New results for QCD
at non-vanishing chemical potential
from Taylor expansion

Edwin Laermann
for the BNL-CCNU-Bielefeld collaboration
Taylor expansion of the pressure

\[
\frac{p}{T^4} = \frac{1}{VT^3} \ln Z(T, \mu_u, \mu_d, \mu_s) = \sum_{i,j,k} \frac{1}{i!j!k!} \chi_{ijk}(T) \left( \frac{\mu_u}{T} \right)^i \left( \frac{\mu_d}{T} \right)^j \left( \frac{\mu_s}{T} \right)^k
\]

with \( \hat{\mu} = \mu/T \)

\[
\chi_{ijk} = \frac{\partial^i}{\partial \hat{\mu}_u^i} \frac{\partial^j}{\partial \hat{\mu}_d^j} \frac{\partial^k}{\partial \hat{\mu}_s^k} \left( \frac{p}{T^4} \right) |_{\hat{\mu}=0}
\]

for instance

\[
\chi_{200} = \frac{1}{VT^3} \left( \frac{1}{4} \left\langle \frac{\partial^2 \ln \det M_u}{\partial \hat{\mu}_u^2} \right\rangle + \frac{1}{16} \left\langle \left( \frac{\partial \ln \det M_u}{\partial \hat{\mu}_u} \right)^2 \right\rangle \right)
\]

with

\[
\frac{\partial^2 \ln \det M_u}{\partial \hat{\mu}_u^2} = \text{tr} \left( M_u^{-1} \frac{\partial^2 M_u}{\partial \hat{\mu}_u^2} \right) - \text{tr} \left( M_u^{-1} \frac{\partial M_u}{\partial \hat{\mu}_u} M_u^{-1} \frac{\partial M_u}{\partial \hat{\mu}_u} \right)
\]

these can be computed on the lattice at \( \hat{\mu} = 0 \) ...

note: \( c_{ijk} = 0 \) for \( i + j + k \) odd because of charge symmetry
... and combined to expansion in terms of $\mu_B, \mu_S, \mu_Q$

$$\frac{p}{T^4} = \sum_{i,j,k} \frac{1}{i!j!k!} \chi_{ijk}^{BSQ}(T) \left( \frac{\mu_B}{T} \right)^i \left( \frac{\mu_S}{T} \right)^j \left( \frac{\mu_Q}{T} \right)^k$$

the generalized susceptibilities $\chi_{ijk}^{BQS}$ are related to cumulants of fluctuations, $\delta N_X = N_X - \langle N_X \rangle$, of net quantum numbers $X = B, S, Q$, e.g.

$$\chi_X^2 = \langle (\delta N_X)^2 \rangle$$
$$\chi_X^4 = \langle (\delta N_X)^4 \rangle - 3\langle (\delta N_X)^2 \rangle^2$$
QCD at $\mu_q \neq 0$ from Taylor expansion

comparison with

**HRG = Hadron Resonance Gas model:** quite successful in describing hadron yields in HIC

$$\frac{p^{\text{HRG}}}{T^4} = \sum_{\text{mesons}} \ln Z_i^M(T, \mu_Q, \mu_S) + \sum_{\text{baryons}} \ln Z_i^B(T, \mu_B, \mu_Q, \mu_S)$$

$$\ln Z_i^{M/B} = \frac{d_i}{\pi^2} \left( \frac{m_i}{T} \right)^2 \sum_{k=1}^{\infty} \frac{(\pm 1)^{k+1}}{k^2} K_2(k m_i/T) \cosh(k(B_i \hat{\mu}_B + Q_i \hat{\mu}_Q + S_i \hat{\mu}_S))$$

$$\simeq \frac{d_i}{\pi^2} \left( \frac{m_i}{T} \right)^2 K_2(m_i/T) \cosh(B_i \hat{\mu}_B + Q_i \hat{\mu}_Q + S_i \hat{\mu}_S)$$

**Boltzmann approximation:** $k = 1$; recall $K_2(x) \sim e^{-x} \rightarrow$ good except for pions

$\Rightarrow$ fluctuations as e.g.

$$\chi_B^n |_{\mu=0} \sim \sum_{\text{baryons}} B_i^n K_2(m_i/T) \simeq K_2(m_i/T)$$
QCD at $\mu_q \neq 0$ from Taylor expansion

Simulation parameters

- $N_F = 2 + 1$: two degenerate u/d quarks + strange quark
- along “line of constant physics” i.e. constant physical $m_K = 500$ MeV and constant $m^G_\pi$
- improved staggered action: hisq
- $m^G_\pi = 160, 140$ MeV corresponds to $m_{u,d}/m_s = 1/20, 1/27$
- $N^3_\sigma \times N_\tau$ lattices with $N_\tau = 6, 8, 12, 16$, $N_\sigma = 4N_\tau$

Note: $a = 1/(TN_\tau)$

Staggered quarks:

16 pion operators in

8 different lattice reps.

