The Slab Method to Measure the Topological Susceptibility

- Topological susceptibility
- Numerical measurement: direct, or in a fixed sector

  Slab method

- Results: 1d O(2) model, 2d O(3) model, 2-flavour QCD

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In a number of important models (with periodic boundary conditions), the configurations occur in top. sectors, labelled by the top. charge $Q \in \mathbb{Z}$.

Continuum (lattice) :
continuous deformations of a conf. can never (only painfully) alter Q.

Top. susceptibility

$$\chi_{\rm t} = \frac{1}{V}\Big(\langle Q^2 \rangle - \langle Q \rangle^2 \Big) \ , \quad \text{here}: \ \langle Q \rangle = 0 \quad (\text{P invariance})$$

Non-perturbative quantity $\Rightarrow$ issue for lattice simulations.

Phenomenological relevance:

Quantitative solution of the U(1) problem [Witten, Veneziano '79]

't Hooft: re-scale strong coupling as $g^2 = g_{\rm s}^2 N_{\rm c}$, for large $N_{\rm c}$, small $g_{\rm s}$

- $N_{\rm f}$ massless quark flavours

$$\chi_{\rm t}^{\rm quenched} \simeq \frac{F_\pi^2 M_{\eta'}^2}{2N_{\rm f}} \; ; \quad F_\pi^2 \propto N_{\rm c} \, , \; M_{\eta'}^2 \propto 1/N_{\rm c} \quad (\text{NGB at } N_{\rm c} \to \infty)$$

predicts $M_{\eta'} \propto 1/\sqrt{N_{\rm c}}$, no SSB for U(1) sym.

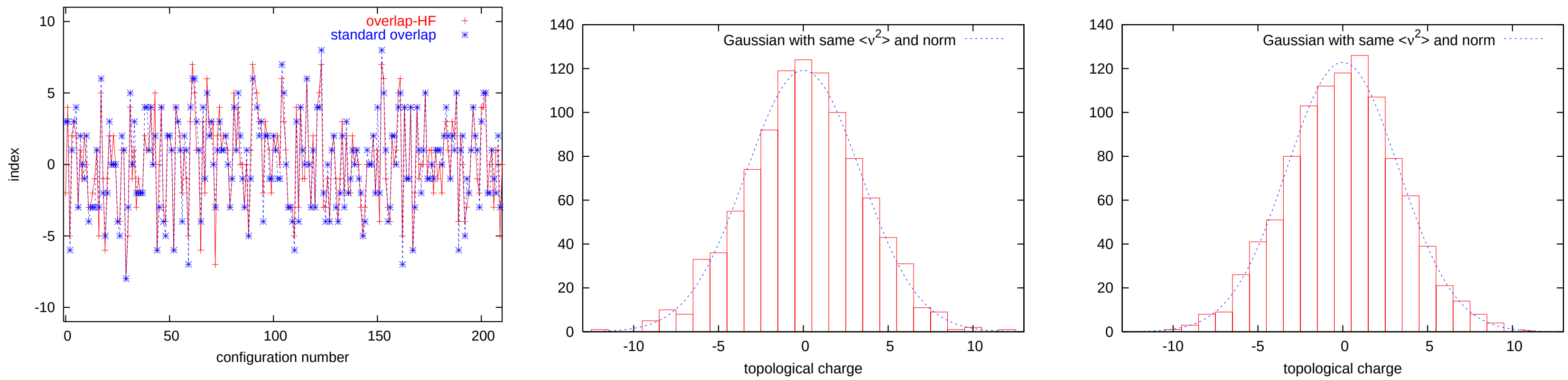
- For $m_u = m_d = 0$, $m_s > 0$:

$$\chi_{\rm t}^{\rm quenched} \simeq \frac{F_\pi^2}{6} \left(M_{\eta'}^2 + M_\eta^2 - 2M_K^2 \right)$$

[Recent analysis: Fukaya et al. '15]

Example for a direct measurement of $\langle Q^2 \rangle$ in quenched QCD [W.B./Shcheredin '06]

