Lattice calculation of the pion transition form factor $\pi^0 \to \gamma^*\gamma^*$

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In collaboration with Harvey Meyer and Andreas Nyffeler

Based on [arXiv:1607.08174]

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Motivations

- One motivation : anomalous magnetic moment of the muon

  $\rightarrow$ one of the most precise tests of the Standard Model

- $\sim 3-4\sigma$ discrepancy between experiment and theory

$$a_\mu = \begin{cases} 116\ 592\ 091(63) \times 10^{-11} & \text{(Exp.)} \\ 116\ 591\ 803(49) \times 10^{-11} & \text{(Theory)} \end{cases}$$

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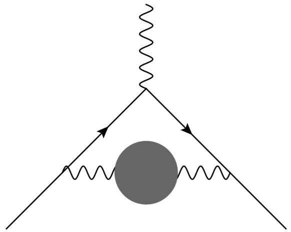

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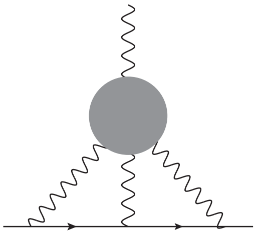

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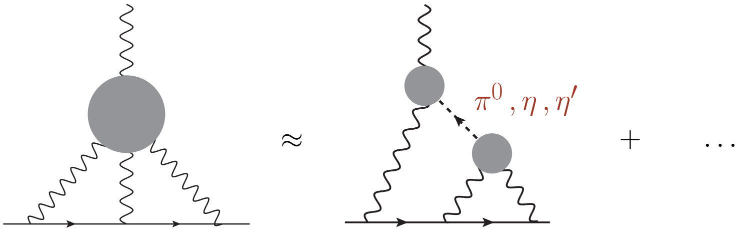

- Pseudoscalar-exchanges dominate numerically (π^0, η, η') : $a_\mu^{\mathrm{HLbL};\pi^0}$

- On input, one needs the $\pi^0 \rightarrow \gamma^*\gamma^*$ form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$ [Jegerlehner, Nyffeler '09]

- Relevant momentum region : $Q^2 \in [0 : 1.5]$ GeV^2 [Nyffeler '16]

- Current estimation are based on models where errors are difficult to estimate

The $\pi^0 \to \gamma^*\gamma^*$ form factor

Form factor : theoretical constraints on $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$

- Low energy ($Q_1^2 \to 0$, $Q_2^2 \to 0$)
- Chiral limit

$\to$ Adler-Bell-Jackiw (ABJ) anomaly :

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0) = \frac{1}{4\pi^2 F_\pi}$$

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- Single-virtual form factor
- Off-shell photon : $Q^2 \to \infty$

$\to$ Brodsky-Lepage behavior :

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2, 0) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{Q^2}$$

Form factor : theoretical constraints on $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)$

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- Large virtualities : $Q^2 \to \infty$

$\to$ OPE prediction :

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2,-Q^2) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{3Q^2}$$

Form factor : theoretical constraints on $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)$

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$\to$ OPE prediction :

$$\mathcal{F}_{\pi^0\gamma^*\gamma}(-Q^2, -Q^2) \xrightarrow[Q^2\to\infty]{} \frac{2F_\pi}{3Q^2}$$

$\to$ Some results in the single-virtual case (dispersive approach [G. Colangelo '14 '15, V. Pauk '14])

$\to$ $a_\mu^{\mathrm{HLbL};\pi^0}$: main contribution comes from the region $Q^2 \in [0, 1.5]\ \mathrm{GeV}^2$

$\to$ Lattice QCD can be used to compute the form factor in the relevant kinematical region

$\to$ Do not rely on any model

Form factor : experimental side

- Decay width : $\Gamma_{\pi^0\gamma\gamma} = 7.82(22)$ eV $\sim 3\%$ [PrimEx '10]

$$\Gamma_{\pi^0\gamma\gamma} = \frac{\pi\alpha_e^2 m_\pi^3}{4}\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$$

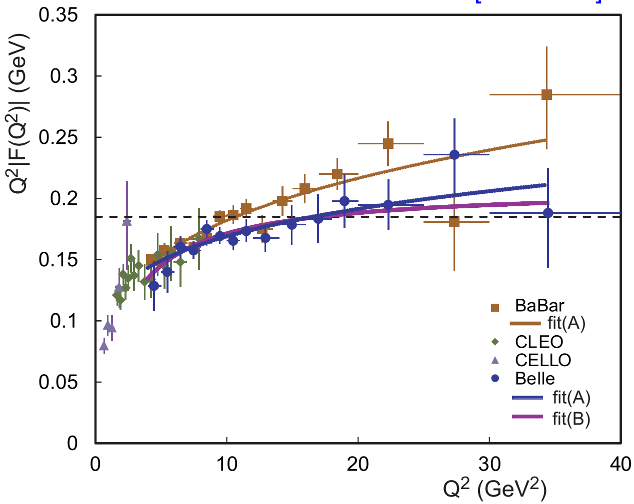
$\rightarrow$ Consistent with current theoretical predictions

$\rightarrow$ Experimental test of the chiral anomaly

- The single-virtual form factor has been measured (CELLO, CLEO, BaBar, Belle)

$$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,0) = F(Q^2)$$

[Belle '12]



$\rightarrow$ Belle data seem to confirm the Brodsky-Lepage behavior $\sim 1/Q^2$.

