Non-perturbative matching of HQET heavy-light axial and vector currents in $N_f = 2$ lattice QCD

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Motivation: Determination of $V_{ub}$

The Cabibbo-Kobayashi-Maskawa matrix $V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$ can be determined from various $b \rightarrow u$ processes:

- Inclusive semi-leptonic (SL) $B \rightarrow X_u \ell \bar{\nu}$ decay
- Exclusive semi-leptonic (SL) $B \rightarrow \pi \ell \bar{\nu}$ decay
  - from lattice QCD: hadronic form factor $f_+(q^2)$
- Leptonic $B \rightarrow \tau \bar{\nu}$ decay
  - from lattice QCD: hadronic decay constant $f_B$

$|V_{ub}| \sim 3\sigma$ tension
$\Rightarrow V_{ub}$ puzzle

- New physics?
- Reliable lattice input needed!

$\begin{array}{c|c|c|c|c|c}
|V_{ub}| & B \rightarrow \pi \ell \bar{\nu} & B \rightarrow \tau \bar{\nu} & B \rightarrow X_u \ell \bar{\nu} \\
0.003 & 0.0035 & 0.004 & 0.0045 & 0.005 & 0.0055 \\
[PDG] & & & & & \\
\end{array}$
$V_{ub}$ via $B \to \pi \ell \bar{\nu}$ in the Standard Model

Experimental and theoretical ($=\text{lattice QCD}$) ingredients:

- $f_+(q^2)$ required to determine $|V_{ub}|$

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2|V_{ub}|^2}{192\pi^3m_B^3} (\lambda(q^2))^{3/2} |f_+(q^2)|^2$$

$$ q = p_B - p_\pi \quad \lambda(q^2) = (q^2 - m_B^2 - m_\pi^2)^2 - 4m_B^2m_\pi^2 $$

- $f_+(q^2)$ can be determined from the semi-leptonic $B \to \pi$ matrix element $\langle \pi(p_\pi)|V^\mu|B(p_B)\rangle$ through

$$\langle \pi(p_\pi)|V^\mu|B(p_B)\rangle = f_+(q^2) \left[ p_B + p_\pi - \frac{m_B^2 - M_\pi^2}{q^2} q^\mu \right]$$

$$ + f_0(q^2) \frac{m_B^2 - M_\pi^2}{q^2} q^\mu $$

- This is what is finally to be computed on the lattice ...
Non-perturbative matching of HQET heavy-light axial and vector currents in $N_f = 2$ lattice QCD

... but as first goal: Form factors in $B_S \to K\ell\nu$ decays

- No experimental data for SL $B_S \to K$ yet $\Rightarrow$ Predictions
- Easier on the lattice, as $m_K = m_K^{\text{phys}}$ (valence) computationally less expensive than with $\pi$, but not far from SL $B \to \pi$ though
- "Just" replace $B$ by $B_S$ and $\pi$ by $K$ in previous formulae

NP Renormalization & Continuum limit taken for the 1st time!
... but as first goal: Form factors in \( B_s \rightarrow K\ell\nu \) decays

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Leading (\( = \text{static} \)) order HQET computation in \( N_f = 2 \) lattice QCD


- NP Renormalization & Continuum limit taken for the 1st time!
- \( f_+(21.22 \text{ GeV}^2) = 1.63(8)(6) \pm 0.24 \Rightarrow \text{Erase by NP HQET @ NLO} \)
Non-perturbative (NP) HQET at $O(1/m_h)$

- Effective theory of QCD for systems with one heavy quark
- Action and operators are expanded in an asymptotic power series of $1/m_h$

Action: $\mathcal{L}^{\text{HQET}} = \frac{\bar{\psi}_h D_0 \psi_h}{\mathcal{L}_{\text{stat}} \sim O(1)} + \frac{\omega_{\text{kin}} \mathcal{O}_{\text{kin}} - \omega_{\text{spin}} \mathcal{O}_{\text{spin}} + \ldots}{\mathcal{L}^{(1)} \sim O(1/m_h)}$

$\mathcal{O}_{\text{kin}} = \bar{\psi}_h D^2 \psi_h$  \hspace{1cm} $\mathcal{O}_{\text{spin}} = \bar{\psi}_h \sigma \cdot \mathbf{B} \psi_h$