$Q :=$ index of $D_{\rm overlap}$ ($V = 12^3 \times 24$, $\beta = 5.85 \;\Rightarrow\; a \simeq 0.123\,{\rm fm}$)



History well de-correlated

histograms for standard overlap and overlap hypercube ≈ Gaussian

width ⇔ $\chi_{\rm t}$

(roughly) compatible with Witten-Veneziano formula at $M_{\eta'} = 958\,{\rm MeV}$

High statistics and cont. extrapolation: Del Debbio et al. '05, Dürr et al. '07

With dynamical quarks: $\chi_{\rm t} \Rightarrow$ axion mass (similar to Witten-Veneziano formula)

Cold Dark Matter candidate ? [e.g. Petreczky/Schadler/Sharma '16]

Fine lattices:
top. sectors separated by high potential walls ($\rightarrow \infty$ in continuum limit).

Markov chain for small update steps:
confined to one sector over a LONG computation time.

Extreme cases with light chiral quarks (also suppress top. transitions, and $|Q|$)
long HMC histories entirely at $Q = 0$. (ergodicity?)

[Fukaya *et al.*, '07, Aoki *et al.*, '08, Borsanyi *et al.*, '15]

Can we still measure χ_t ? Yes, we can, by indirect methods !

• Brower-Chandrasekharan-Negele-Wiese '03: formula for observable $\mathcal{O}$ measured at fixed Q (expansion in $1/(V\chi_{\rm t}) = 1/\langle Q^2\rangle$):

$$\langle \mathcal{O} \rangle|_Q \simeq \langle \mathcal{O} \rangle + \frac{\rm const.}{V\chi_{\rm t}}\Big(1 - \frac{Q^2}{V\chi_{\rm t}}\Big)$$

Requires $\langle \mathcal{O} \rangle|_Q$ in several $|Q|$ and $V \overset{\rm fit}{\Longrightarrow}$ $\langle \mathcal{O} \rangle, \ \chi_{\rm t}$ (if $\langle Q^2 \rangle > 1, \ |Q| \leq 1 \,{\rm or}\, 2)^*$

Works well for $\langle \mathcal{O} \rangle$, but large uncertainties for $\chi_{\rm t}$ [W.B. *et al.*, '16]

• Aoki-Fukaya-Hashimoto-Onogi '07: formula: exclusively for $\chi_{\rm t}$:

$$\langle q_0 \, q_x \rangle \big|_{|Q|, \, {\rm large} \, |x|} \simeq -\frac{\chi_{\rm t}}{V} + \frac{Q^2}{V^2} = -\frac{\chi_{\rm t}}{V}\Big(1 - \frac{Q^2}{V\chi_{\rm t}}\Big)$$

q_x: top. charge density, plateau value of correlation $\Rightarrow$ $\chi_{\rm t}$

Successful in a suitable regime*; large V $\to$ tiny signal, to be extracted by all-to-all correlations. [Tests in σ-models and 4d SU(2) Yang-Mills theory: Bautista *et al.*, '15]

Slab Method

idea first expressed by de Forcrand et al. '99

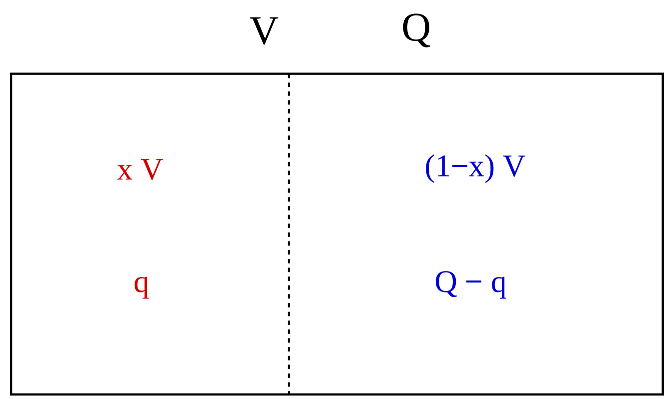
[similar for instanton liquids: Shuryak/Verbaarschot '95]