$\rightarrow$ Belle and Babar results are quite different

$\rightarrow$ no measurement at low $Q^2 < 0.5\ \mathrm{GeV}^2$

- No result yet for the double-virtual form factor

$\hookrightarrow$ but measurement planned at BESIII in the range $[0.3-3]\ \mathrm{GeV}^2$

Lattice computation

Lattice calculation

In Minkowski space-time :

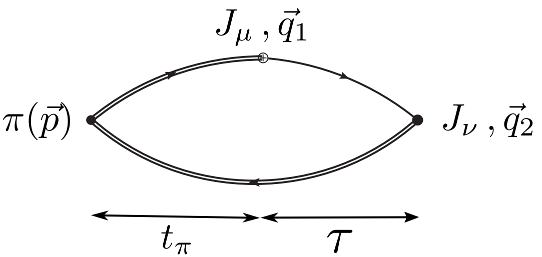
$$\epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha\, q_2^\beta\; \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) = i \int \mathrm{d}^4x\, e^{iq_1x}\, \langle \Omega | T\{J_\mu(x) J_\nu(0)\} | \pi^0(p)\rangle = M_{\mu\nu}(p, q_1)$$

- $J_\mu(x)$ hadronic component of the electromagnetic current : $J_\mu(x) = \frac{2}{3}\overline{u}(x)\gamma_\mu u(x) - \frac{1}{3}\overline{d}(x)\gamma_\mu d(x) + \dots$

In Euclidean space-time : [Cohen et al. '08] [Feng et al. '12]

$$M^E_{\mu\nu}(p, q_1) = -\int \mathrm{d}\tau\, e^{\omega_1\tau} \int \mathrm{d}^3z\, e^{-i\vec{q}_1\vec{z}}\, \langle 0| T\left\{ J_\mu(\vec{z}, \tau) J_\nu(\vec{0}, 0)\right\} |\pi(p)\rangle$$

- Analytical continuation
- We must kept $q_{1,2}^2 < M_V^2 = \min(M_\rho^2, 4m_\pi^2)$ to avoid poles
- $q_1 = (\omega_1, \vec{q}_1)$



The main object to compute is the three-point correlation function :

$$C^{(3)}_{\mu\nu}(\tau, t_\pi; \vec{p}, \vec{q}_1, \vec{q}_2) = \sum_{\vec{x},\vec{z}} \langle T\left\{ J_\nu(\vec{0}, t_f) J_\mu(\vec{z}, t_i) P(\vec{x}, t_0)\right\} \rangle\, e^{i\vec{p}\vec{x}}\, e^{-i\vec{q}_1\vec{z}}$$

Lattice calculation

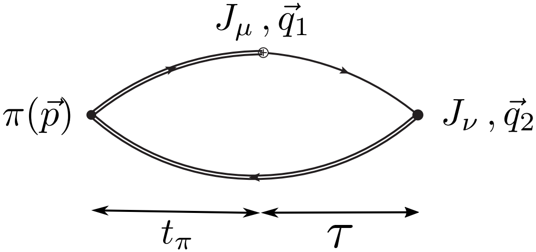
$$M^E_{\mu\nu}(p,q_1) = \lim_{t_\pi\to\infty} \frac{2E_\pi}{Z_\pi} \left(\int_{-\infty}^{0} \mathrm{d}\tau\, e^{\omega_1\tau}\, e^{-E_\pi(\tau-t_\pi)}\, C_{\mu\nu}(\tau,t_\pi;\vec{p},\vec{q}_1) + \int_0^\infty \mathrm{d}\tau\, e^{\omega_1\tau}\, e^{E_\pi t_\pi}\, C_{\mu\nu}(\tau,t_\pi;\vec{p},\vec{q}_1) \right)$$

- E_π and Z_π (overlap with our interpolating field) are extracted from the two-point correlation function :

$$C^{(2)}(t) = \sum_{\vec{x}} \langle P(\vec{x},t)P^\dagger(\vec{0},0)\rangle\, e^{-i\vec{p}\vec{x}} \xrightarrow[t\to\infty]{} \frac{|Z_\pi|^2}{2E_\pi}\left(e^{-E_\pi t} + e^{-E_\pi(T-t)}\right) ,$$

- We choose $\vec{p}=\vec{0}$ (pion at rest)

$$\begin{cases} q_1^2 = \omega_1^2 - |\vec{q}_1|^2 \\ q_2^2 = (m_\pi - \omega_1)^2 - |\vec{q}_1|^2 \end{cases}$$



→ Momenta are discrete : $|\vec{q}_1|^2 = \left(\frac{2\pi}{L}\right)^2 |\vec{n}|^2 \quad , \quad |\vec{n}|^2 = 1,2,3,4,5,6,8,\ldots$

→ ω_1 is a free parameter : $q_1 = (\omega_1, \vec{q}_1)$

→ By varying continuously ω_1 we have access to different values of (q_1^2, q_2^2)

→ But ω_1 such that $q_{1,2}^2 < M_V = \min(M_\rho^2, 4m_\pi^2)$ (below hadronic threshold)

Kinematic reach in the photon virtualities

Photons momenta :

$$q_1^2 = \omega_1^2 - |\vec{q}_1|^2$$
$$q_2^2 = (m_\pi - \omega_1)^2 - |\vec{q}_1|^2$$

$\Rightarrow |\vec{q}_1|^2 = (2\pi/L)^2 |\vec{n}|^2 \quad , \quad |\vec{n}|^2 = 1, 2, 3, 4, 5, \ldots$

