Complex Langevin Dynamics for a Random Matrix Model of QCD at finite density

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Work in collaboration with J. Glesaaen (Frankfurt U.), O. Philipsen (Frankfurt U.), J. Verbaarschot (Stony Brook U.)

Many approaches to attack the sign problem

- Conventional/Monte Carlo based methods
  - Reweighting
  - Taylor expansion
  - Imaginary μ
  - Strong Coupling Expansion
  - Mean Field analyses
- Alternative methods
  - Stochastic Quantization-Complex Langevin
  - Lefschetz Thimble
  - Canonical ensembles
  - Dual variables
  - Density of States

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Stochastic quantization as an alternative

- consider the trivial "QFT" given by the partition function
- $\mathcal{Z} = \int e^{-S(x)} dx$
- in the real Langevin formulation
- $x(t + \delta t) = x(t) - \partial_x S(x(t))\delta t + \delta\xi$
- stochastic variable $\delta\xi$ with zero mean and variance given by $2\delta t$
- generalization to complex actions Parisi(1983), Klauder (1983)
- $x \to z = x + iy$
- $z(t + \delta t) = z(t) - \partial_z S(z(t))\delta t + \delta\xi$
- one can study gauge theories with complex actions Aarts, James, Seiler, Sexty, Stamatescu, ...

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- proof relating Langevin dynamics to the path integral quantization-no longer holds
- simulations are not guaranteed to converge to "the correct solution"
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RMT

- focus on a much simpler theory than QCD. Random Matrix Theory
- same flavor symmetries with QCD which uniquely determine (in the ϵ-regime)
- mass dependence of the chiral condensate $\langle \bar{\eta}\eta \rangle = \partial_m \log Z$
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The Stephanov Model

- $\mathcal{Z} = \int DW e^{-n\Sigma^2 \mathrm{Tr} WW^\dagger} det^{N_f} \begin{pmatrix} m & iW + \mu \\ iW^\dagger + \mu & m \end{pmatrix}$

  Stephanov (1996)

- solve via bosonization
- $\mathcal{Z}(m, \mu) = \int d\sigma d\sigma^* e^{-n\sigma^2} (\sigma\sigma^* + m(\sigma + \sigma^*) + m^2 - \mu^2)^n$
- where σ is an $N_f \times N_f$ matrix
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The phase transition

- in the thermodynamic limit evaluate $\mathcal{Z}$ via a saddle point approximation
- there is a phase transition separating a phase with zero and non-zero baryon density
- In the chiral limit $\mu_c = 0.527$ for $\mu \in \mathbb{R}$
- $\mu_c = i$ for $\mu \in \mathbb{I}$
- we can compute $\Sigma(m, \mu)$ and $n_B(m, \mu)$ and compare it with the Complex Langevin simulation
- first attempts in the Osborn model Mollgaard and Splittorff(2013-2014), Nagata, Nishimura, Shimasaki (2015-2016)

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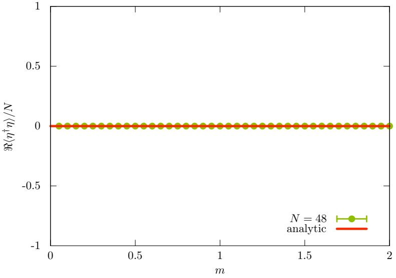
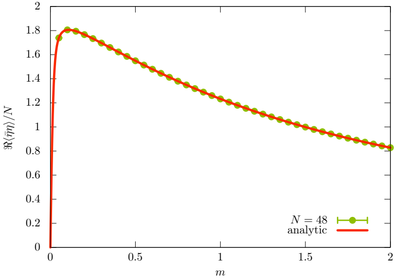
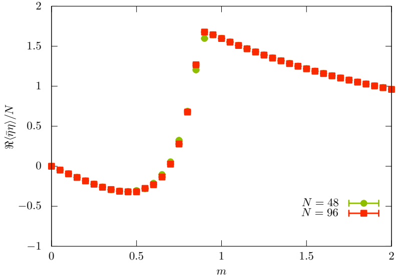