Assume Gaussian distribution of top. charges,

$$p(Q) \propto e^{-Q^2/(2\chi_t V)}$$

well confirmed, up to lattice artifacts (see below)

Split volume V into sub-volumes := **slabs** of sizes xV, $(1-x)V$ $\quad (0 < x < 1)$



Fixed total Q $\Rightarrow$ slab charges $\sum_x q_x$: $q,\ Q-q \in \mathbb{R}$ (face between slabs non-periodic)

Probability distribution (at fixed V, x, Q):

$$p_1(q)\,p_2(Q-q) \quad\propto\quad \exp\Big(-\frac{q^2}{2\chi_t V x}\Big)\;\cdot\;\exp\Big(-\frac{(Q-q)^2}{2\chi_t V(1-x)}\Big)$$

$$\propto\quad \exp\Big(-\frac{1}{2\chi_t V}\,\frac{q'^{\,2}}{x(1-x)}\Big)\,, \qquad q' := q - xQ$$

$$\langle q\rangle = xQ \quad\Rightarrow\quad \langle q'^{\,2}\rangle = \langle q^{\,2}\rangle - x^2Q^2$$

Measure $\langle q^{\,2}\rangle$, $\langle q'^{\,2}\rangle$ at various x, fit to parabola $\Rightarrow$ χ_t

Quantum rotor (1d XY model)

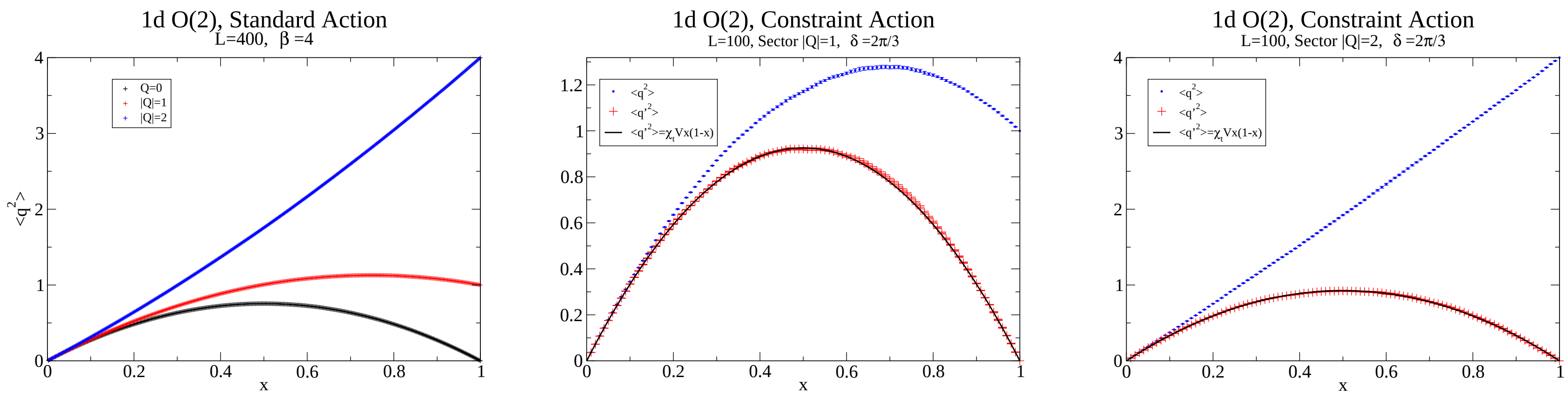
Periodic time lattice $t = 1\ldots L$, angular variable ϕ_t . Geometrical definition:

$$Q[\phi] = \frac{1}{2\pi}\sum_t \Delta\phi_t\,, \quad \Delta\phi_t = (\phi_{t+1}-\phi_t) \bmod 2\pi \in (-\pi,\pi]$$