$\Rightarrow \omega_1$ is a (real) free parameter

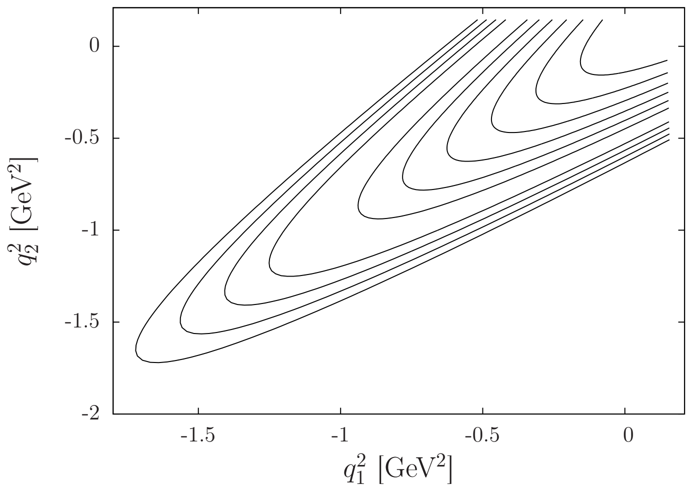


Figure: $48^3 \times 96$ at $a = 0.065$ fm.

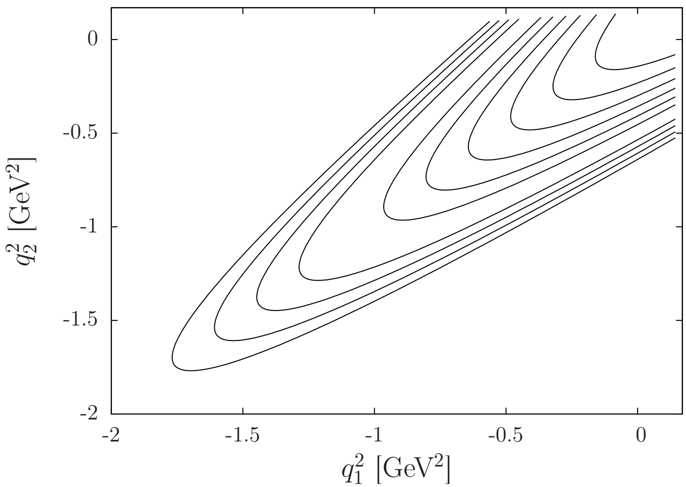


Figure: $64^3 \times 128$ at $a = 0.048$ fm.

Lattice setup

- $N_f = 2$ dynamical quarks ($\mathcal{O}(a)$-improved Wilson-Clover Fermions)
- Pion masses in the range [190:440] MeV
- 3 lattice spacings
  $\rightarrow a = 0.075$ fm
  $\rightarrow a = 0.065$ fm
  $\rightarrow a = 0.048$ fm

CLS
based

- Photons virtualities up to $Q^2_{1,2} \sim 1.5\ \mathrm{GeV}^2$
  $\rightarrow$ results are averaged over equivalent momenta (no new inversion needed)
  $\rightarrow$ results are averaged over Lorentz components
- $C^{(3)}_{\mu\nu}$: one local and one conserved vector current

$$J^l_\mu(x) = \sum_f Q_f\ \overline{\psi}_f(x)\gamma_\mu\psi_f(x)\,,$$

$$J^c_\mu(x) = \sum_f \frac{Q_f}{2}\left(\overline{\psi}_f(x+a\hat{\mu})(1+\gamma_\mu)U^\dagger_\mu(x)\psi_f(x) - \overline{\psi}_f(x)(1-\gamma_\mu)U_\mu(x)\psi_f(x+a\hat{\mu})\right)\,.$$

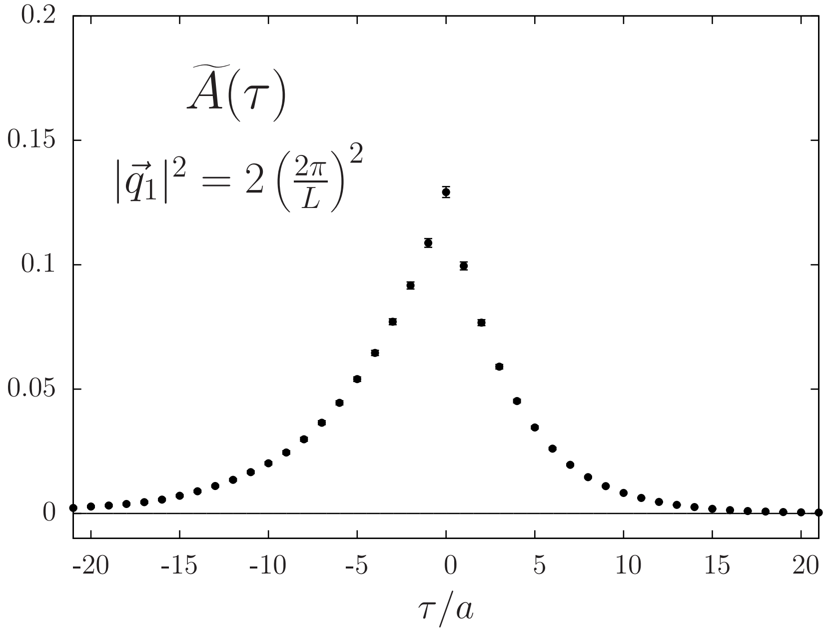
Results

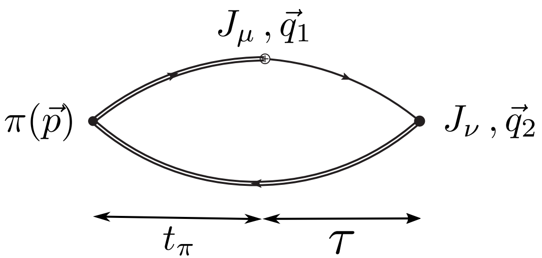
Shape of the integrand for F7 ($a = 0.065$ fm and $m_\pi = 270$ MeV)

