Progress on three-particle quantization condition

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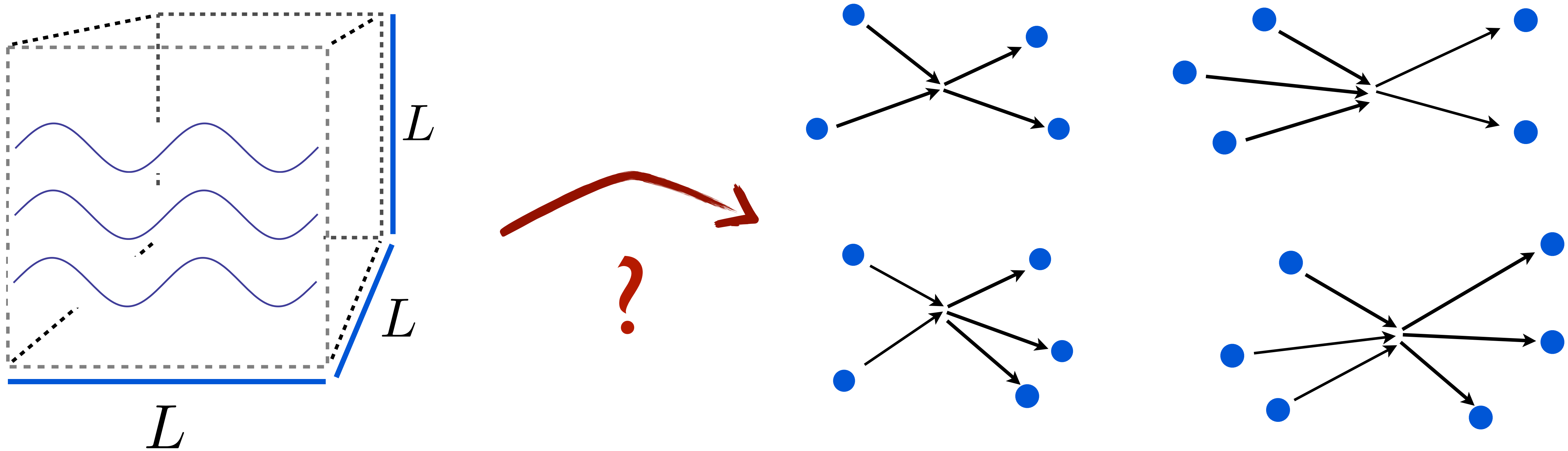
Outline

- Motivation
- Original formalism (relativistic, model independent)
- Extensions underway
- A new test: volume dependence of Efimov-like 3-particle bound state (compared to NRQM)

The fundamental issue

- Lattice QCD can calculate energy levels of multiple-particle systems in a box
- How are these related to scattering amplitudes?



Potential applications

- Studying resonances with three particle decay channels

$$\omega(782) \to \pi\pi\pi \qquad K^* \longrightarrow K\pi\pi \quad N(1440) \to N\pi\pi$$

- Calculating weak decay amplitudes/form factors involving 3 particles, e.g. $K \to \pi\pi\pi$

- Determining NNN interactions
  - Input for effective field theory treatments of larger nuclei & nuclear matter
  - Similarly, $\pi\pi\pi$, $\pi K\overline{K}$, … interactions needed for study of pion/kaon condensation

Previous result [Hansen & SS 1408.5933]

- **<u>Step I</u>**: FV spectrum is determined (for given L, **P**) by solutions of

Infinite volume 3-particle scattering quantity; depends on cutoff function H

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$$

Known kinematical quantity: essentially the same as in 2-particle analysis

$$F_3 = \frac{F_2}{2\omega L^3}\left[\frac{1}{3} - \mathcal{M}_{2,L} F_2 - \mathcal{D}_L^{(u,u)} \frac{F_2}{2\omega L^3}\right]$$

Depends on $\mathcal{M}_2$ and F_2

Depends on $\mathcal{M}_2$ and on new kinematical quantity G

- Superficially similar to 2-particle form ...

$$\det\left[F_2^{-1} + \mathcal{K}_2\right] = 0$$

... but F_3 lives in a larger space, and contains both kinematical, finite-volume quantities (F_2 & G) and the dynamical, infinite-volume quantity $\mathcal{K}_2$

