O(4) scaling analysis in two-flavor QCD at finite temperature and density with improved Wilson quarks

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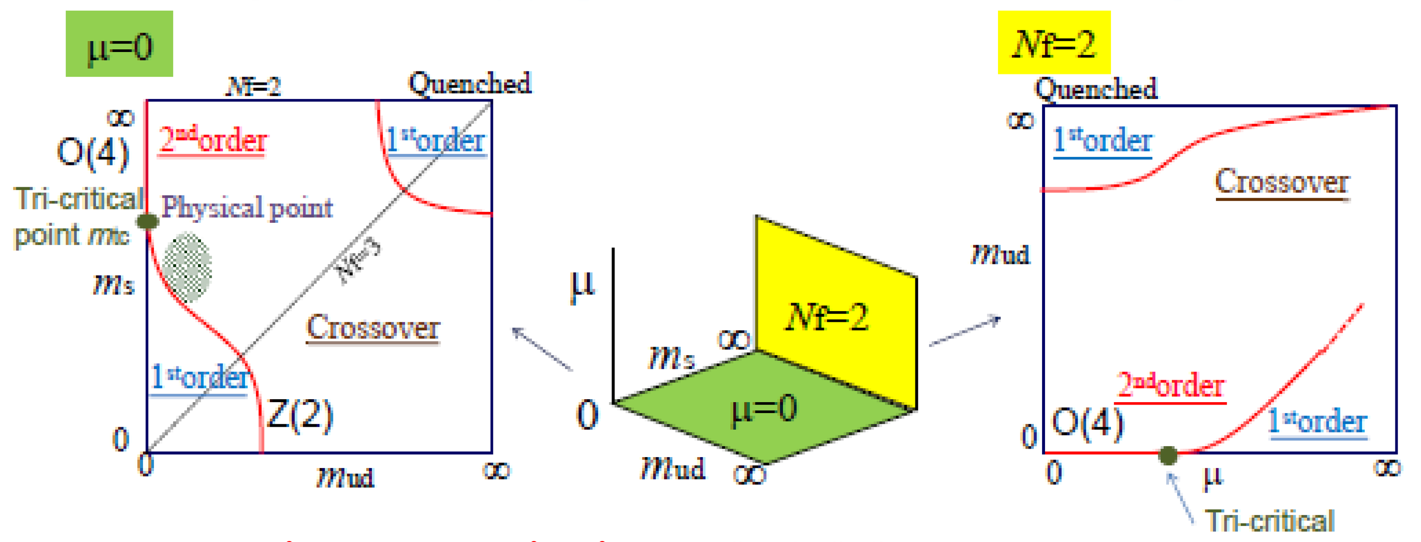
We study scaling behavior of a chiral order parameter in the low density region, performing a simulation of two-flavor lattice QCD with improved Wilson quarks. It has been confirmed that the scaling behavior of the chiral order parameter defined by a Ward-Takahashi identity agrees with the scaling function of the three-dimensional O(4) spin model at zero chemical potential. We discuss the scaling properties of the chiral phase transition at finite density, applying the reweighting method and calculating derivatives of the chiral order parameter with respect to the chemical potential. In the comparison between the scaling functions of the O(4) spin model and QCD at low density, there is a fit parameter which can be interpreted as the curvature of the chiral phase transition curve in the QCD phase diagram with respect to temperature and chemical potential. We determine the curvature of the phase boundary by the fitting. The physical scale is set by the gradient flow.

1. Introduction

Quark mass dependence of QCD phase transition

Expected nature of phase transitions

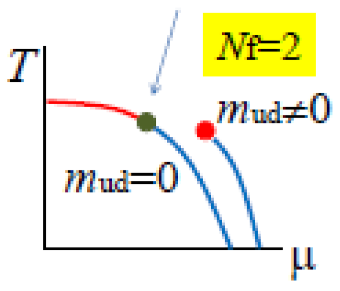
Nf=2 QCD and O(4) spin model: same universality class?