Lattice actions:

$$S_{\text{standard}}[\phi] \quad = \quad \beta \sum_t (1 - \cos(\Delta\phi_t)) \; , \qquad S_{\text{Manton}}[\phi] = \frac{\beta}{2} \sum_t (\Delta\phi_t)^2$$

$$S_{\text{constraint}}[\phi] \quad = \quad \begin{cases} 0 & \Delta\phi_t < \delta \quad \forall t \\ +\infty & \text{otherwise} \end{cases} \qquad \text{Cont. limit} : \beta \to \infty, \; \delta \to 0$$

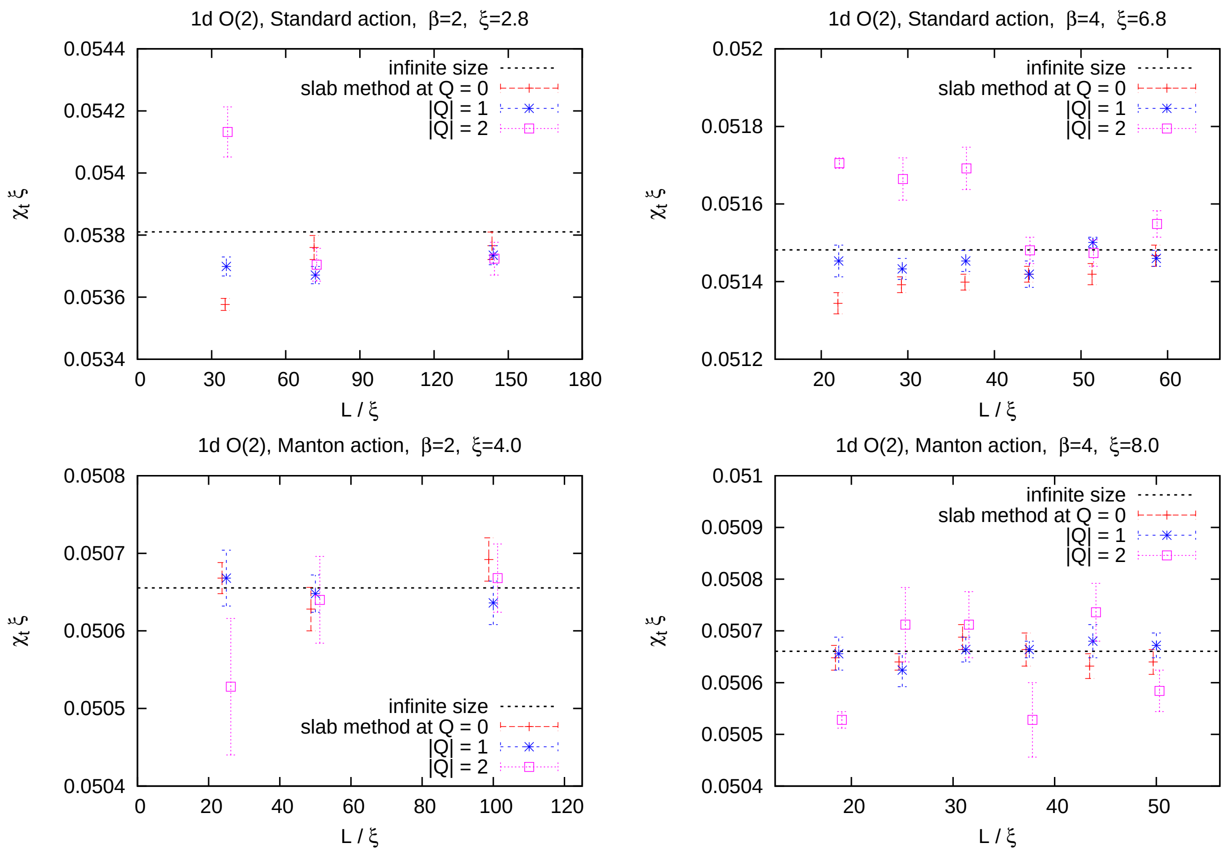


$\langle q^2 \rangle$ at $\mathbf{Q} = \mathbf{0}$ $|\mathbf{Q}| = \mathbf{1}$ $|\mathbf{Q}| = \mathbf{2}$ $\qquad |Q| = 1, \, 2 \; : \; \langle q^2 \rangle \;\; \langle {q'}^2 \rangle$ **fit**