Previous result [Hansen & SS 1504.04248]

- **Step 2**: Relate unphysical scattering quantity to physical scattering amplitude

Involve only $\mathcal{M}_2$ and G
so "known"

$$i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L\; i\mathcal{K}_{\mathrm{df},3\to3}\frac{1}{1 - iF_3\; i\mathcal{K}_{\mathrm{df},3\to3}}\mathcal{R}_L\right]$$

$$i\mathcal{M}_{3\to3} = \lim_{L\to\infty}\Big|_{i\epsilon}\; i\mathcal{M}_{L,3\to3}$$

Symmetrization

Sums go over to integrals with iε pole prescription

- Result is an integral equation giving $\mathcal{M}_3$ in terms of $\mathcal{K}_{\mathrm{df},3}$

- Confirms that finite volume spectrum is determined by infinite-volume scattering amplitudes [Polejaeva & Rusetsky, 14] in a relativistic analysis

Limitations of previous result

1. Assumes Z_2 symmetry: no $2 \leftrightarrow 3$, $1 \leftrightarrow 2$, … vertices

2. No resonances allowed in two-particle subchannels in kinematic range considered

Extensions underway

[Briceño, Hansen & SS]

- Conjectured result without Z_2 symmetry (derivation nearly complete):

$$\det\left[1 - \begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix} \begin{pmatrix} \mathcal{K}_2 & \mathcal{K}_{23} \\ \mathcal{K}_{32} & \mathcal{K}_{\mathrm{df},3} \end{pmatrix}\right] = 0$$

Lüscher zeta-function

3↔2 K-matrix; Requires no long-distance subtraction

  - Relation of K-matrices to $\mathcal{M}_2$, $\mathcal{M}_{23}$, $\mathcal{M}_{32}$ & $\mathcal{M}_3$ to be determined

- Including above-threshold 2-particle K-matrix poles:
  - Approach based on factorization of poles in early stages

Testing the formalism

- Threshold expansion [Hansen & SS, 1602.00324]
  - Matches $1/L^3$—$1/L^5$ terms from NRQM [Beane, Detmold & Savage 07; Tan 08]
  - Matches $1/L^3$—$1/L^6$ terms from relativistic φ^4 theory up to $O(\lambda^3)$ [Hansen & SS, 1509.07929]
- New result presented here for finite-volume dependence of Efimov-like 3-particle bound state: [Hansen & SS, in prep.]
  - Matches NRQM result [Meissner, Rios & Rusetsky, 1412.4969]

NRQM result

[Meissner, Rios & Rusetsky, 1412.4969 + erratum]

$$E_B \equiv 3m - \frac{\kappa^2}{m}$$

Efimov state

$$\Delta E_L = c\frac{\kappa^2}{m}\frac{1}{(\kappa L)^{3/2}}\exp\left(-\frac{2\kappa L}{\sqrt{3}}\right)\left[1+\mathcal{O}\left(\frac{1}{\kappa L}\right)\right]$$

FV energy shift

Constants from Efimov wavefunction

$$c = -36\times 3^{3/4}\pi^{7/2}C_0|A|^2\sinh^2(s_0\pi/2)$$

- Assumes two-body potential, unitary limit (scattering length a→∞), **P=0**
- Aim to reproduce exponent, leading power & overall constant from relativistic formalism

Reproducing MRR: step 1

- Assume $K_{df,3}$=0 (no local 3-particle interaction)

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0 \qquad \Rightarrow F_3 \longrightarrow \infty$$

$$F_3 = \frac{F_2}{2\omega L^3}\left[\frac{1}{3} - \mathcal{M}_{2,L}F_2 - \mathcal{D}_L^{(u,u)}\frac{F_2}{2\omega L^3}\right] \qquad \Rightarrow D_L^{(u,u)} \longrightarrow \infty$$