Interesting properties in N_f=2+1 QCD

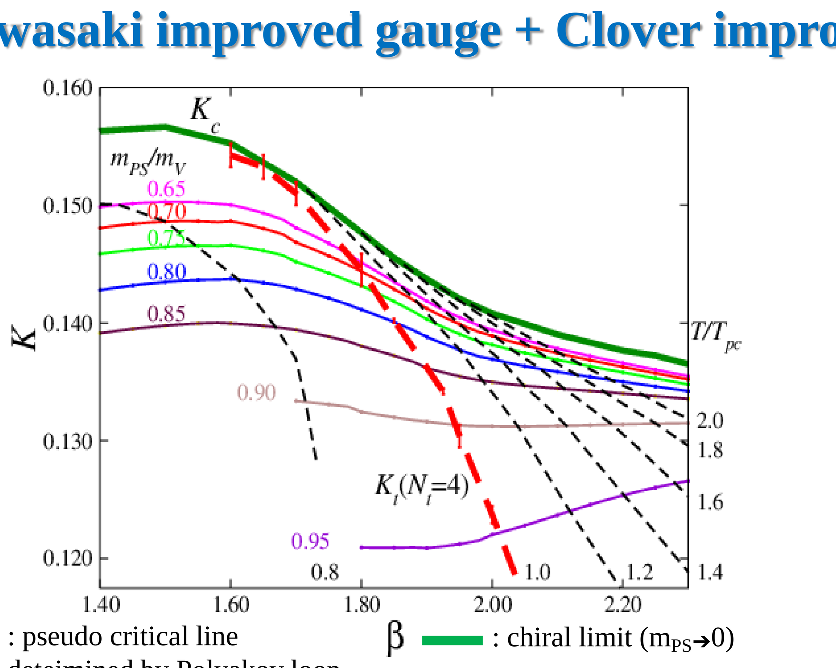
- O(4) scaling for m_s>>m_{tc}
- Tri-critical scaling at m_{tc}
- First order transition at m_s<m_{tc}
- O(4) scaling at small μ
- Tri-critical point at μ≠0

This study



2. Wilson quark simulation

Iwasaki improved gauge + Clover improved Wilson quark action



: pseudo critical line deteimined by Polyakov loop; : chiral limit (m_{PS}→0)

#conf.=500-5500, #noise=150 for each color and spin indices

β	κ	β	κ	β	κ
1.50	0.150290	1.50	0.143480	1.50	0.148750
1.55	0.150251	1.55	0.143654	1.55	0.148500
1.60	0.150030	1.60	0.143749	1.60	0.148800
1.65	0.149328	1.65	0.143416	1.65	0.147900
1.70	0.148086	1.70	0.142871	1.70	0.147100
1.75	0.146763	1.75	0.142105	1.75	0.145800
1.80	0.145127	1.80	0.141139	1.80	0.144100
1.85	0.143502	1.85	0.140070		
1.90	0.141849	1.90	0.138817		
1.95	0.140472	1.95	0.137716		
2.00	0.139411	2.00	0.136931		

: m_{PS}/m_V =0.65 (orange); : m_{PS}/m_V =0.80 (blue); : m_{PS}/m_V =0.70 (red)

$$Z=\int\prod_{x,\mu}dU_\mu(x)(\det M)^{N_f}e^{-S_g}\qquad W^{n\times m}_{\mu\nu}\text{ : nxm Wilson loop}$$

$$S_g=-\beta\sum_x\Big(c_0\sum_{\mu<\nu}W^{1\times1}_{\mu\nu}(x)+c_1\sum_{\mu\neq\nu}W^{1\times2}_{\mu\nu}(x)\Big)$$

$$\beta=6/g^2,\quad c_0=-0.331,\quad c_1=1-8c_0,$$

$$c_{SW}=(1-0.8412\beta^{-1})^{-3/4},\ \mu_q\text{ : chemical potential}$$

$$M_{x,y}=\delta_{x,y}-\kappa\sum_i\left[(1-\gamma_i)U_i\delta_{x+\hat i,y}+(1+\gamma_i)U_i^\dagger\delta_{x-\hat i,y}\right]$$
$$-\kappa\left[e^{\mu_q a}(1-\gamma_4)U_4\delta_{x+\hat 4,y}+e^{-\mu_q a}(1+\gamma_4)U_4^\dagger\delta_{x-\hat 4,y}\right]$$
$$+\delta_{x,y}c_{SW}\kappa\sum_{\mu<\nu}\sigma_{\mu\nu}F_{\mu\nu}\qquad\text{: clover term}$$