‘taste violations’

$$M^{\text{RMS}}_\pi = \sqrt{\sum_{i=1}^{16} m^2_{\pi,i}}$$

at $T \simeq T_c$: hisq with much reduced taste violations:
QCD at $\mu_q \neq 0$ from Taylor expansion

Equation of State for (2+1) flavor QCD

first at $\mu_B = 0$: pressure, entropy and energy density

- consistent between different lattice groups
  - different actions
  - continuum limit
  - physical quark masses

- comparison with HRG model
  - up to crossover region: QCD agrees quite well with HRG
  - however: QCD systematically above HRG
next: \( \mu_B \neq 0 \), but \( \mu_Q = \mu_S = 0 \) \( \Rightarrow \)

\[
p(T, \mu_B) = \sum_i \frac{1}{i!} \chi_{i0}^{BSQ}(T) \left( \frac{\mu_B}{T} \right)^i
\]

\[
\frac{\Delta(T, \mu_B)}{T^4} = \frac{p(T, \mu_B) - p(T, 0)}{T^4} = \frac{\chi_B^2}{2} \left( \frac{\mu_B}{T} \right)^2 \left[ 1 + \frac{1}{12} \frac{\chi_B^4}{\chi_B^2} \left( \frac{\mu_B}{T} \right)^2 \right]
\]

\[\text{variance} \quad \text{kurtosis / variance} \quad : \frac{\kappa_B}{\sigma_B^2}\]

- leading and next to leading order agree well with HRG for \( T < 150 \text{ MeV} \)
- but: in crossover region large deviations from HRG e.g. \( \sim 40\% \) for \( T \simeq 165 \text{ MeV} \)
- 4th order contribution rapidly becomes small at large \( T \)
\[
\frac{\Delta(T, \mu_B)}{T^4} = \frac{p(T, \mu_B) - p(T, 0)}{T^4} = \frac{\chi_B^2}{2} \left( \frac{\mu_B}{T} \right)^2 \left[ 1 + \frac{1}{12} \frac{\chi_B^4}{\chi_B^2} \left( \frac{\mu_B}{T} \right)^2 \right]
\]

estimating next order correction
\[
\frac{1}{360} \frac{\chi_B^6}{\chi_B^2} \left( \frac{\mu_B}{T} \right)^4
\]

⇒ pressure well under control for \( \mu_B/T \leq 2 \)
finally: all $\mu_B, \mu_Q, \mu_S \neq 0$

- for instance: match experimental/initial conditions in heavy ion collision
  - strangeness neutrality: $\langle n_S \rangle = 0$
  - isospin asymmetry: $\langle n_Q \rangle = r\langle n_B \rangle$ with $r = 0.4$ for Au-Au and Pb-Pb

- expand in powers of $\mu_B, \mu_Q, \mu_S$ and solve for $\mu_Q, \mu_S$ to given order $\Rightarrow$

$$\hat{\mu}_Q(T, \mu_B) = q_1(T)\hat{\mu}_B + q_3(T)\hat{\mu}_B^3$$
$$\hat{\mu}_S(T, \mu_B) = s_1(T)\hat{\mu}_B + s_3(T)\hat{\mu}_B^3$$

$\Rightarrow$ only 2 parameters $T, \mu_B$

LO: continuum extrapolated

NLO: small cut-off dependence
NLO corrections $< 10\%$
QCD at $\mu_q \neq 0$ from Taylor expansion

Pressure coefficients

$$\frac{p(T, \mu_B) - p(T, 0)}{T^4} = \frac{1}{2} p_2 \left( \frac{\mu_B}{T} \right)^2 + \frac{1}{24} p_4 \left( \frac{\mu_B}{T} \right)^4 + ...$$

- considerable strangeness dependence
- mild isospin dependence
\[
\frac{X(T, \mu_B) - X(T, 0)}{T^4} = \frac{1}{2} X_2 \left( \frac{\mu_B}{T} \right)^2 + \frac{1}{24} X_4 \left( \frac{\mu_B}{T} \right)^4 + \ldots
\]

\[\mu_Q = \mu_S = 0\]

\[n_S = 0 \text{ and } n_Q/n_B = 0.4\]
QCD at $\mu_q \neq 0$ from Taylor expansion

finally

pressure

energy

$\mu_Q = \mu_S = 0$

$n_S = 0$ and $n_Q/n_B = 0.4$
Summary

- used Taylor coefficients for Equation of State at $\mu_B \neq 0$
  - at $\mu_S = 0 = \mu_Q$ as well as
  - at $\mu_S \neq 0 \neq \mu_Q$ especially at $n_S = 0$ and $n_Q/n_B = 0.4$ (RHIC, LHC)

- expansion under control up to $\mu_B/T \lesssim 2$ ($\sqrt{s_{NN}} \gtrsim 12$ GeV)