$\mathcal{M}_2$ and F_2 do not diverge below threshold

Reproducing MRR: step 1

- Assume $K_{df,3}$=0 (no local 3-particle interaction)

$$i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \xcancel{i\mathcal{K}_{\mathrm{df},3\to3}\frac{1}{1 - iF_3\, i\mathcal{K}_{\mathrm{df},3\to3}}} \mathcal{R}_L\right]$$

Pole in $\mathcal{M}_{L,3}$ requires
pole in $\mathcal{D}_L=\mathcal{S}[\mathcal{D}_L^{(u,u)}]$

Reproducing MRR: step 1

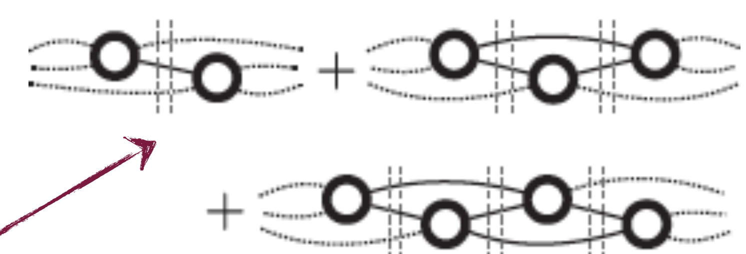
- Assume $K_{df,3}$=0 (no local 3-particle interaction)

$$i\mathcal{M}_{L,3\to3} = i\mathcal{D}_L + \mathcal{S}\left[\mathcal{L}_L \cancel{i\mathcal{K}_{\mathrm{df},3\to3}\frac{1}{1-iF_3\, i\mathcal{K}_{\mathrm{df},3\to3}}}\mathcal{R}_L\right]$$

Pole in $\mathcal{M}_{L,3}$ requires
pole in $\mathcal{D}_L=\mathcal{S}[\mathcal{D}_L^{(u,u)}]$

- Assume pure s-wave 2-particle interaction (same approx. as [MRR])

$$i\mathcal{D}_L^{(u,u)} \equiv i\mathcal{M}_{2,L}\, iG\, i\mathcal{M}_{2,L}\frac{1}{1-iGi\mathcal{M}_{2,L}}[2\omega L^3] =$$

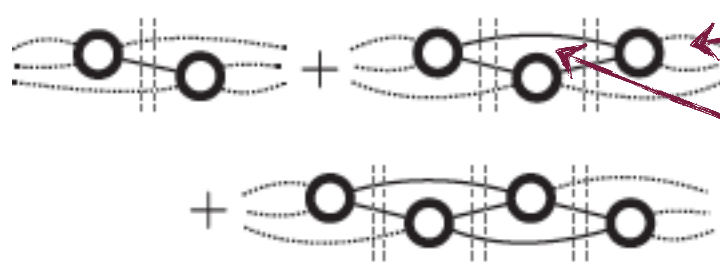


$$G^s_{p,k} = \frac{H(\vec{p})H(\vec{k})}{2\omega_{kp}(E-\omega_p-\omega_k-\omega_{kp})}\frac{1}{2\omega_k L^3}$$

Matrix indices are momentum of spectator

Reproducing MRR: step 2

- Dominant FV corrections to $\mathcal{D}_L^{(u,u)}$



- Replace sums with integrals + sum-integrals (non-trivial since loops not independent)

$$i\mathcal{D}_L^{s,(u,u)}(\vec{p},\vec{k}) \approx i\mathcal{D}^{s,(u,u)}(\vec{p},\vec{k}) + \int_{r,t} \frac{1}{2\omega_r}\frac{1}{2\omega_t} i\mathcal{D}^{s,(u,u)}(\vec{p},\vec{r}) iI(\vec{r},\vec{t}) i\mathcal{D}^{s,(u,u)}(\vec{t},\vec{k}) + \dots$$

$$iI(\vec{p},\vec{k}) \equiv \left[\frac{1}{L^3}\sum_s - \int_s\right] \frac{1}{2\omega_s} iG^{s,\infty}(\vec{p},\vec{s}) i\mathcal{M}_2^s(\vec{s}) iG^{s,\infty}(\vec{s},\vec{k})$$