$\langle q^2 \rangle(x)$: Parabola from $\langle q^2 \rangle(0) = 0$ to $\langle q^2 \rangle(1) = Q^2$

$\langle {q'}^2 \rangle(x)$: $L\chi_t \, x(1-x)$, vanishes at $x = 0$ and $x = 1$

Results for scaling quantity $\chi_t\, \xi$ (ξ: correlation length)

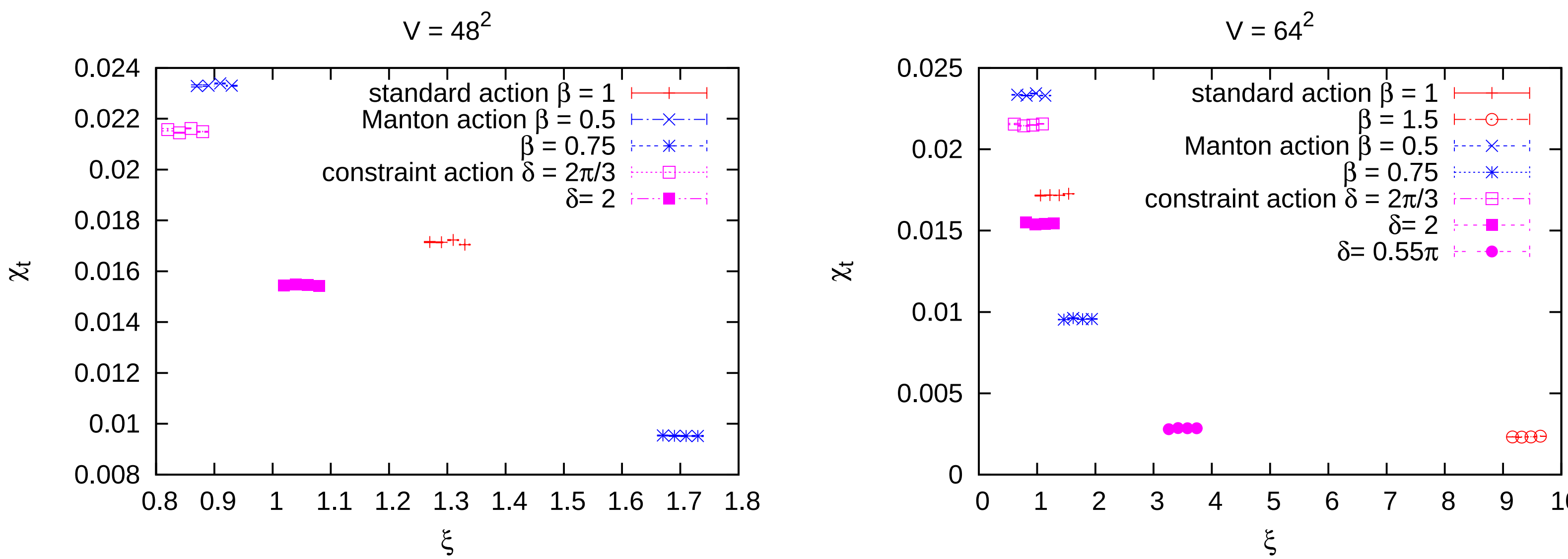


Thermodynamic limit ($L \to \infty$) known analytically for all three lattice actions; converges slowly, best for for small $|Q|$.

