E_n}\frac{\langle\pi(\mathbf{p})|J_\mu|n,\mathbf{k}\rangle\langle n,\mathbf{k}|H_W|K(\mathbf{k})\rangle}{E_K(\mathbf{k})-E_n}\left(1-e^{(E_K(\mathbf{k})-E_n)T_a}\right)$$
$$+\sum_m \frac{1}{2E_m}\frac{\langle\pi(\mathbf{p})|H_W|m,\mathbf{p}\rangle\langle m,\mathbf{p}|J_\mu|K(\mathbf{k})\rangle}{E_m-E_\pi(\mathbf{p})}\left(1-e^{-(E_m-E_\pi(\mathbf{p}))T_b}\right)$$

- When $E_n < E_K(\mathbf{k})$, the exponential in T_a diverges. For physical masses this is true for $n=\pi$, $n=\pi\pi$ or $n=\pi\pi\pi$.

Removal of Single Pion Divergence: Method 1

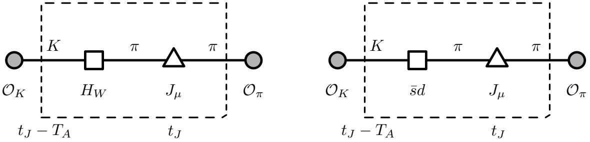


- For our exploratory studies with $M_\pi \simeq 430$ MeV, $M_K \simeq 620$ MeV only the the π state is divergent:

$$D_\mu\left(T_a, \mathbf{k}, \mathbf{p}\right) = -a \frac{\left\langle \pi\left(\mathbf{p}\right) \middle| J_\mu \middle| \pi\left(\mathbf{k}\right)\right\rangle \left\langle \pi\left(\mathbf{k}\right) \middle| H_W \middle| K\left(\mathbf{k}\right)\right\rangle}{2E_\pi\left(\mathbf{k}\right)\left(e^{a\left(E_\pi(\mathbf{k}) - E_K(\mathbf{k})\right)} - 1\right)} e^{\left(E_K(\mathbf{k}) - E_\pi(\mathbf{k})\right) T_a}$$

- The energies and matrix elements and can be extracted from 2pt and 3pt functions.

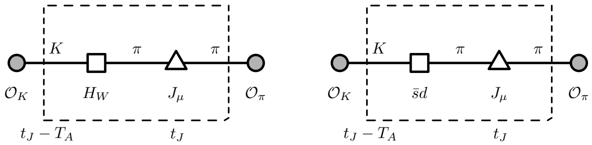
Removal of Single Pion Divergence: Method 2



- Alternatively we can remove it by performing the scalar shift

$$H_W \to H'_W = H_W + c_s \bar{s}d,$$

Removal of Single Pion Divergence: Method 2



- Alternatively we can remove it by performing the scalar shift

$$H_W \to H'_W = H_W + c_s \bar{s}d,$$

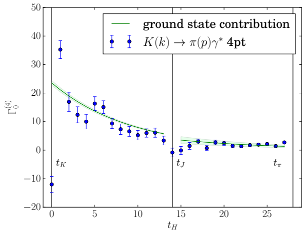
- This shift is unphysical owing to the chiral Ward Identity $i(m_s - m_d)\bar{s}d = \partial_\mu V^\mu_{\bar{s}d}$.
- We tune the parameter c_s to achieve $\langle \pi(\mathbf{k}) | H'_W | K(\mathbf{k}) \rangle = 0$.

[9] RBC-UKQCD Collaboration, *Phys. Rev. Lett.* 133 (2014) 112003. arxiv:1406.0916

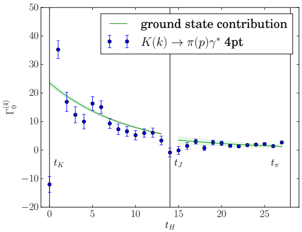
Results

$K^+ \to \pi^+\gamma^*$

- We have simulated the decay $K(k) \to \pi(p) + \gamma^*(q)$ on a $2+1$ flavor $24^3 \times 64$ lattice using Domain Wall Fermions with Iwasaki gauge action, and an inverse lattice spacing of $a^{-1} \simeq 1.78$ GeV, $M_\pi \simeq 430$ MeV, $M_K \simeq 620$ MeV, $m_c \simeq 533$ MeV, using 128 configurations.
- $t_K = 0$, $t_\pi = 28$, the current is inserted at $t_J = 14$. Below is the result for $\mathbf{k} = (0,0,0)$, $\mathbf{p} = (1,0,0)$, $\mu = 0$.



