Four-Fermion Theories with Exact Chiral Symmetry in Three Dimensions

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1. Introduction to Four-Fermion Theories
2. Two Four-Fermion Interactions
3. Fierz Identities for the Thirring Model
4. Fermion Bag Approach
5. Summary
What are Four-Fermion Theories?

QFTs of fermions with 4th power of fermion fields as interaction:

$$\mathcal{L} = \bar{\psi}_j (\partial + m) \psi_j + \sum_\alpha \frac{g^2_\alpha}{2N_f} (\bar{\psi}_j \Gamma_\alpha \psi_j)^2 \quad j = 1, \ldots, N_f$$

Main Models

<table>
<thead>
<tr>
<th>Theory</th>
<th>Year</th>
<th>Description</th>
<th>Gamma Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thirring</td>
<td>1958</td>
<td>soluble fermionic theory in 2D</td>
<td>$\Gamma_\alpha = \gamma_\mu$</td>
</tr>
<tr>
<td>Nambu &amp; Jona-Lassinio</td>
<td>1961</td>
<td>dynamical mass generation in 4D</td>
<td>$\Gamma_1 = \mathbb{1}, \Gamma_2 = i\gamma_5$</td>
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<tr>
<td>Gross &amp; Neveu</td>
<td>1974</td>
<td>asymptotic freedom, chiral symmetry breaking in 2D</td>
<td>$\Gamma = \mathbb{1}$</td>
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</table>
What is the Thirring model?

QFT with $N_f$ flavours of massless fermions with current interaction

Euclidean Lagrangian

$$\mathcal{L} = \bar{\psi}_j \slashed{D} \psi_j + \frac{g^2}{2N_f} \sum_{\mu=1}^{3} (\bar{\psi}_j \gamma^\mu \psi_j)^2 \quad j = 1, \ldots, N_f$$

- 3-dimensional euclidean spacetime
- representation of Clifford algebra:
  - irreducible 2-dimensional (later in this talk)
  - reducible 4-dimensional (now)

Motivation

Similarity to QED$_3$ and possible applications in superconductors, graphene, ...
Why is the Thirring model interesting?

Symmetries

- chiral symmetry, generated by \{1, \gamma_4, \gamma_5, i\gamma_4\gamma_5\}
- flavour symmetry

Result: \( U(N_f, N_f) \), can be spontaneously broken to \( U(N_f) \otimes U(N_f) \)
\( \Rightarrow \) chiral condensate \( \langle \bar{\psi}\psi \rangle \neq 0 \)

Chiral Symmetry Breaking

“\( N_f = 0.5 \)” irreducible representation for \( N_{f,\text{irr}} = 1 \) corresponds to Gross-Neveu model with chiral symmetry breaking

\( N_f \to \infty \) no chiral symmetry breaking

\( \Rightarrow \) There is a \( N_f^{cr} \) where chiral behaviour changes.
What is the value of $N_f^{cr}$?

Results for $N_f^{cr}$ from Schwinger-Dyson equations, $\frac{1}{N_f}$-expansion, functional renormalization group, Lattice simulation with staggered fermions:

- Kondo (1995)
- Gomes et al. (1991)
- Itoh et al. (1995)
- Del Debbio, Hands et al. (1996-1999)
- Kim & Kim (1996)
- Hands et al. (2007)
- Hong & Park (1994)
- Jansen & Gies (2012)
Chiral Symmetry on the Lattice

Nielsen-Ninomiya Theorem

It is not possible to have a chiral, local and translational invariant Dirac operator with correct continuum limit without doublers.

older results: staggered fermions with mass
- mass breaks symmetry explicitly
- still doublers and no full chiral symmetry
- symmetry correct in the continuum limit?

our approach: SLAC derivative
- in momentum space: multiplication by $i\gamma^\mu p_\mu$
- exact chiral symmetry
- not local: need to do Fourier transformation
Technical Problems

- Chiral condensate always zero due to exact chiral symmetry and integration over fermions.
- Peak in susceptibility may indicate lattice artefact phase.

Coupling to Global Model [PoS(LATTICE 2015)050]

- Can obtain nice histograms of $\langle \bar{\psi} \psi \rangle$.
- Hard to recover Thirring model.

$\Rightarrow$ No reliable conclusion regarding $N_f^{cr}$, but likely $N_f^{cr} \leq 2$. 
Why Coupled Four-Fermion Interactions?

- Easy to study chiral symmetry breaking in Gross-Neveu model.
  \[ \Rightarrow \text{Gain new insights into Thirring model and the larger theory space by coupling these two models.} \]
- Test predictions from functional renormalization group.