$$M^E_{\mu\nu} = \frac{2E_\pi}{Z_\pi} \int_{-\infty}^{\infty} \mathrm{d}\tau \, \widetilde{A}_{\mu\nu}(\tau) \, e^{\omega_1 \tau} \, e^{-E_\pi \tau}$$

$$A_{\mu\nu}(\tau) = \lim_{t_\pi \to \infty} C_{\mu\nu}(\tau, t_\pi) e^{E_\pi t_\pi}$$

$$\widetilde{A}_{\mu\nu}(\tau) = \begin{cases} A_{\mu\nu}(\tau) & \tau > 0 \\ A_{\mu\nu}(\tau)\, e^{-E_\pi \tau} & \tau < 0 \end{cases}$$





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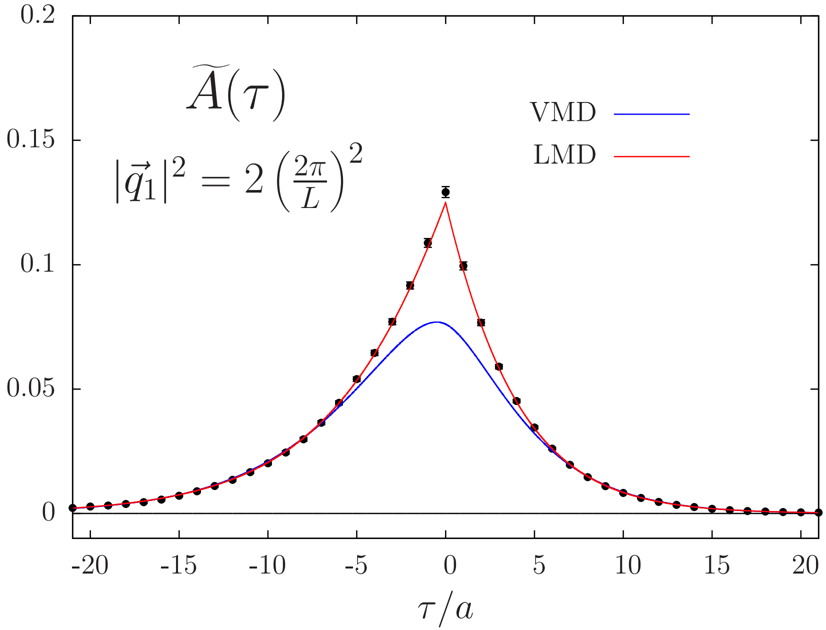
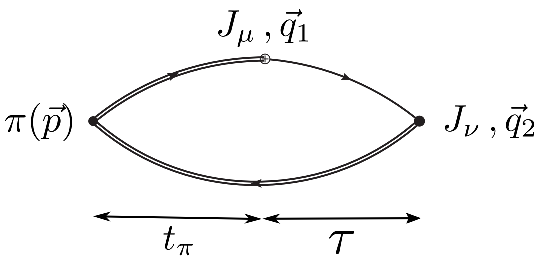




- Finite time extent of the lattice :

→ Fit the data using a vector meson dominance (VMD) model or lowest meson dominance (LMD) model :

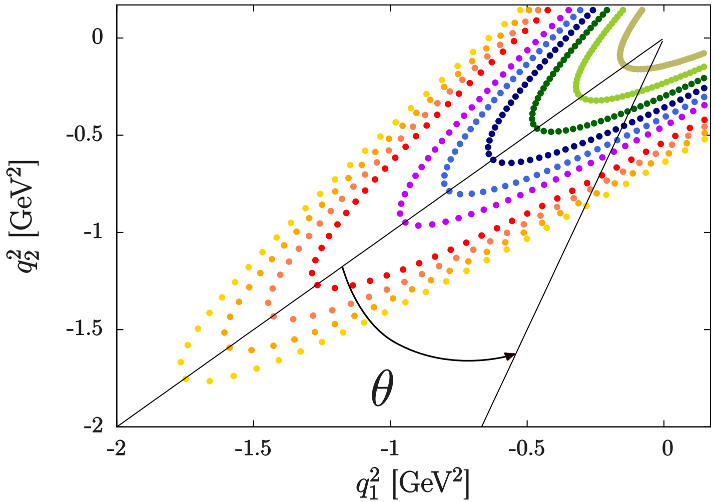
$$\mathcal{F}^{\mathrm{VMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

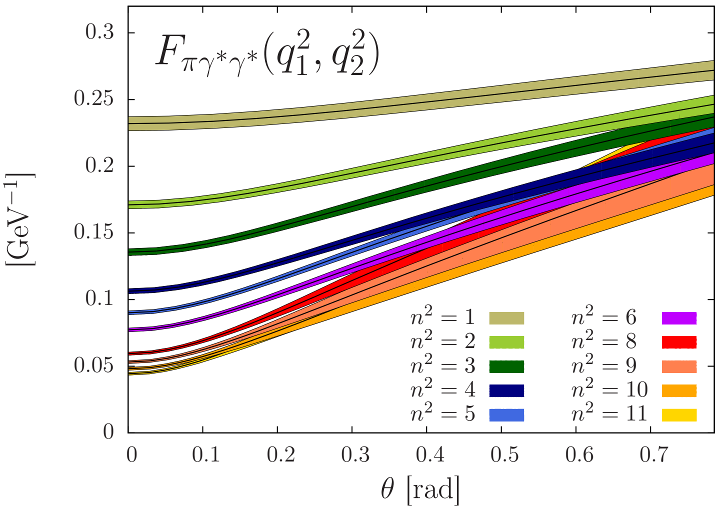
$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

→ Introduces a cut-off $\tau_c \gtrsim 1.3$ fm

Form factor : $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$

- For each value of $|\vec{q}_1|^2 = |\vec{n}|^2(2\pi/L)^2$, one gets a curve by varying continuously the value of ω_1 (or θ).
- $\tan(\theta - \pi/4) = q_1^2/q_2^2$





$\longrightarrow$ Comparison with phenomenological models?