Operators: $\mathcal{O}_R^{\text{HQET}} = Z_{O}^{\text{HQET}} \left[ \mathcal{O}^{\text{stat}} + \sum_i c_{\mathcal{O}_i} \mathcal{O}_i \right]$
Non-perturbative (NP) HQET at $O\left(1/m_h\right)$

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Action:

$$\mathcal{L}^{\text{HQET}} = \bar{\psi}_h D_0 \psi_h - \omega_{\text{kin}} \mathcal{O}_{\text{kin}} - \omega_{\text{spin}} \mathcal{O}_{\text{spin}} + \ldots$$

$$\mathcal{L}_{\text{stat}} \sim O(1)$$

$$\mathcal{L}^{(1)} \sim O\left(1/m_h\right)$$

 Operators:

$$\mathcal{O}^{\text{HQET}}_R = Z^{\text{HQET}}_{\mathcal{O}} \left[ \mathcal{O}^{\text{stat}} + \sum_i c_{\mathcal{O}_i} \mathcal{O}_i \right]$$

- Parameters: $(m_{\text{bare}}, \omega_{\text{kin}}, \omega_{\text{spin}}, Z^{\text{HQET}}_{\mathcal{O}}, c_{\mathcal{O}_1}, \ldots)$
- All $\mathcal{O}_i$ with the same quantum numbers and correct dimension must be taken into account
- $1/m_h$ – terms $\triangleq$ local operator insertions in CFs (via expanding the functional integral weight in $1/m_h$, directly on the lattice)

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}[\varphi] e^{-S_{\text{rel}} - a^4 \sum_x \mathcal{L}_{\text{stat}}(x)} \mathcal{O} \left\{ 1 - a^4 \sum_x \mathcal{L}^{(1)}(x) + \ldots \right\} \Rightarrow \text{renormalizable}$$
Non-perturbative matching between HQET and QCD


A finite-volume, recursive strategy:

Matching volume: \( L_1 \approx 0.5 \text{ fm} \rightarrow am_h \ll 1 \), relativistic b-quark feasible
Operators considered

Previously: $A_0$ (= time component of heavy-light axial current)

- Its (B-to-vacuum) matrix element enters the computation of the B-meson decay constant $f_B$

$$A_{0,R}^{\text{HQET}} = Z_{A_0}^{\text{HQET}} \left[ A_0^{\text{stat}} + \sum_{i=1}^{2} c_{A_0,i} A_{0,i} \right]$$

$$A_{0,1} = \bar{\psi}_l \gamma_5 \gamma_i \frac{1}{2} \left( \nabla_i - \stackrel{\leftarrow}{\nabla}_i \right) \psi_h, \quad A_{0,2} = \bar{\psi}_l \gamma_5 \gamma_i \frac{1}{2} \left( \nabla_i + \stackrel{\leftarrow}{\nabla}_i \right) \psi_h$$

- $A_{0,2}$ vanishes due to the sum over $x$, if BCs are periodic
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- 5 HQET parameters left: $(m_{bare}, \omega_{kin}, \omega_{spin}, Z_{A_0}^{HQET}, c_{A_0,1})$

→ published $N_f = 2$ results by LPHA Collaboration:

- 5 HQET parameters of $L^{HQET}, A_{0,R}^{HQET}$

- Application #1: $b$-quark mass

- Application #2: $f_B, f_{B_s}$

[Blossier et al., JHEP 09 (2012) 132]

[Bernadoni et al., PLB 730 (2014) 171]

[Bernadoni et al., PLB 735 (2014) 349]
Operators considered

Now: \( A_0, A_k, V_0, V_k \) (\( \equiv \) all axial & vector current components)

- Application in mind: Computation of form factor \( f_+ \) incl. \( O(\frac{1}{m_h}) \)
- 14 new parameters appear, e.g. in the vector channel from

\[
V_{k,R}^{\text{HQET}} = Z_{V_k}^{\text{HQET}} \left[ V_k^{\text{stat}} + \sum_{i=1}^{4} c_{V_k,i} V_k,i \right]
\]

\[
\begin{align*}
V_{k,1} &= \bar{\psi}_l \gamma_k \gamma_i \frac{1}{2} \left( \nabla_i - \overleftrightarrow{\nabla}_i \right) \psi_h , \\
V_{k,2} &= \bar{\psi}_l \frac{1}{2} \left( \nabla_k - \overleftrightarrow{\nabla}_k \right) \psi_h \\
V_{k,3} &= \bar{\psi}_l \gamma_k \gamma_i \frac{1}{2} \left( \nabla_i + \overleftrightarrow{\nabla}_i \right) \psi_h , \\
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\end{align*}
\]
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$$V_{k,3} = \bar{\psi}_l \gamma_k \gamma_i \frac{1}{2} \left( \nabla_i + \nabla_i \right) \psi_h, \ V_{k,4} = \bar{\psi}_l \frac{1}{2} \left( \nabla_k + \nabla_k \right) \psi_h$$