2d O(3) model (Heisenberg model)

$\chi_t\,\xi^2$ diverges logarithmically in cont. limit $\Rightarrow$ consider just χ_t at finite ξ (lattice units)

Again: geometric formulation of Q, analogous three lattice actions

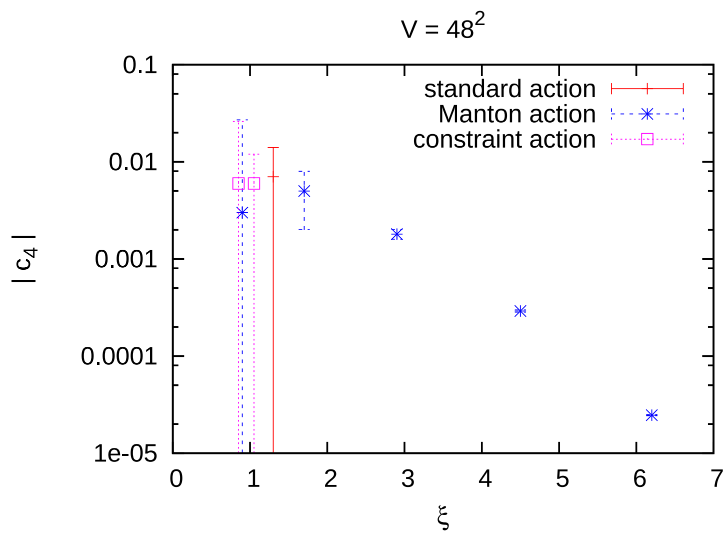


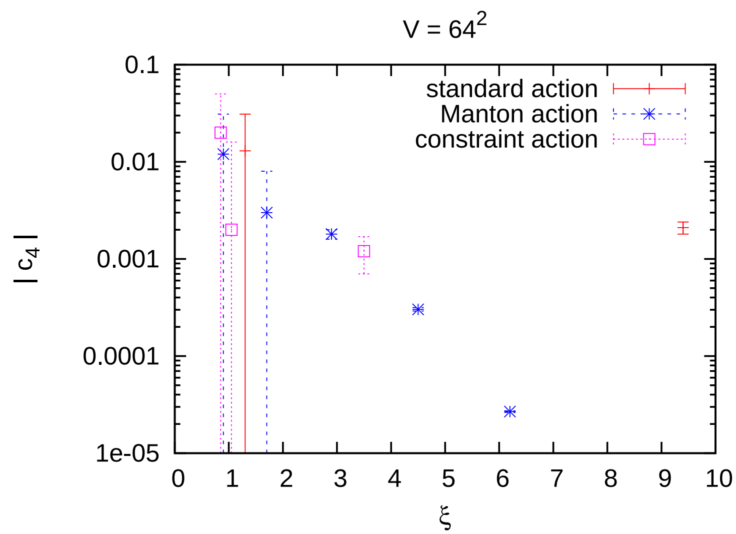
Each set of data points shows (from left to right): χ_t directly measured (feasible with cluster algorithm), and with slab method at $|Q| = 0,\ 1,\ 2$ [data in JHEP 12(2015)070]

Kurtosis

$$c_4 = \frac{1}{V}\Big(3\langle Q^2\rangle^2 - \langle Q^4\rangle\Big)$$

measures deviation from a Gauss distribution (Gaussian: $c_4 = 0$)





$c_4 \to 0$ in the cont. limit $\xi \to \infty$; hardly depends on V

Manton action: fastest convergence, exponential (1d: classically perfect; 2d: new)

2-flavour QCD

- Wilson gauge action, q_x from lattice field strength tensor

  after smoothing, $\sum_x q_x$ is slightly re-scaled and rounded to $Q \in \mathbb{Z}$

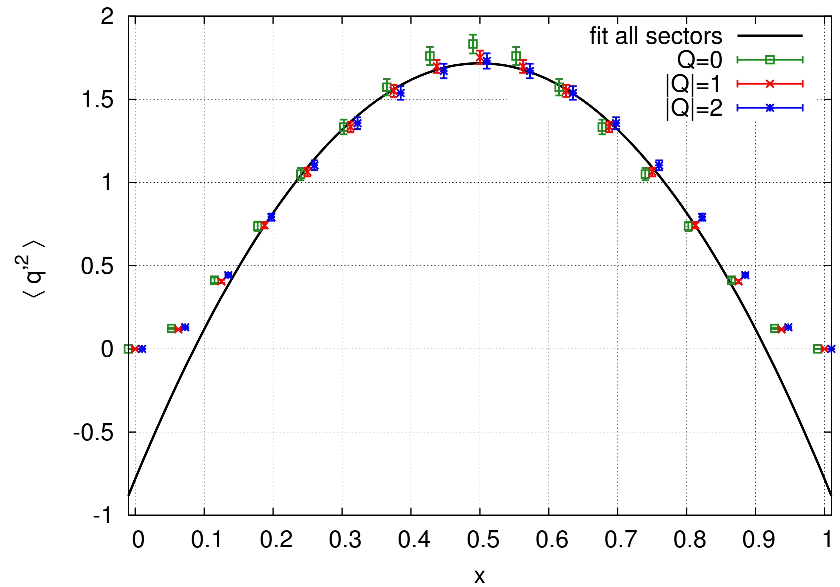
- twisted mass quarks (full twist, $\mu = 0.015$); $M_\pi \simeq 650\,\mathrm{MeV}$