Comparison with phenomenological models (1) : VMD (vector meson dominance)

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	$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$	$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,0)$	$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2)$
VMD	α	$\alpha M_V^2/Q^2$	$\alpha M_V^4/Q^4$
Theory	$1/(4\pi^2 F_\pi)$	$2F_\pi/Q^2$	$2F_\pi/(3Q^2)$
	✓	(Brodsky-Lepage) ✓	(OPE) X

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Two fitting procedures :

- Local fit : each ensemble is fitted independently ($\alpha(a, m_\pi^2)$, $M_V(a, m_\pi^2)$) + chiral and continuum extrapolation
- Global fit : combined chiral and continuum extrapolation $\rightarrow$ 6 fit parameters

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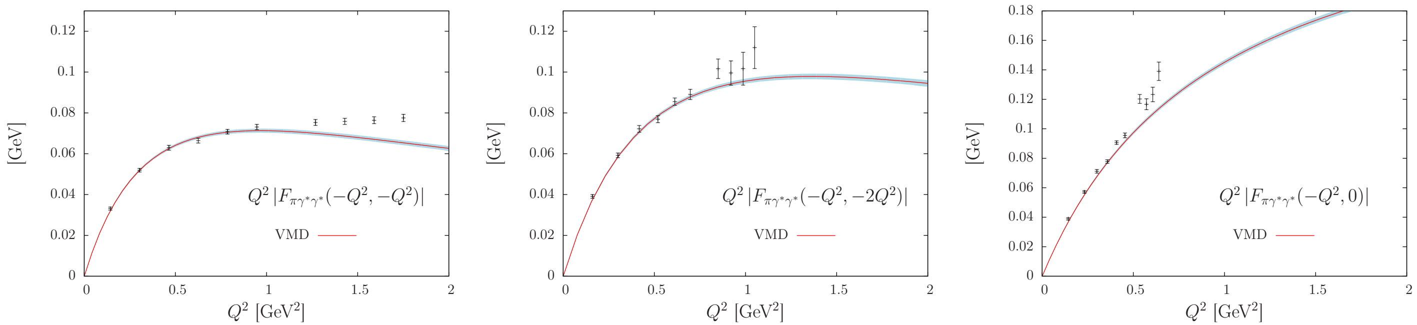
$$\alpha^{\mathrm{VMD}} = 0.243(18)\ \mathrm{GeV}^{-1}\,, \quad M_V^{\mathrm{VMD}} = 0.944(34)\ \mathrm{GeV}\,.$$



↪ The model fails to describe our data ! We do not recover the anomaly result : $\alpha \neq \alpha_{\mathrm{th}} = 0.274\ \mathrm{GeV}^{-1}$

Comparison with phenomenological models (2) : LMD (Lowest meson dominance)

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta(q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

(inspired from the large-NC approximation to QCD)
[hep-ph/9407402] [hep-ph/9908283]

	$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$	$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,0)$	$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2)$
LMD	α	$-\beta/M_V^2$	$-2\beta/Q^2$
Theory	$1/(4\pi^2 F_\pi)$	$2F_\pi/Q^2$	$2F_\pi/(3Q^2)$
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Theory	$1/(4\pi^2 F_\pi)$	$2F_\pi/Q^2$	$2F_\pi/(3Q^2)$
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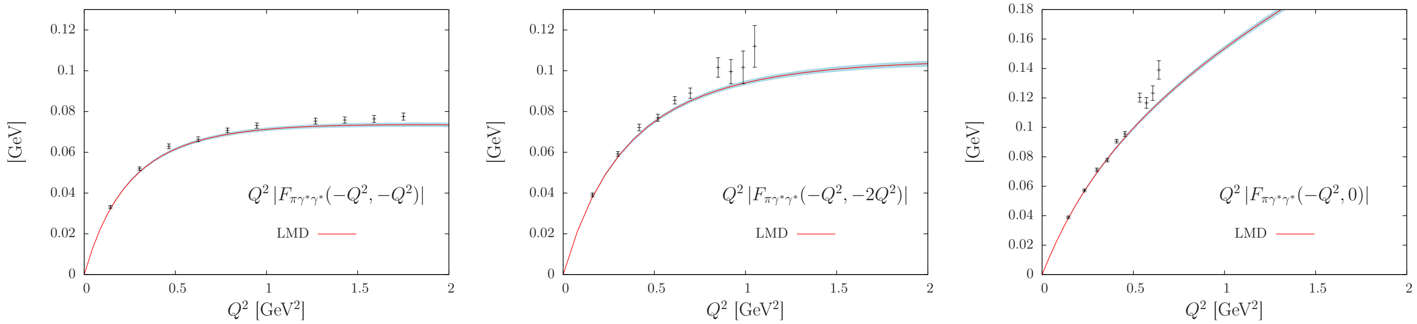
$$\alpha^{\mathrm{LMD}} = 0.275(18)\ \mathrm{GeV}^{-1} \quad , \quad \beta = -0.028(4)\ \mathrm{GeV} \quad , \quad M_V^{\mathrm{LMD}} = 0.705(24)\ \mathrm{GeV}\,.$$