Interaction with \( i \gamma_4 \gamma_5 \)

\[ (\bar{\psi} i \gamma_4 \gamma_5 \psi)^2 \] is interesting because

- same symmetry (and problems) as Thirring, while
- expected to be Gross-Neveu-like, corresponds to irreducible Gross-Neveu model.
Gross-Neveu and Thirring

\[ \lambda = \frac{1}{2g^2}, \text{ Lattice } 8 \times 7 \times 7, N_f = 1 \]

Chiral Condensate

Susceptibility
Gross-Neveu and $i\gamma_4\gamma_5$

$$\lambda = \frac{1}{2g^2}, \text{ Lattice } 12 \times 11 \times 11, \ N_f = 1$$

$\lambda_{\text{GN}}$ vs $\lambda_{\text{Th}}$

Chiral Condensate

$\lambda_{\text{GN}}$ vs $\lambda_{\text{Th}}$

Susceptibility

pure $i\gamma_4\gamma_5$ limit

pure Gross-Neveu limit
\[ \lambda = \frac{1}{2g^2}, \text{ Lattice } 12 \times 11 \times 11, \ N_f = 2 \]
Gross-Neveu and $i\gamma_4\gamma_5$

\[ \lambda = \frac{1}{2g^2}, \text{ Lattice } 12 \times 11 \times 11, \ N_f = 3 \]

\[ \rightarrow \text{ pure } i\gamma_4\gamma_5 \text{ limit } \]

Chiral Condensate

Susceptibility
Gross-Neveu and $i\gamma_4\gamma_5$

$$\lambda = \frac{1}{2g^2}, \text{ Lattice } 12\times11\times11, \ N_f = 4$$

→ pure $i\gamma_4\gamma_5$ limit ←

Chiral Condensate

Susceptibility
Fierz Identity (1) for Thirring Interaction

Goal: rewrite interaction term, use irreducible representation:
2-component spinors $\chi^a$, $a = 1, \ldots, 2N_f := N_{f,irr}$

\[
(\bar{\chi}^a \sigma_\mu \chi^a)(\bar{\chi}^b \sigma^\mu \chi^b) = - (\bar{\chi}^a \chi^a)(\bar{\chi}^b \chi^b) - 2 (\bar{\chi}^a \chi^b)(\bar{\chi}^b \chi^a) \quad (1)
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**Lagrangian after Hubbard-Stratonovich Transformations**

$\phi$ real scalar field, $T^{ab}$ hermitian, traceless matrix field

\[
\mathcal{L}_1 = \bar{\chi}_a \left[ (\phi + \phi) \delta^{ab} + T^{ab} \right] \chi_b + \frac{N_{f, \text{irr}}}{4g^2} T_{ab} T^{ba} + \frac{N_{f, \text{irr}}^2}{2g^2 (2 + N_{f, \text{irr}})} \phi^2
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\]

$D_1^{ab}$ has no special properties $\Rightarrow$ complex eigenvalues
Sign Problem after Fierz Rearrangement

\[ \lambda_{Th}/N_{f,irr} \]

- **Thirring** has no sign problem
- **Fierz (1)** has a complex phase with \[ \langle \omega \rangle := \langle e^{-i \text{Im} S} \rangle \approx 0 \] for \[ \lambda_{Th} \in [0.2, 0.3] \]
- **Fierz (2)** another identity, imaginary eigenvalues, switches sign on nearly every update \[ \Rightarrow \langle \omega \rangle \approx 0 \ \forall \lambda_{Th} \]
Current approach to get informations about chiral symmetry breaking and solving the sign problem of Fierz (1):

⇒ Introduce a spin field $k_{\chi i}^{ab} \in \{0, 1\}$, integrate fermions, $T^{ab}$ and $\phi$:

Final Partition Sum

$$Z(\lambda) \propto \sum_k (-\lambda)^{-\frac{k}{2}} \det (i\partial[k]) 2^{\tilde{n}_x^2} \prod_x f(n_x^1, n_x^2)$$

- $\partial[k]$ is the SLAC operator matrix with columns and rows deleted corresponding to $k$.
- $n_x^1$, $n_x^2$ and $\tilde{n}_x^2$ count certain entries of $k_{\chi i}^{ab}$
- $f(a, b)$ is a product of gamma- and confluent hypergeometric functions
Simple Metropolis simulation for $N_{f,\text{irr}} = 1$:
- Agreement with analytical calculation and original formulation.
- Deviations for Fierz(1) (sign problem!).

$N_{f,\text{irr}} = 1, 2 \times 3 \times 3$

\[
\langle k \rangle \propto \left\langle \frac{d \ln Z(\lambda)}{d \lambda} \right\rangle
\]
The Thirring model shows chiral symmetry breaking for $N_f < N_f^{cr}$, for which there are many different predictions.

Simulations with multiple four-fermion couplings may allow new insights.

We get a sign problem after Fierz transformation.

Fierz Identities

Fermion bag approach may solve the sign problem and allow access to the chiral condensate.

Thank you for your attention!