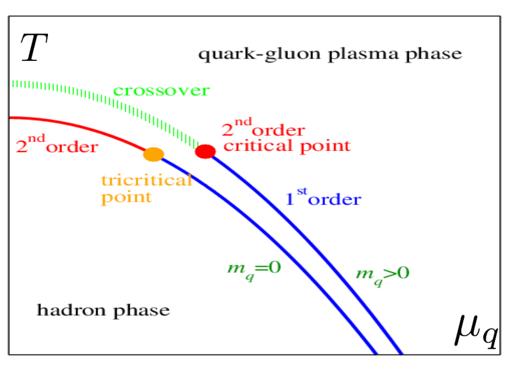
3. Scaling behavior of chiral order parameter at μ≠0

3-1. O(4) scaling test for N_f=2 QCD

3-D O(4) spin model: magnetization(M), external field(h), reduced temperature(t)

Nf=2 QCD : chiral order parameter($\langle\bar\psi\psi\rangle$), ud-quark mass($m_{ud}$), critical β ($\beta_{ct}$)

$$M=\langle\bar\psi\psi\rangle,\quad h=2m_q a,\quad t=\begin{cases}\beta-\beta_{ct}+\frac{c'}{2}\left(\frac{\mu_q}{T}\right)^2\\ \frac{T-T_{ct}}{T_{ct}}+\frac{c}{2}\left(\frac{\mu_q}{T}\right)^2\end{cases}$$



scaling function:

$$\boxed{M/h^{1/\delta}=f(t/h^{1/\gamma})}$$

The second derivative ($t\propto T$) :

$$\frac{d^2M}{d(\mu_q/T)^2}\bigg|_{\mu_q=0}=c\frac{dM}{dt}\bigg|_{\mu_q=0},\quad \frac{dM/dt}{h^{1/\delta-1/\gamma}}\bigg|_{\mu_q=0}=\frac{df(x)}{dx}\bigg|_{x=t/h^{1/\gamma}}$$

Note: c is the curvature of the critical line in the (T, μ_q) plane, c' is in the (β, μ_q) plane,

We calculate the curvature c' from this equation and change c' to c (See: 3-6-1.).

3-2. Chiral order parameter defined by Ward-Takahashi identies

Ward-Takahashi identities in the continuum limit [Bochichio et al., NPB262, 331 (1985)].

$$\langle\partial_\mu A_\mu(x)P(y)\rangle-2m_{ud}a\langle P(x)P(y)\rangle=\delta(x-y)\langle\bar\psi\psi(x)\rangle,\ P=\bar\psi\gamma_5\psi,\ A_\mu=\bar\psi\gamma_\mu\gamma_5\psi,$$

$$2m_{ud}a=-m_\pi\frac{\langle A_4(t)P(0)\rangle}{\langle P(t)P(0)\rangle},\ \langle\bar\psi\psi\rangle^{WI}=\frac{2m_{ud}a(2\kappa)^2}{N_s^3N_t}\sum_{x,y}\langle P(x)P(y)\rangle$$

Because the chiral symmetry is explicitly broken for Wilson quarks, we define $m_q a$ and $\langle\bar\psi\psi\rangle^{WI}$ by WT identities.

3-3. Reweighting method for the chiral order parameter at μ≠0

$$\langle\bar\psi\psi\rangle^{WI}=\frac{2m_{ud}a(2\kappa)^2}{N_s^3N_t}\sum_{x,y}\langle P(x)P(y)\rangle=\frac{2m_{ud}a(2\kappa)^2}{N_s^3N_t}\sum_{x,y}\langle\mathrm{tr}\,(M^{-1}\gamma_5M^{-1}\gamma_5)\rangle$$

The expectation value at μ≠0 is computed by the reweighting method with O(μ^3) error.

$$\langle\mathrm{tr}\,(M^{-1}\gamma_5M^{-1}\gamma_5)\rangle_{(\beta,\mu)}=\frac{1}{Z}\int\mathcal{D}U\,\mathrm{tr}\,(M^{-1}\gamma_5M^{-1}\gamma_5)\,(\det M(\mu))^{N_f}e^{-S_g(\beta)}=\frac{\left\langle\mathrm{tr}\,(M^{-1}\gamma_5M^{-1}\gamma_5)\,e^{N_f[\ln\det M(\mu)-\ln\det M(0)]}\right\rangle_{(\beta_0,0)}}{\left\langle e^{N_f[\ln\det M(\mu)-\ln\det M(0)]}\right\rangle_{(\beta_0,0)}}$$