- The data are well describe by this model
- α^{LMD} is compatible with the theoretical prediction $\alpha^{\mathrm{th}} = 1/(4\pi^2 F_\pi) = 0.274\ \mathrm{GeV}^{-1}$ $\rightarrow$ (accuracy 7%)
- β^{LMD} is compatible with the OPE prediction $\beta^{\mathrm{OPE}} = -F_\pi/3 = -0.0308\ \mathrm{GeV}$
- Might be surprising as the form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,0)$ has the wrong asymptotic behavior ...

Comparison with phenomenological models (3) : LMD+V

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2 + q_2^2) + \widetilde{h}_1 (q_1^2 + q_2^2)^2 + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2 + q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2 - q_1^2)(M_{V_2}^2 - q_1^2)(M_{V_1}^2 - q_2^2)(M_{V_2}^2 - q_2^2)}$$

	$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$	$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,0)$	$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2)$
LMD+V	α	$-\widetilde{h}_5/Q^2$	$-2\widetilde{h}_0/Q^2$
Theory	$1/(4\pi^2 F_\pi)$	$2F_\pi/Q^2$	$2F_\pi/(3Q^2)$
	✓	(Brodsky-Lepage) ✓	(OPE) ✓

- Refinement of the LMD model (include a second vector resonance, ρ') [hep-ph/0106034]
- All the theoretical constraints are satisfied (if one sets $\widetilde{h}_1 = 0$)

Comparison with phenomenological models (3) : LMD+V

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2+q_2^2) + \widetilde{h}_1 (q_1^2+q_2^2)^2 + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2+q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2 - q_1^2)(M_{V_2}^2 - q_1^2)(M_{V_1}^2 - q_2^2)(M_{V_2}^2 - q_2^2)}$$

	$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0,0)$	$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,0)$	$\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q^2,-Q^2)$
LMD+V	α	$-\widetilde{h}_5/Q^2$	$-2\widetilde{h}_0/Q^2$
Theory	$1/(4\pi^2 F_\pi)$	$2F_\pi/Q^2$	$2F_\pi/(3Q^2)$
	✓	(Brodsky-Lepage) ✓	(OPE) ✓

- Refinement of the LMD model (include a second vector resonance, ρ') [hep-ph/0106034]
- All the theoretical constraints are satisfied (if one sets $\widetilde{h}_1 = 0$)
- But the number of parameters also increases (local fits are unstable, global fit only)
- Assumptions:
  - $\widetilde{h}_1 = 0$
  - $M_{V_1} = m_\rho^{\mathrm{exp}} = 0.775\ \mathrm{GeV}$ in the continuum and chiral limit (but chiral corrections are taken into account in the fit)
  - Constant shift in the spectrum: $M_{V_2}(\widetilde{y}) = m_{\rho'}^{\mathrm{exp}} + M_{V_1}(\widetilde{y}) - m_\rho^{\mathrm{exp}}$ with $m_{\rho'}^{\mathrm{exp}} = 1.465\ \mathrm{GeV}$

Comparison with phenomenological models (3) : LMD+V

$$\mathcal{F}^{\mathrm{LMD+V}}_{\pi^0\gamma^*\gamma^*}(q_1^2,q_2^2) = \frac{\widetilde{h}_0\, q_1^2 q_2^2 (q_1^2+q_2^2) + \widetilde{h}_1 (q_1^2+q_2^2)^2 + \widetilde{h}_2\, q_1^2 q_2^2 + \widetilde{h}_5\, M_{V_1}^2 M_{V_2}^2\, (q_1^2+q_2^2) + \alpha\, M_{V_1}^4 M_{V_2}^4}{(M_{V_1}^2-q_1^2)(M_{V_2}^2-q_1^2)(M_{V_1}^2-q_2^2)(M_{V_2}^2-q_2^2)}$$

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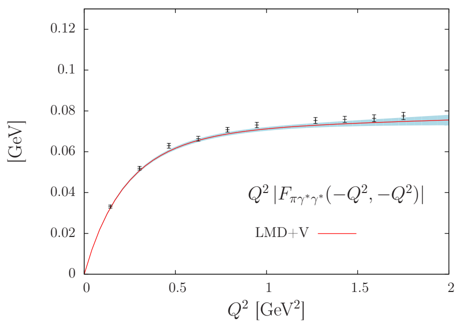
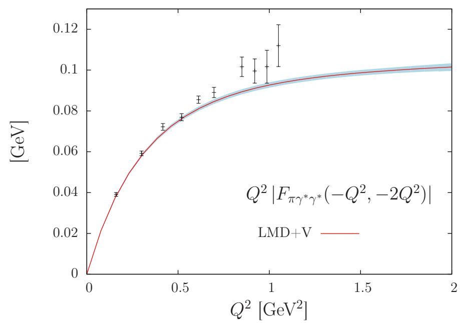
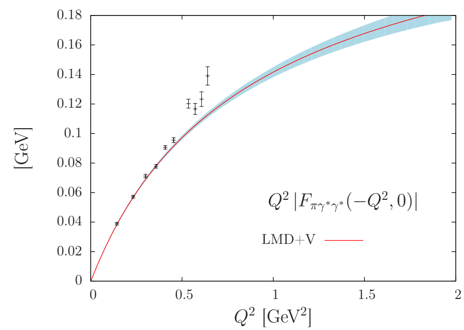
Theory	$1/(4\pi^2 F_\pi)$	$2F_\pi/Q^2$	$2F_\pi/(3Q^2)$
	✓	(Brodsky-Lepage) ✓	(OPE) ✓

$$\alpha^{\mathrm{LMD+V}} = 0.273(24)\ \mathrm{GeV}^{-1} \quad , \quad \widetilde{h}_2 = 0.345(167)\ \mathrm{GeV}^3 \quad , \quad \widetilde{h}_5 = -0.195(70)\ \mathrm{GeV}\,.$$