- All in all, the complete set of 19 HQET parameters is:

$$\left( m_{\text{bare}}, \omega_{\text{kin}}, \omega_{\text{spin}}, c_{A_0,1}, c_{A_0,2}, Z_{A_0}^{\text{HQET}}, c_{A_k,1}, c_{A_k,2}, c_{A_k,3}, c_{A_k,4}, Z_{A_k}^{\text{HQET}}, c_{V_0,1}, c_{V_0,2}, Z_{V_0}^{\text{HQET}}, c_{V_k,1}, c_{V_k,2}, c_{V_k,3}, c_{V_k,4}, Z_{V_k}^{\text{HQET}} \right)$$
Reminder: NP finite-volume matching strategy

Why non-perturbative?

- Operator mixing in the lattice effective theory induces power divergences
  \[(am_h)^{-n} : \frac{g_0^{2l}}{a^n} \sim \frac{1}{\ln^l(a\Lambda_{QCD}) a^n} , \ n = 1, 2\]
  that must be subtracted \textit{NP’ly} to have a continuum limit
  (otherwise, truncated terms in the perturbative series would spoil it ...)

- Power \((1/m_h)\) corrections are only defined, when the leading term is computed non-perturbatively
  \[(\alpha(m_h))^{l} \sim \left\{ \frac{1}{2b_0 \ln(m_h/\Lambda_{QCD})} \right\}^{l} \xrightarrow{m_h \rightarrow \infty} \frac{\Lambda_{QCD}}{m_h}\]
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\(\Rightarrow\) \textit{NP HQET} takes care of this, no predictions are lost \(\Rightarrow\) Idea:

Equate (in small volume) QCD " = " HQET in the sense of

\[ \Phi_i^{\text{QCD}}(L, m_h, 0) \overset{l}{=} \Phi_i^{\text{HQET}}(L, m_h, a) \quad m_h = M: \text{ RGI mass} \]
Determine parameters via *matching* HQET & QCD by NP’ly imposing

\[
\Phi_i^{\text{QCD}}(L, m_h, 0) \overset{!}{=} \Phi_i^{\text{HQET}}(L, m_h, a) \quad m_h = M: \text{ RGI mass}
\]

\[
\equiv \eta_i(L, a) + \varphi_i^j(L, a) \omega_j(M, a) + O\left(\frac{1}{m_h^2}\right)
\]

**Matching conditions** for the complete set of HQET parameters in Lagrangian & all heavy-light flavour currents:

- Choose \(i = 1, \ldots, n_{\text{par}} = 19\) suitable observables \(\Phi_i\) that are sensitive to the HQET parameters and possess a linear HQET expansion
- The matching equations above thus consist of ...
  - ... renormalized QCD quantities \(\Phi_i^{\text{QCD}}\)
  - ... bare HQET correlators \(\varphi_i^j\)
  - ... static-order (parameter-free) HQET terms \(\eta_i\)
  - ... and the *HQET parameters* \(\omega_i\)
In more convenient matrix-vector notation:


- Collect all $i = 1, \ldots, n_{\text{par}} = 19$ parameters in a column vector

$$\omega = \begin{pmatrix} m_{\text{bare}}, \omega_{\text{kin}}, \omega_{\text{spin}}, \\
C_{A,0,1}, C_{A,0,2}, Z_{A}^{\text{HQET}}, C_{A,k,1}, C_{A,k,2}, C_{A,k,3}, C_{A,k,4}, Z_{A_k}^{\text{HQET}}, \\
C_{V,0,1}, C_{V,0,2}, Z_{V}^{\text{HQET}}, C_{V,k,1}, C_{V,k,2}, C_{V,k,3}, C_{V,k,4}, Z_{V_k}^{\text{HQET}} \end{pmatrix}^T$$
In more convenient matrix-vector notation:


1. Collect all $i = 1, \ldots, n_{\text{par}} = 19$ parameters in a column vector

$$\omega = (m_{\text{bare}}, \omega_{\text{kin}}, \omega_{\text{spin}},$$

$$c_{A_0,1}, c_{A_0,2}, Z_{A_0}^{\text{HQET}}, c_{A_k,1}, c_{A_k,2}, c_{A_k,3}, c_{A_k,4}, Z_{A_k}^{\text{HQET}},$$

$$c_{V_0,1}, c_{V_0,2}, Z_{V_0}^{\text{HQET}}, c_{V_k,1}, c_{V_k,2}, c_{V_k,3}, c_{V_k,4}, Z_{V_k}^{\text{HQET}})^T$$

2. Column vector of $n_{\text{par}}$ observables $\Phi = (\Phi_1, \ldots, \Phi_{n_{\text{par}}})$, such that its linear HQET expansion implies a matrix of structure

$$\varphi = \begin{pmatrix}
\varphi_1^1 & * & * & 0 & 0 \\
0 & * & 0 & 0 & 0 \\
0 & * & * & 0 & 0 \\
0 & * & 0 & * & 0 \\
0 & * & 0 & 0 & *
\end{pmatrix}$$

for bare HQET correlation functions


$$\Phi_{\text{QCD}}^{L, M, 0} = \Phi_{\text{HQET}}^{L, M, a} = \eta(L, a) + \varphi(L, a) \cdot \omega(M, a)$$
Complete list of HQET parameters and their meaning & origin:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\omega_i$</th>
<th>origin</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2, 3</td>
<td>$m_{\text{bare}}, \omega_{\text{kin}}, \omega_{\text{kin}}$</td>
<td>$\mathcal{L}^{\text{HQET}}$</td>
</tr>
<tr>
<td>4, ..., 6</td>
<td>$c_{A_{0,1}}, c_{A_{0,2}}, \ln Z_{A_0}^{\text{HQET}}$</td>
<td>$A_0^{\text{HQET}}$</td>
</tr>
<tr>
<td>7, ..., 11</td>
<td>$c_{A_{k,1}}, c_{A_{k,2}}, c_{A_{k,3}}, c_{A_{k,4}}, \ln Z_{A_{k}}^{\text{HQET}}$</td>
<td>$A_{k}^{\text{HQET}}$</td>
</tr>
<tr>
<td>12, ..., 14</td>
<td>$c_{V_{0,1}}, c_{V_{0,2}}, \ln Z_{V_0}^{\text{HQET}}$</td>
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</tr>
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</tr>
</tbody>
</table>

$\Rightarrow$ 19 observables for 19 free parameters are needed
How does the matching proceed in practice?

Starting point is the foregoing equation

$$\Phi^{\text{QCD}}(L, M, o) \overset{!}{=} \Phi^{\text{HQET}}(L, M, a) = \eta(L, a) + \varphi(L, a) \cdot \omega(M, a)$$
In small volume ($L_1 \approx 0.5$ fm):
Compute observables $\Phi^{QCD}(L_1, M, a)$ in relativistic QCD and extrapolate them to the continuum limit, i.e.

$$\Phi^{QCD}(L_1, M, 0) = \lim_{a \to 0} \Phi^{QCD}(L_1, M, a)$$

CL $a \to 0$ can be taken in QCD (l.h.s.) due to small volume!
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Match with HQET ($a \lesssim 0.05$ fm) to solve for parameters in $L_1$:

$$
\Phi^{\text{HQET}}(L_1, M, a) = \eta(L_1, a) + \varphi(L_1, a) \cdot \tilde{\omega}(M, a) \\
\Rightarrow \quad \tilde{\omega}(M, a) = \varphi^{-1} \left[ \Phi^{\text{QCD}}(L_1, M, 0) - \eta \right]
$$
Non-perturbative matching of HQET heavy-light axial and vector currents in $N_f = 2$ lattice QCD

- Step scaling $L_1 \rightarrow L_2 = 2L_1 \approx 1$ fm:

  Employ determined parameters $\tilde{\omega}(M, a)$ in the computation of HQET observables $\Phi^{HQET}(L_2, M, a)$ in larger volume and extrapolate them to the continuum, too.
Extract the parameters $\omega(M, a)$ for larger $a$ (i.e., those typically encountered in large-volume simulations) by solving the associated matrix-vector equation in $L_2$:

$$\omega(M, a) = \varphi(L_2, a)^{-1} \left[ \Phi_{HQET}^{L_2}(L_2, M, 0) - \eta(L_2, a) \right]$$
Once determined in this way, the $n_{\text{par}}$ HQET parameters $\omega(M, a) = \{\omega_i(M, a)\}$, which NP’ly absorb the log. & power divergences of the effective theory and inherit the NP QCD quark mass dependence, can finally be used to calculate the desired large-volume HQET observables at $1/M$ for $M = M_b$. 
Schrödinger Functional (SF)