$$\ln\det M(\mu)-\ln\det M(0)=\mu\frac{\partial\ln\det M}{\partial\mu}+\frac{\mu^2}{2}\frac{\partial^2\ln\det M}{\partial\mu^2}+\mathcal{O}(\mu^3)$$

$$\frac{\partial\ln\det M}{\partial\mu}=\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}\right),\ \frac{\partial^2\ln\det M}{\partial\mu^2}=\mathrm{tr}\left(M^{-1}\frac{\partial^2M}{\partial\mu^2}\right)+\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\right),\ \frac{\partial\mathrm{tr}\,(M^{-1}\gamma_5M^{-1}\gamma_5)}{\partial\mu}=-2\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\gamma_5M^{-1}\gamma_5\right)$$

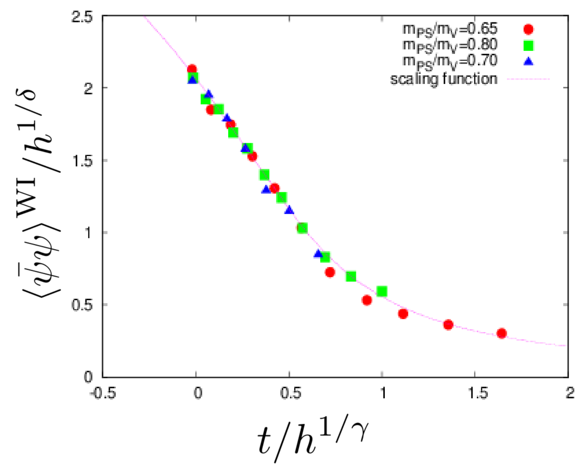
$$\frac{\partial^2\mathrm{tr}\,(M^{-1}\gamma_5M^{-1}\gamma_5)}{\partial\mu^2}=-2\mathrm{tr}\left(M^{-1}\frac{\partial^2M}{\partial\mu^2}M^{-1}\gamma_5M^{-1}\gamma_5\right)+4\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\gamma_5M^{-1}\gamma_5\right)+4\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\gamma_5M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\gamma_5\right)$$

$$\left(\frac{\partial^nM}{\partial\mu^n}\right)_{(x,y)}=\begin{cases}-\kappa\left[(1-\gamma_4)U_4\delta_{x+\hat4,y}-(1+\gamma_4)U_4^\dagger\delta_{x-\hat4,y}\right] & \text{for n=odd}\\ -\kappa\left[(1-\gamma_4)U_4\delta_{x+\hat4,y}+(1+\gamma_4)U_4^\dagger\delta_{x-\hat4,y}\right] & \text{for n=even}\end{cases}$$

・These operators can be calculated by the random noise method.

3-4. Scaling plot at μ=0 and O(4) scaling function

We fit the data of $\langle\bar\psi\psi\rangle^{WI}$ by the O(4) scaling function.



$$\boxed{M/h^{1/\delta}=f(t/h^{1/\gamma})}$$

$$t=\beta-\beta_{ct}+\frac{c'}{2}\left(\frac{\mu_q}{T}\right)^2$$

Symbols: results at μ=0

Lines: O(4) scaling function

J. Engels, T. Mendes, Nucl. Phys. B 572 (2000) 289
J. Engels, L. Fromme, M. Seniuch, Nucl. Phys. B 675 (2003) 533.

We conform scaling behavior of a chiral order parameter at μ=0, performing a simulation of Nf=2 lattice QCD with improved Wilson quarks. This scaling behavior agrees with the scaling function of the 3d O(4) spin model at μ=0 .

Consistent with O(4) scaling function.