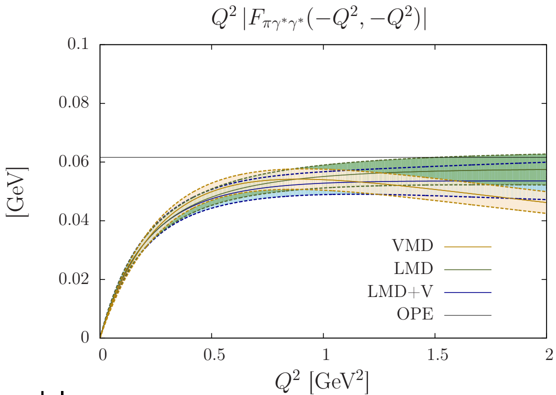


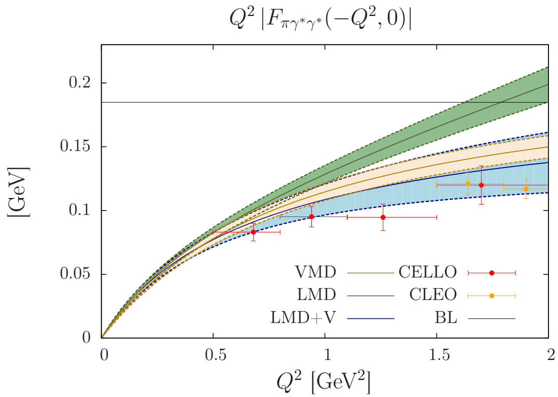




- $\alpha^{\mathrm{LMD+V}}$ is again compatible with the theoretical prediction $\alpha^{\mathrm{th}} = 1/(4\pi^2 F_\pi) = 0.274\ \mathrm{GeV}^{-1}$ $\rightarrow$ (accuracy 9%)
- The data are well describe by this model

Final results for the form factor (at the physical point)





LMD model :

$$\alpha^{\mathrm{LMD}} = 0.275(18)(3)\ \mathrm{GeV}^{-1}\,, \quad \beta = -0.028(4)(1)\ \mathrm{GeV}\,, \quad M_V^{\mathrm{LMD}} = 0.705(24)(21)\ \mathrm{GeV}$$

LMD+V model :

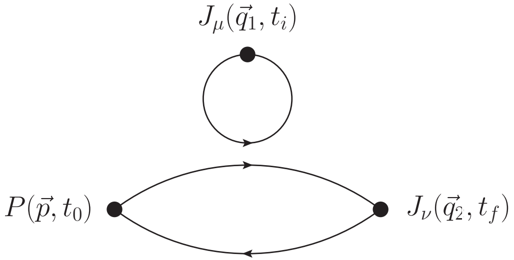
$$\alpha^{\mathrm{LMD+V}} = 0.273(24)(7)\ \mathrm{GeV}^{-1}\,, \quad \widetilde{h}_2 = 0.345(167)(83)\ \mathrm{GeV}^3\,, \quad \widetilde{h}_5 = -0.195(70)(34)\ \mathrm{GeV}$$

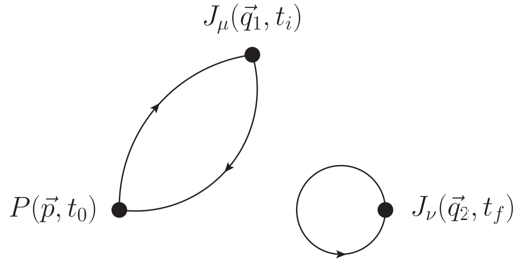
where $\widetilde{h}_0 = -F_\pi/3 = -0.0308$ GeV, $M_{V_1} = 0.775$ GeV and $M_{V_2} = 1.465$ GeV are fixed at the physical point.

Systematic errors :

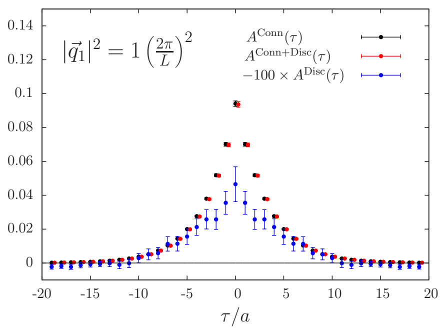
- Finite-time extent of the lattice
- Finite-size effects (no dedicated study, but data suggest small rather effect).
- Disconnected contributions