- Observables defined in the QCD Schrödinger functional setup
- Finite volume $T \times L^3$
- Dirichlet BCs in time, periodic BCs in space
- Additional periodicity phase angle vector $\theta$ for fermion fields:
  $$\psi(x_0, x + n \cdot L) = e^{i \theta \cdot n} \cdot \psi(x)$$
- Boundary fields $\zeta$ used to build CFs
- Well known renormalization properties
  - Bare parameters $(L/a, \beta, \kappa_1, \kappa_h)$ to define lines of constant physics in matching volume $L_1^4$, keeping $\bar{g}_{SF}^2$ and (RGI) light & heavy quark masses fixed
Non-perturbative matching of HQET heavy-light axial and vector currents in $N_f = 2$ lattice QCD

SF correlation functions

Example (#1, boundary–bulk CFs)

\[ f_{A_k}(x_0, \theta_l, \theta_h) = i \frac{a^6}{6L^3} \sum_{x,y,z,k} \langle \bar{\psi}_l(x) \gamma_k \gamma_5 \psi_h(x) \times A_k(x) \times \bar{\zeta}_h(y) \gamma_5 \zeta_l(z) \rangle \]

\[ k_{V_k}(x_0, \theta_l, \theta_h) = -\frac{a^6}{6L^3} \sum_{x,y,z,k} \langle \bar{\psi}_l(x) \gamma_k \psi_h(x) \times V_k(x) \times \bar{\zeta}_h(y) \gamma_k \zeta_l(z) \rangle \]

Example (#2, boundary–boundary CF)

\[ F_1(\theta_l, \theta_h) = -\frac{a^{12}}{2L^6} \sum_{u,v,y,z} \langle \bar{\zeta}'_l(u) \gamma_5 \zeta'_h(v) \bar{\zeta}_h(y) \gamma_5 \zeta_l(z) \rangle \]
Three-point SF correlation functions

- 3–point correlators for alternative matching conditions
- Perturbative studies (at tree-level & 1–loop) favor observables constructed from 3–point CFs: yield flatter $1/m_h$–dependence
  
  [Hesse & Sommer, JHEP 02 (2013) 115; Della Morte et al., JHEP 05 (2014) 060; Korcyl]

- To be checked non-perturbatively: in progress ...

\[
J_{A_1}^1(x_0, \theta_L, \theta_H) = -\frac{a^{15}}{2L^6} \sum_{uvyzx} \langle \bar{\zeta}'_l(u) \gamma_1 \zeta'_l(v) \times \\
\times \bar{\psi}_l(x) \gamma_1 \gamma_5 \psi_h(x) \bar{\zeta}_h(z) \gamma_5 \zeta'_l(y) \rangle
\]

Similarly in the vector meson channel:

\[
F_{V_0}(x_0, \theta_L, \theta_H) = -\frac{a^{15}}{2L^6} \sum_{uvyzx} \langle \bar{\zeta}'_l(u) \gamma_5 \zeta'_l(v) \times \\
\times \bar{\psi}_l(x) \gamma_0 \psi_h(x) \bar{\zeta}_h(z) \gamma_5 \zeta'_l(y) \rangle
\]
Choice of observables

- The observables in $\Phi$ shall ...
  - ... be sensitive to the parameters $\omega$
  - ... entail small $1/m_h$–terms to expect $O(1/m_h^2)$ to be negligible

- $\theta_l, \theta_h$: Most important free (kinematical) parameters
  $\rightarrow$ Great flexibility in choice of obs. / matching conditions

- 1 "natural" set proposed and its feasibility first demonstrated at tree-level [Della Morte, Dooling, H., Hesse & Simma, JHEP 05 (2014) 060]