Reproducing MRR: step 3

$$i\mathcal{D}_L^{s,(u,u)}(\vec{p},\vec{k}) \approx i\mathcal{D}^{s,(u,u)}(\vec{p},\vec{k}) + \int_{r,t} \frac{1}{2\omega_r}\frac{1}{2\omega_t} i\mathcal{D}^{s,(u,u)}(\vec{p},\vec{r}) iI(\vec{r},\vec{t}) i\mathcal{D}^{s,(u,u)}(\vec{t},\vec{k}) + \ldots$$

- Infinite-volume bound state ⇒ pole in $\mathcal{D}^{(u,u)}$

$$i\mathcal{D}^{(u,u)}(\vec{p},\vec{k}) = -i\frac{\Gamma(\vec{p})\bar{\Gamma}(\vec{k})}{E^2 - E_B^2} + \text{non-pole}$$

- Insert into $\mathcal{D}^{(u,u)}{}_L$, and resum:

$$\mathcal{D}_L^{s,(u,u)}(\vec{p},\vec{k}) \approx -\frac{\Gamma^{(u)}(\vec{p})}{E^2 - E_B^2}\left[1 + \frac{J}{E^2 - E_B^2} + \frac{J^2}{(E^2 - E_B^2)^2} + \ldots\right] \overline{\Gamma}^{(u)}(\vec{k})$$

$$J = \int_{r,t} \frac{\overline{\Gamma}^{(u)}(\vec{r})}{2\omega_r} I(\vec{r},\vec{t}) \frac{\Gamma^{(u)}(\vec{t})}{2\omega_t}$$

- Leads to pole in $\mathcal{D}^{(u,u)}{}_L$ at shifted energy

$$\Delta E_L = J/(2E_B)$$

Reproducing MRR: step 4

$$J = -\left[\frac{1}{L^3}\sum_s - \int_s\right]\left(\int_r \frac{\overline{\Gamma}^{(u)}(\vec{r})\; G^{s,\infty}(\vec{r},\vec{s})}{2\omega_r}\right)\left(\frac{\mathcal{M}_2^s(\vec{s})}{2\omega_s}\right)\left(\int_t \frac{G^{s,\infty}(\vec{s},\vec{t})\;\Gamma^{(u)}(\vec{t})}{2\omega_t}\right)$$

- Can simplify using analog of Bethe-Salpeter equation

$$i\mathcal{D}^{(u,u)}(\vec{p},\vec{k}) = i\mathcal{M}_2(\vec{p})iG^\infty(\vec{p},\vec{k})i\mathcal{M}_2(\vec{k}) + \int_s \frac{1}{2\omega_s} i\mathcal{M}_2(\vec{p})iG^\infty(\vec{p},\vec{s})i\mathcal{D}^{(u,u)}(\vec{s},\vec{k})$$

$$\Rightarrow \qquad \Gamma^{(u)}(\vec{p}) \;= -\mathcal{M}_2(\vec{p})\int_s \frac{1}{2\omega_s} G^\infty(\vec{p},\vec{s})\;\Gamma^{(u)}(\vec{s})$$

$$\Rightarrow \qquad \boxed{J = -\left[\frac{1}{L^3}\sum_s - \int_s\right]\bar{\Gamma}^{(u)}(s)\frac{1}{2\omega_s\mathcal{M}_2^s(s)}\Gamma^{(u)}(s) \;= 2E_B\Delta E_L}$$

General result valid for any bound state if $\mathcal{K}_{\mathrm{df},3}$=0

Reproducing MRR: step 5

$$J = -\left[\frac{1}{L^3}\sum_s - \int_s\right] \bar{\Gamma}^{(u)}(s) \frac{1}{2\omega_s \mathcal{M}_2^s(s)} \Gamma^{(u)}(s) \;\; = 2E_B \Delta E_L$$

- To evaluate need to know the form of Γ for Efimov bound state
- Can show (for instantaneous V_{ij}, by extending [Feldman & Fulton, 82])

$$\Gamma = 4\sqrt{3}m^2 \left(E_B - 3m - \sum_{i=1}^{3} \frac{\vec{p}_i^{\,2}}{2m} \right) \psi \Big|_{\text{on shell}}$$