3-5. Derivatives of the chiral order parameter at μ=0

3-5-1. method 1 (reweighting method)

We fit the data of $\langle\bar\psi\psi\rangle^{WI}$ at μ≠0 by $\langle\bar\psi\psi\rangle^{WI}(\mu)=x+y(\mu/T)^2$, where x and y are the fit parameters. (The first derivative is zero due to the symmetry: μ→−μ)

$$x=\langle\bar\psi\psi\rangle^{WI},\ y=\frac12\frac{\partial^2\langle\bar\psi\psi\rangle^{WI}}{\partial(\mu_q/T)^2}\quad\text{at }\mu=0$$

3-5-2. method 2 (Tayler method)

We calculate the expectation value of the following operators and obtain the derivatives.

$$\langle\bar\psi\psi\rangle^{WI}=\frac{2m_{ud}a(2\kappa)^2}{N_s^3N_t}F_0,\ \frac{\partial^2\langle\bar\psi\psi\rangle^{WI}}{\partial(\mu_q/T)^2}=\frac{2m_{ud}a(2\kappa)^2}{N_s^3N_t}(F_2-F_0A_2)$$

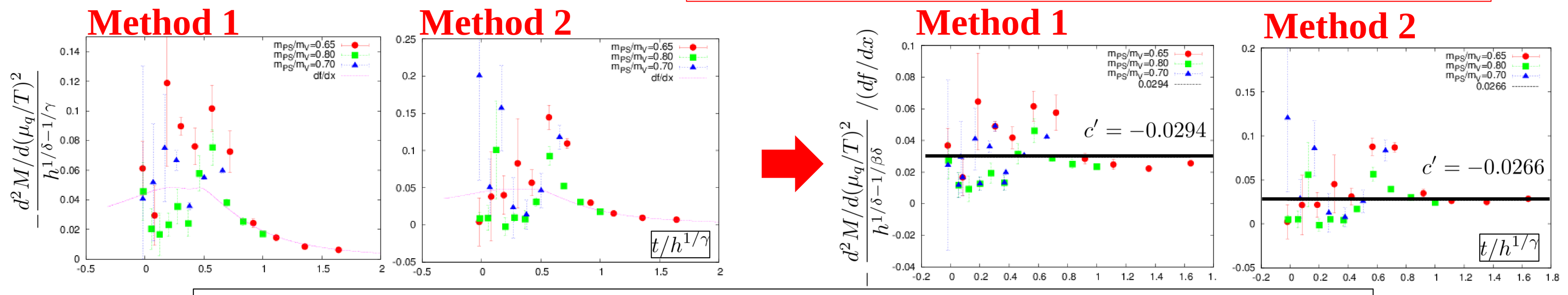
$$F_0=\langle\mathrm{tr}\,(M^{-1}\gamma_5M^{-1}\gamma_5)\rangle,\ \mu\equiv\mu_q a=(\mu_q/T)/N_t$$

$$A_2=\left\langle N_f\frac{\partial^2\ln\det M}{\partial\mu^2}\right\rangle+\left\langle N_f\left(\frac{\partial\ln\det M}{\partial\mu}\right)^2\right\rangle$$

$$F_2=\left\langle\frac{\partial^2\mathrm{tr}\,(M^{-1}\gamma_5M^{-1}\gamma_5)}{\partial\mu^2}\right\rangle+2\left\langle\frac{\partial\mathrm{tr}\,(M^{-1}\gamma_5M^{-1}\gamma_5)}{\partial\mu}N_f\frac{\partial\ln\det M}{\partial\mu}\right\rangle+2\left\langle\mathrm{tr}\,(M^{-1}\gamma_5M^{-1}\gamma_5)\,N_f\frac{\partial^2\ln\det M}{\partial\mu^2}\right\rangle+2\left\langle\mathrm{tr}\,(M^{-1}\gamma_5M^{-1}\gamma_5)\left(N_f\frac{\partial\ln\det M}{\partial\mu}\right)^2\right\rangle$$

Second derivatives of $\langle\bar\psi\psi\rangle^{WI}$ for Nf=2 QCD:

$$\frac{\partial^2\langle\bar\psi\psi\rangle^{WI}/\partial(\mu_q/T)^2}{h^{1/\delta-1/\gamma}}=c'\frac{df(x)}{dx}\bigg|_{x=t(\beta)/h^{1/\gamma}}=c\frac{df(x)}{dx}\bigg|_{x=t(T)/h^{1/\gamma}}$$



O(4) scaling plot of the second derivative of the chiral order parameter(left), and the second derivative of the curvature c' in the (β, μ_q/T) plane(right) with two method. These values are similar.