- 4 additional sets of observables [Korcyl & Simma] investigated, composed to minimize (cont.) tree-level $1/m_h^2$–corrections
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Recall matrix $\varphi$:

$$
\varphi = \begin{pmatrix}
    m_{\text{bare}} & \omega_{\text{kin}}, \omega_{\text{spin}} & c_{A_0,1}, c_{A_0,2}, Z_{A_0}^{\text{HQET}} & c_{A_k,1}, \ldots & \vdots \\
    * & * & * & 0 & \vdots \\
    0 & * & 0 & 0 & \vdots \\
    0 & * & * & 0 & \vdots \\
    0 & * & 0 & * & \vdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix}
$$
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Elements of $\varphi \doteq $ Static correlators with (local) $1/m_h$–insertions:

$$\langle O \rangle = \langle O \rangle_{\text{stat}} + \omega_{\text{kin}} a^4 \sum_x \langle OO_{\text{kin}}(x) \rangle_{\text{stat}} + \omega_{\text{spin}} a^4 \sum_x \langle OO_{\text{spin}}(x) \rangle_{\text{stat}}$$

$$\equiv \langle O \rangle_{\text{stat}} + \omega_{\text{kin}} \langle O \rangle_{\text{kin}} + \omega_{\text{spin}} \langle O \rangle_{\text{spin}}$$

$$\langle O \rangle_{\text{stat}} = \frac{1}{Z} \int_{\text{fields}} O \exp \left\{ - a^4 \sum_x \left[ \mathcal{L}_{\text{light}}(x) + \mathcal{L}_{\text{h}}^{\text{stat}}(x) \right] \right\}$$
Example (#1, for matching the axial current $A_k$)

- $\Phi_7$ is sensitive to $c_{A_{k,1}}$ and $c_{A_{k,2}}$:

$$\Phi_{QCD}^7 = \ln \left( \frac{f_{A_k}(T/2, \theta_1, \theta_1)}{f_{A_k}(T/2, \theta_2, \theta_2)} \right)$$

$$\Phi_{HQET}^7 = \Phi_{stat}^7 \underbrace{+ \omega_{\text{kin}} \Phi_{kin}^7 + \omega_{\text{spin}} \Phi_{spin}^7 + c_{A_{k,1}} \Phi_{7,1} + c_{A_{k,2}} \Phi_{7,2}}_{\in \eta} \in \varphi \cdot \omega$$
Example (#1, for matching the axial current $A_k$)

- $\Phi_7$ is sensitive to $c_{A_{k,1}}$ and $c_{A_{k,2}}$:

$$\Phi_{QCD}^7 = \ln \left( \frac{f_{A_k}(T/2, \theta_1, \theta_1)}{f_{A_k}(T/2, \theta_2, \theta_2)} \right)$$

$$\Phi_{HQET}^7 = \Phi_{stat}^7 + \omega_{kin} \Phi_{kin}^7 + \omega_{spin} \Phi_{spin}^7 + c_{A_{k,1}} \Phi_{7,1} + c_{A_{k,2}} \Phi_{7,2}$$

Example (#2, for matching the vector current $V_k$)

- $\Phi_{15}$ is sensitive to $c_{V_{k,1}}$ and $c_{V_{k,2}}$:

$$\Phi_{QCD}^{15} = \ln \left( \frac{k_{V_k}(T/2, \theta_1, \theta_1)}{k_{V_k}(T/2, \theta_2, \theta_2)} \right)$$

$$\Phi_{HQET}^{15} = \Phi_{stat}^{15} + \omega_{kin} \Phi_{kin}^{15} + \omega_{spin} \Phi_{spin}^{15} + c_{V_{k,1}} \Phi_{15,1} + c_{V_{k,2}} \Phi_{15,2}$$
Example (#3, for matching the vector current $V_0$)

- $\Phi_{14}$ is sensitive to $c_{V_0,1}$ and $Z_{V_0}^{HQET}$:

$$\Phi_{14}^{QCD} = \ln \left( \frac{F_{V_0}(T/2, \theta_1, \theta_1)}{\sqrt{F_1(\theta_1, \theta_1) \times F_1^{ll}(\theta_1, \theta_1)}} \right)$$

$$\Phi_{14}^{HQET} = \Phi_{14}^{stat} + \omega_{kin} \Phi_{14}^{kin} + \omega_{spin} \Phi_{14}^{spin} + c_{V_0,1} \Phi_{14} + \ln Z_{V_0}^{HQET}$$