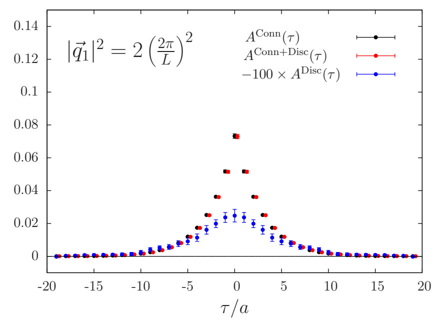
Disconnected contributions

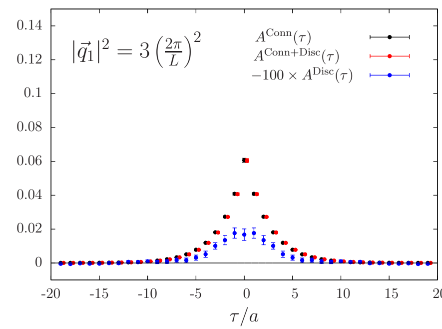




- Disconnected contribution has been computed on E5 only ($m_\pi = 440$ MeV, $a = 0.065$ fm)
- Loops : 75 stochastic sources with full-time dilution and a generalized Hopping Parameter Expansion.
- Two-point functions : 7 stochastic sources with full-time dilution
- $|\vec{q}_1|^2 = |\vec{n}|^2(2\pi/L)$ with $|\vec{n}|^2 = 1, 2, 3$.





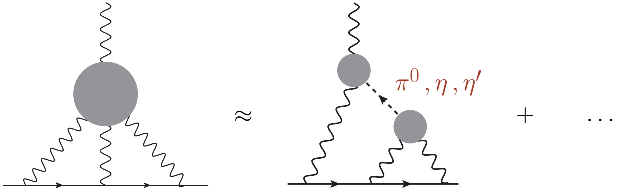


$\rightarrow$ The disconnected contribution is below 1%.
$\rightarrow$ But the pion mass dependence could be large ...

Phenomenology : pion-pole contribution to $a_\mu^{\mathrm{HLbL};\pi^0}$

[F. Jegerlehner '09]

$$a_\mu^{\mathrm{HLbL};\pi^0} = \left(\frac{\alpha_e}{\pi}\right)^3 \left(a_\mu^{\mathrm{HLbL};\pi^0(1)} + a_\mu^{\mathrm{HLbL};\pi^0(2)}\right) ,$$



where

$$a_\mu^{\mathrm{HLbL};\pi^0(1)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0)\,,$$

$$a_\mu^{\mathrm{HLbL};\pi^0(2)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_2(Q_1,Q_2,\tau)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2)\; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0)\,.$$

$\rightarrow$ $w_{1,2}(Q_1,Q_2,\tau)$ are some model-independent weight functions (concentrated at small momenta below 1 GeV)

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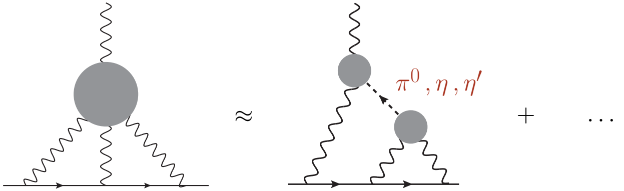

where

$$a_\mu^{\mathrm{HLbL};\pi^0(1)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_1(Q_1,Q_2,\tau) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1+Q_2)^2) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) ,$$

$$a_\mu^{\mathrm{HLbL};\pi^0(2)} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \; w_2(Q_1,Q_2,\tau) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \; \mathcal{F}_{\pi^0\gamma^*\gamma^*}(-(Q_1+Q_2)^2, 0) .$$

$\rightarrow$ $w_{1,2}(Q_1,Q_2,\tau)$ are some model-independent weight functions (concentrated at small momenta below 1 GeV)

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = (65.0 \pm 8.3) \times 10^{-11}$$

$\rightarrow$ most model calculations yield results in the range

$$a_\mu^{\mathrm{HLbL};\pi^0} = (50 - 80) \times 10^{-11}$$

Model	$a_\mu^{\mathrm{HLbL};\pi^0} \times 10^{11}$
LMD (this work)	68.2(7.4)
LMD+V (this work)	65.0(8.3)
LMD (theory)	73.7
LMD+V (theory + phenomenology)	62.9

Λ [GeV]	LMD		LMD+V	
0.25	14.6	(21.4%)	14.4	(22.1%)
0.5	37.9	(55.5%)	37.2	(57.2%)
0.75	50.7	(74.4%)	49.5	(76.1%)
1.0	57.3	(84.0%)	55.5	(85.4%)
1.5	62.9	(92.3%)	60.6	(93.1%)
2.0	65.1	(95.5%)	62.5	(96.1%)
5.0	67.7	(99.2%)	64.6	(99.4%)
20.0	68.2	(100%)	65.0	(100%)

Conclusion

- We have performed a calculation of the pion transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$ with two dynamical quarks.
- The VMD model fails to describe our data, especially in the double virtual case.
- However, the LMD and LMD+V models describe our data successfully.
- In particular we recover the anomaly results ($\alpha^{\mathrm{th}} = 0.274~\mathrm{GeV}^{-1}$) in the continuum and chiral limit

$$\alpha^{\mathrm{LMD}} = 0.275(18)(3)~\mathrm{GeV}^{-1} \quad , \quad \alpha^{\mathrm{LMD+V}} = 0.273(24)(7)~\mathrm{GeV}^{-1}$$

  $\rightarrow 7-9\%$ accuracy

- Disconnected contributions have been computed on one lattice ensemble.
- Provides a first lattice estimate of the pion-pole contribution to the hadronic light-by-light scattering in the $g-2$ of the muon

$$a_{\mu;\mathrm{LMD+V}}^{\mathrm{HLbL};\pi^0} = (65.0 \pm 8.3) \times 10^{-11}$$

- Experimental results for the double–virtual form factor should be available soon (BES III).