- 10^5 confs, $V = 16^3 \times 32$, $\beta = 3.9 \;\Rightarrow\; a \simeq 0.079\,\mathrm{fm}$

  Gradient flow with Runge-Kutta integration in flow time t
  (results for $\epsilon = 0.01$ and 0.001 agree)

  Lüscher's reference scale $t_0^2 \, \langle E \rangle_{\rm clover} = 0.3$, here: $t_0 = 2.42$

  Slabs: $16^3 \times 32x$ and $16^3 \times 32(1-x)$



$\langle {q'}^2 \rangle(x)$ for $|Q| = 0,\ 1,\ 2$ at $t = 5t_0$

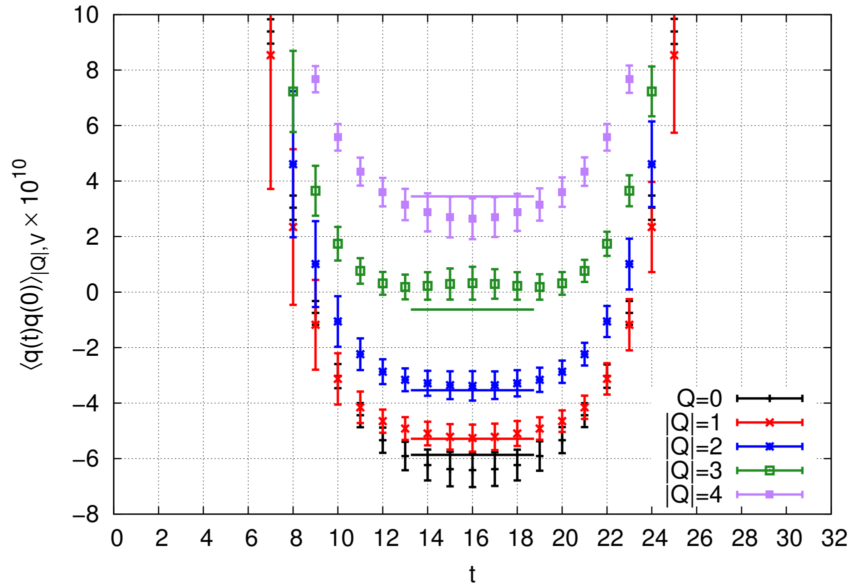
$x \gtrsim 0$ and $x \lesssim 1$: flat slabs involved, does not follow parabola;
at $x \gtrsim 0$: $|\text{deviation}| \propto e^{-c(t)x}$ thin slab effect

For $0.2 \lesssim x \lesssim 0.8$ matches well (joint) fit to:

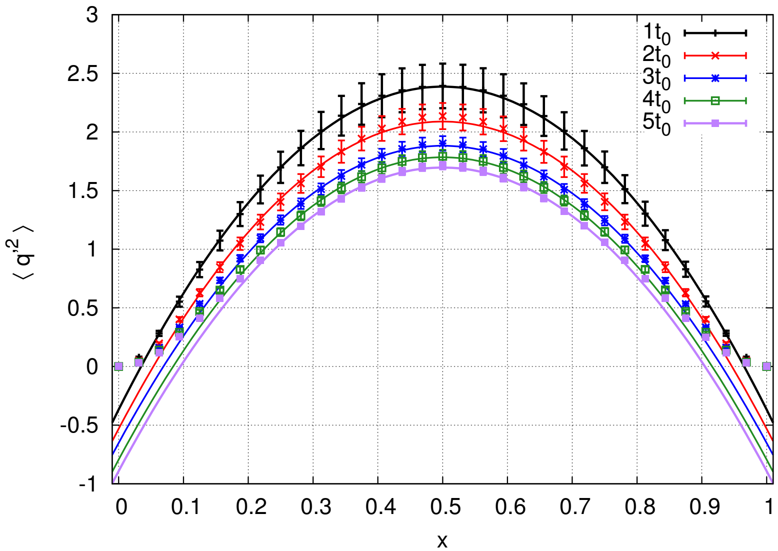
$$\langle {q'}^2 \rangle(x) = \chi_{\rm t} V x(1-x) + \text{const}$$

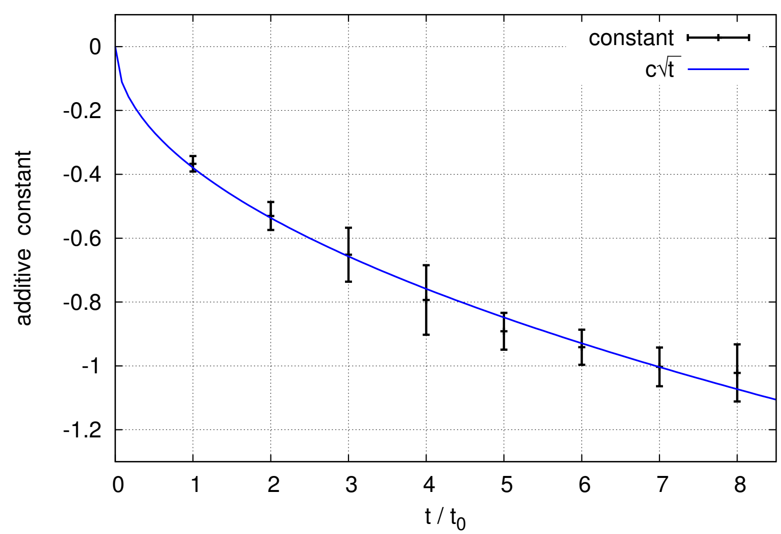
Result consistent with other methods:

$$\chi_t\, a^4 = \begin{cases} 7.76(20) \cdot 10^{-5} & \text{direct} \\ 7.63(14) \cdot 10^{-5} & \text{slab method for } |Q| \le 2 \\ 7.69(22) \cdot 10^{-5} & \text{AFHO method '' ''} \end{cases}$$



Data for $\langle q_0\, q_t \rangle|_{|Q|} \simeq -\frac{\chi_t}{V} + \frac{Q^2}{V^2}$ at flow time $6t_0$





Left: $\langle {q'}^2 \rangle(x)$ in the sector $|Q| = 1$, at $t = t_0 \ldots 5t_0$

longer flow time: reduces stat. errors, enhances deviations at extreme x, additive constant becomes more negative

Right: additive constant $\propto \sqrt{t}$ (behaviour of the diffusion range)

Fit to $c_1\sqrt{t} + c_2 \;\Rightarrow\; c_2 = 0.003(18)$ *i.e.* compatible with $\mathrm{const}|_{t=0} = 0$

Conclusions

Slab method:

Simple approach to measure $\chi_{\rm t}$ within a single top. sector, best at small $|Q|$

Only assumption: Gauss-distribution of top. charges
(well confirmed, up to lattice artifacts).

Precision not affected by "topological slowing down",
but persistent finite-size effects (often polynomial at fixed topology)

Successful tests in

- non-linear σ-models: straight application
2d O(3) model: %-level precision, 1d O(2): far beyond

- 2-flavour QCD: %-level, after gradient flow time $\approx 5t_0$

Requires additive constant, and discarding very narrow slabs ($x \gtrsim 0$ or $x \lesssim 1$).