- Alternatively, fix $Z_{V_0}^{HQET}$ using 2–point functions only:

$$\Phi_{14}^{QCD} = \ln \left( \frac{k_{V_0}(T/2, \theta_1, \theta_1)}{\sqrt{K_1(\theta_1, \theta_1)}} \right)$$

$$\Phi_{14}^{HQET} = \Phi_{14}^{stat} + \omega_{kin} \Phi_{14}^{kin} + \omega_{spin} \Phi_{14}^{spin} + c_{V_0,1} \Phi_{14} + \ln Z_{V_0}^{HQET}$$
Choice of observables: Tree-level example

Figure: $c_{V_{k,1}}$ for different $\theta$’s vs. inverse mass [from: JHEP 05 (2014) 060]
Non-perturbative matching computation

Simulations (resp. measurements of correlators) re-use the available SF ensembles from earlier matching of HQET action & $A_0$

- $N_f = 2$, NP’ly $O(a)$ improved, $\theta = (0.5, 0.5, 0.5)$ for sea quarks

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Non-perturbative matching of HQET heavy-light axial and vector currents in $N_f = 2$ lattice QCD

Non-perturbative matching computation

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- Measurements at 7 different renormalized masses in $L = L_1$ ($z \equiv 1/(LM) \equiv 1/(LM_h) \in \{3.0, 4.0, 6.15, 12.75, 13.25, 13.75, 20.0\}$)
- Various combinations of $\theta$’s of light and heavy quarks (to support several sets of observables resp. matching strategies)

Jochen Heitger
Preliminary results from analysis of the NP data

- For now:
  - Choice "So" among the matching observables / "strategies"
  - Analysis with the others to follow ...

- QCD side:
  Continuum extrapolations in \((a/L_1)^2\) for all \(z\)–values

- HQET side:
  - Results for "HYP1/2" static actions available, here "HYP2" only
  - Continuum extrapolations \(linear in a^2\) for the static pieces, whereas \(linear in a\) for the \(O(1/m_h)\) ones
  - So far, only observables from 2–point functions included

- (Preliminary) Jackknife error analysis, "UWerr" still to be done
Non-perturbative matching of HQET heavy-light axial and vector currents in $N_f = 2$ lattice QCD

- Combined continuum extrapolation of QCD observable #1, needed for fixing $m_{\text{bare}}$ (strategy "So")

![Graph showing the combined continuum extrapolation of QCD observables](image-url)
Heavy quark mass dependence of QCD observable #1 in $L_1$, $z = 1/(L_1 M)$
Continuum extrapolation of HQET observable #1 in $L_2$, having plugged in the HQET parameters determined in $L_1$ \[\rightarrow \text{CL exists} \iff 1/a\text{–divergence correctly cancelled!}\]
Heavy quark mass dependence of HQET observable #1 in $L_2$, $z = 1/(L_1 M)$

→ Last missing step: Solve the full linear system in $L_2$ for $\omega(M, a)$ and interpolate them to $\beta$’s used in large-volume simulations
Combined continuum extrapolation of QCD observable #15, needed for fixing $c_{V,k,1}$ (strategy "So")
Heavy quark mass dependence of $QCD$ observable #15 in $L_1$, $z = 1/(L_1 M)$
continuum extrapolation of the static part of HQET observable #15 in $L_1$ (linearly in $(a/L_1)^2$; employing "HYP2" data only)
Continuum extrapolation of HQET observable #15 in $L_2$, having plugged in the HQET parameters determined in $L_1$
Heavy quark mass dependence of HQET observable #15 in $L_2$, $z = 1/(L_1 M)$

→ Last missing step: Solve the full linear system in $L_2$ for $\omega(M, a)$ and interpolate them to $\beta$’s used in large-volume simulations
Summary & Outlook

NP HQET at $O(1/m_h)$ works in practice → Status of B-physics:

- Determination of HQET parameters of action and $A_0$ for $N_f = 2$
- $N_f = 2$ Results for ...
  - ... b-quark mass & leptonic B-meson decay constants, ...
  - ... HQET form factor in $B_s \rightarrow K\ell\nu$ semi-leptonic decays: so far, leading-order (i.e., static) only, but *continuum limit*!
    (→ Aside: $V_{ub}$ puzzle remains)
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- NP matching of action & *all HQET heavy-light currents*
  - Strategy exists; tree-level & 1–loop investigations
  - Decision on kinematical parameters & Simulations
  - Final analysis to extract the HQET parameters in progress (large-volume simulations to obtain $1/m_h$–parts in $f_+(q^2)$ in parallel)

Jochen Heitger