Schrödinger wavefunction

$$\boxed{\Gamma^{(u)} = 4\sqrt{3}m^2 \left(E_B - 3m - \sum_{i=1}^{3} \frac{\vec{p}_i^{\,2}}{2m} \right) \phi_3 \Big|_{\text{on shell}}}$$

$$\psi = \phi_1 + \phi_2 + \phi_3$$

Fadeev wavefunction

Reproducing MRR: step 6

$$\boxed{\Gamma^{(u)} = 4\sqrt{3}m^2 \left(E_B - 3m - \sum_{i=1}^{3} \frac{\vec{p}_i^{\,2}}{2m} \right) \phi_3 \Big|_{\text{on shell}}}$$

- Using standard result for φ3 for Efimov bound state find

known constant

$s_{\rm th} = 2\kappa/\sqrt{3}$

$$\Gamma^{(u)}(\vec{s}) = \frac{c_\Gamma}{\sqrt{1+\vec{s}^{\,2}/s_{\rm th}^2}} \left[1 + \mathcal{O}(1+\vec{s}^{\,2}/s_{\rm th}^2) \right]$$

- Also need $\mathcal{M}_2$ in unitary limit

$$\frac{1}{\mathcal{M}_2^s(s)} = \frac{\kappa}{32\pi m} \left[\sqrt{1+\vec{s}^{\,2}/s_{\rm th}^2} - \cancelto{0}{1/(\kappa a)} \right]$$

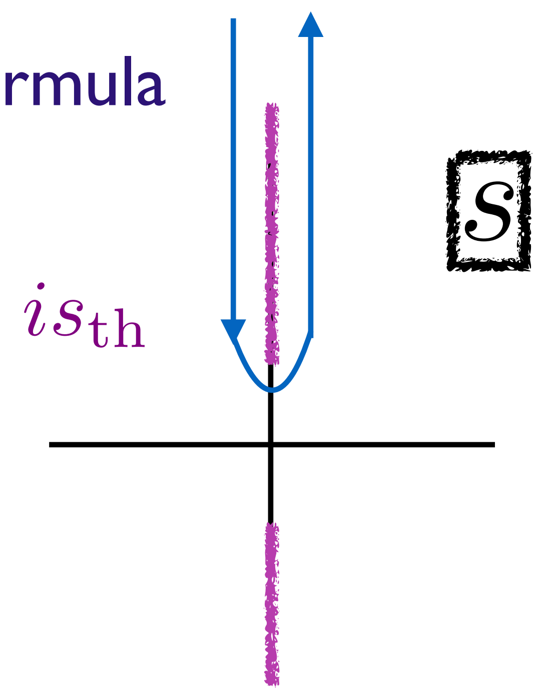
Reproducing MRR: step 7

$$J = -\left[\frac{1}{L^3}\sum_s - \int_s\right] \bar{\Gamma}^{(u)}(s) \frac{1}{2\omega_s \mathcal{M}_2^s(s)} \Gamma^{(u)}(s) \;\; = 2E_B \Delta E_L$$

- Evaluate sum-integral difference using Poisson summation formula

$$J = -c_\Gamma^2 \frac{\kappa}{64\pi m^2} 6 \int \frac{d^3 s}{(2\pi)^3} e^{iL\hat{x}\cdot\vec{s}} \frac{1}{\sqrt{1+s^2/s_{\rm th}^2}}$$

$$= -c_\Gamma^2 \frac{\kappa}{64\pi m^2} 6 e^{-s_{\rm th} L} \left(\frac{s_{\rm th}}{2\pi L}\right)^{3/2} [1 + \mathcal{O}(1/[\kappa L])]$$



- Final result agrees with NRQM!

$$\Delta E_L \;\; = -A^2(\kappa^2/m) \frac{e^{-s_{\rm th} L}}{(\kappa L)^{3/2}} \times 36 \cdot 3^{3/4} \pi^{7/2} C_0 \sinh^2(\pi s_0/2) \; [1 + \mathcal{O}(1/[\kappa L])]$$

Completing the check

- $\mathcal{K}_{df,3}$=0 approximation cannot exactly reproduce NRQM, since introduces dependence on cutoff function
- We think it is straightforward to relax this approximation while maintaining the final result