3-6. Relationship c and c'

3-6-1. Scale setting by using O(a^2) improved gradient flow

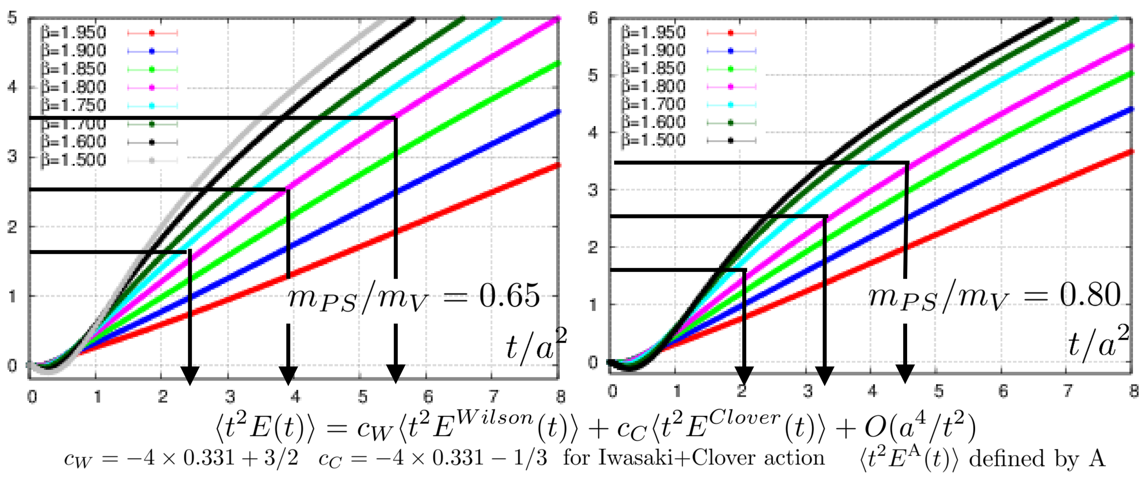
We calculate scales ($\sqrt{t_{1.5}}/a$, $\sqrt{t_{2.5}}/a$, $\sqrt{t_{3.5}}/a$) for each simulation parameter by using O(a^2) improved gradient flow. Then, we take extra simulations to estimate the scale in chiral limit (cyan table).

β	κ
1.50	0.151290
1.50	0.152040
1.50	0.152290
1.50	0.152540
1.50	0.152790
1.50	0.153040
1.50	0.153290

β	κ
1.60	0.151030
1.60	0.151280
1.60	0.151530
1.60	0.151780

β	κ
1.70	0.150086

: m_{PS}/m_V→0



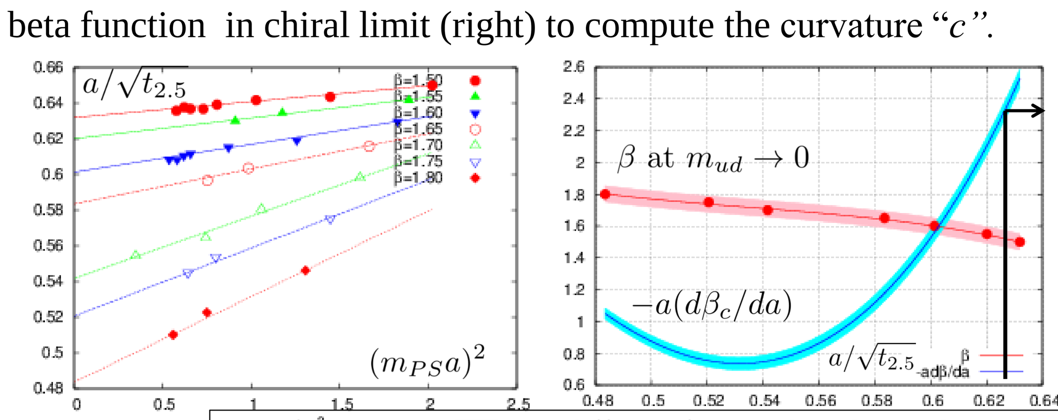
$$\langle t^2E(t)\rangle=c_W\langle t^2E^{Wilson}(t)\rangle+c_C\langle t^2E^{Clover}(t)\rangle+O(a^4/t^2)$$

$c_W=-4\times0.331+3/2\quad c_C=-4\times0.331-1/3$ for Iwasaki+Clover action; $\langle t^2E^A(t)\rangle$ defined by A

The scale $\sqrt{t_X}/a$ means the value at which $\langle t^2E(t)\rangle|_{t=t_X}=X$. Changing the hopping parameter κ, we will estimate the relationship between these scale $\sqrt{t_X}/a$ and gauge coupling β.