Method 1



Method 2

$K^+ \to \pi^+ \gamma^*$

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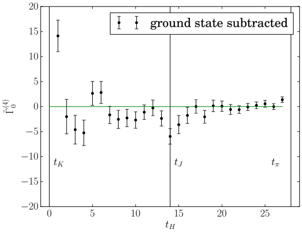

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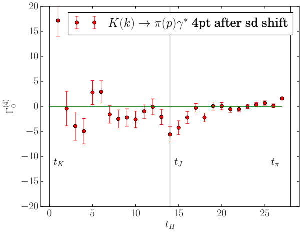
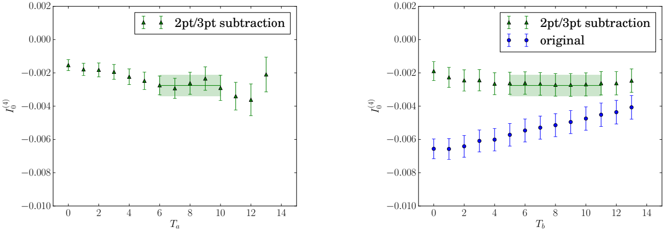


Method 2

Matrix Element

- We integrate both sides of the integral independently and fit to

$$I_\mu^{(4)} = A_\mu + c_\mu^1 e^{(E_K(\mathbf{k}) - E_\pi(\mathbf{k}))T_a} + c_\mu^2 e^{-(E_K(\mathbf{p}) - E_\pi(\mathbf{p}))T_b}$$



$I_\mu = \int_{t_J - T_A}^{t_J+8} dt_H \tilde{\Gamma}_\mu^{H_W}$ $I_\mu = \int_{t_J - 7}^{t_J+T_B} dt_H \tilde{\Gamma}_\mu^{H_W}$

- In lattice units we obtain $A_\mu = -0.0028(6)$.

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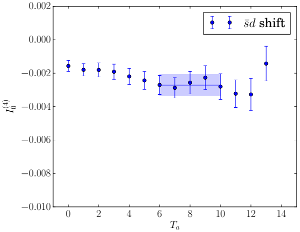
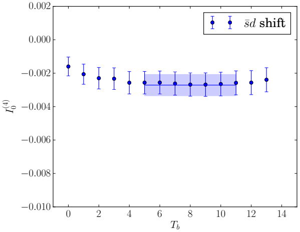


$$I_\mu = \int_{t_J - T_A}^{t_J+8} dt_H \left(\tilde{\Gamma}_\mu^{H_W} - c_s \tilde{\Gamma}_\mu^{\bar{s}d}\right)$$



$$I_\mu = \int_{t_J - 7}^{t_J+T_B} dt_H \left(\tilde{\Gamma}_\mu^{H_W} - c_s \tilde{\Gamma}_\mu^{\bar{s}d}\right)$$

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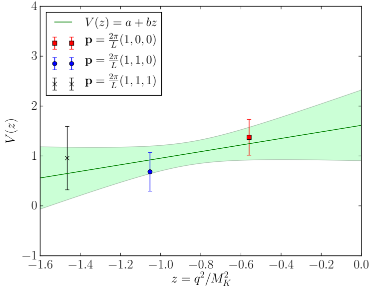
Form Factor

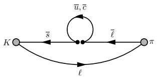
- We simulated the decay for 3 different pion final momenta to obtain the form factor dependence on $z = q^2/M_K$.
  - $A_\mu\left(q^2\right) \equiv G_F \dfrac{V(z)}{(4\pi)^2}\left(q^2 (k+p)_\mu - \left(M_K^2 - M_\pi^2\right) q_\mu\right),$
- Fit to $V(z) = a + bz$
  - $a = 1.6(7),\ b = 0.7(8).$



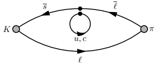
- a and b from fits to experimental spectra: $a = -0.578(16)$, $b = -0.779(66)$.

Charm Mass Dependence

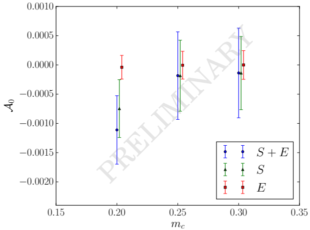
- Preliminary study of mass dependence of charm for $\mathbf{p} = \frac{2\pi}{L}(1,0,0)$.
  - $m_c^{\overline{\mathrm{MS}}}(\mu) = 533$ MeV, $m_c^{\overline{\mathrm{MS}}}(\mu) = 667$ MeV, $m_c^{\overline{\mathrm{MS}}}(\mu) = 800$ MeV, with $\mu = 2$ GeV.



Saucer



Eye



- Unable to resolve charm mass dependence.

	S	E	$S+E$
$m^{533} - m^{667}$	0.00056(61)	0.00003(24)	0.00057(76)
$m^{533} - m^{800}$	0.00061(62)	0.00004(25)	0.00062(77)
$m^{667} - m^{800}$	0.000045(41)	0.000004(19)	0.000046(47)

Summary

- We have developed theoretical techniques for calculating the long distance contributions to $K^+ \to \pi^+\ell^+\ell^-$.
- Our exploratory calculations show that we can successfully extract the matrix elements in practice.

- Outlook
  - Push to physical pion and kaon masses.
  - Physical charm?

Thank you!