3-6-2. Beta function in chiral limit

We calculate scales ($\sqrt{t_{1.5}}/a$, $\sqrt{t_{2.5}}/a$, $\sqrt{t_{3.5}}/a$) in the chiral limit (left) and beta function in chiral limit (right) to compute the curvature "c".



$$c'=\frac{d^2\beta_{ct}}{d(\mu_q/T)^2}\bigg|_{\mu_q=0}=a\frac{d\beta_{ct}}{da}\times c,$$

$$a\frac{d\beta_{ct}}{da}=\begin{cases}-4.11(1)&(\sqrt{t_{1.5}}/a)\\-2.38(5)&(\sqrt{t_{2.5}}/a)\\-2.295(8)&(\sqrt{t_{3.5}}/a)\end{cases}$$

Beta function at $\sqrt{t_{1.5}}/a$ is different from at the other scales. Beta function at $\sqrt{t_{X>2.5}}/a$ become stable.

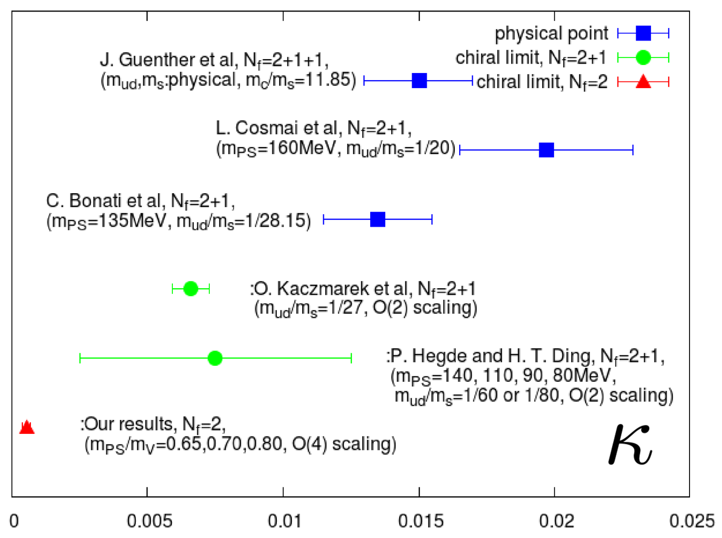
3-7. Curvature of critical line in the chiral limit

$$\frac{T_c(\mu_B)}{T_c}=1-\kappa\left(\frac{\mu_B}{T}\right)^2+\mathcal{O}(\mu_B^4),\ \kappa\text{ : the curvature of critical line}$$

$$c'=-\frac{d^2\beta_{ct}}{d(\mu_q/T)^2},\ c=\frac{d^2T_{ct}}{d(\mu_q/T)^2}=a\frac{d\beta}{da}\times c',\ \boxed{\kappa=\frac{c}{18}}$$

We estimate the curvature κ by 2 method and 2 scales ($\sqrt{t_{2.5}}/a$, $\sqrt{t_{3.5}}/a$).

	$\sqrt{t_{2.5}}/a$	$\sqrt{t_{3.5}}/a$
Method 1	0.0007(1)	0.0007(1)
Method 2	0.0006(2)	0.0006(2)



Our results are different from the other results. Please pay attention,

1. Blue results show the curvature at the physical point and green results show the curvature at the chiral limit.
2. $\langle\bar\psi\psi\rangle^{WI}$ with Wilson quarks restores O(4) symmetry exactly.

●Physical point (blue point)
J. Guenther et al: PoS(Lattice 2015) 142, L. Cosmai et al: PoS(Lattice 2015) 143,
C. Bonati et al: PoS(Lattice 2015) 144
●Chiral limit (green point)
O. Kaczmarek et al: Phys. Rev. D **83**, 014504 (2012), P. Hegde: PoS(Lattice 2015) 141
●Chiral limit (red point) ･･･ Our results

3-8. Summary

1) We discussed the scaling property of the chiral phase transition in the low density region.
2) We calculated the chiral order parameter and its second derivative performing a simulation of Nf=2 QCD with Wilson quarks and compared the results with an expected scaling function.
3) The scaling behavior is roughly consistent with our expectation.
4) The curvature of the critical line in the (T, μ) plane was estimated.

presented by Takashi Umeda (Hiroshima Univ.)