$\langle \eta^\dagger \eta \rangle$ for $\mu = 0$



$\langle \bar{\eta} \eta \rangle$ for $\mu = 0$

Numerical Validity-Matrix Size



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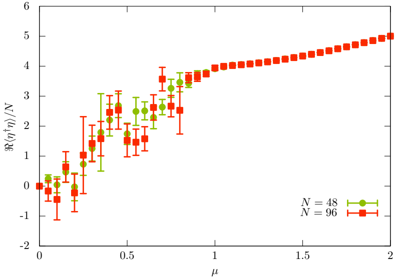

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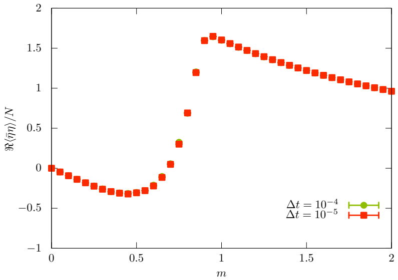
Numerical Validity-Step Size



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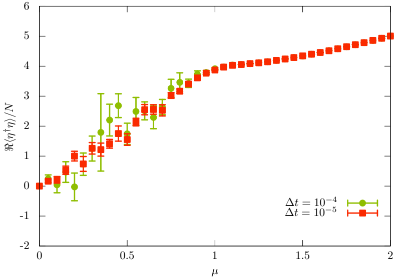

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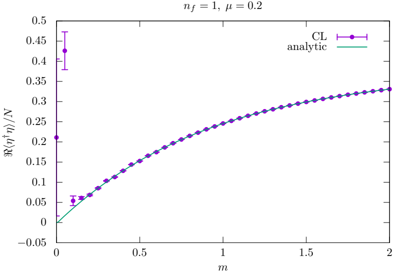
m-scan for $\mu = 0.2$



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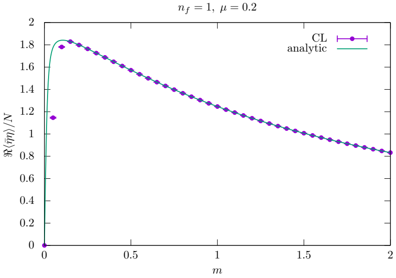

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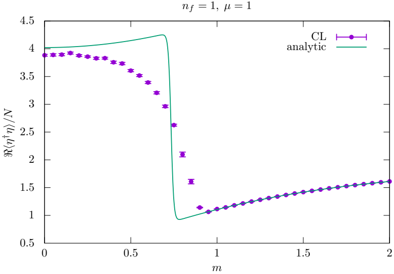

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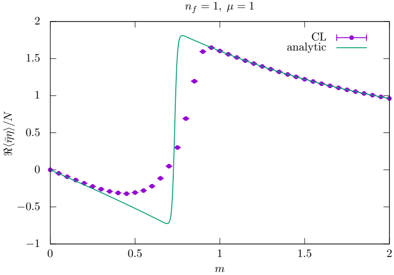

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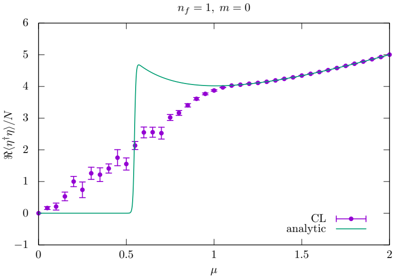

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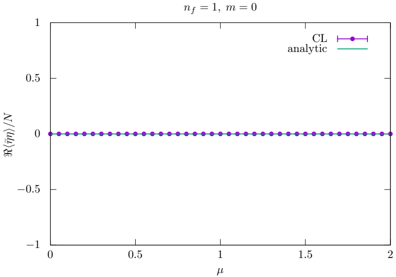

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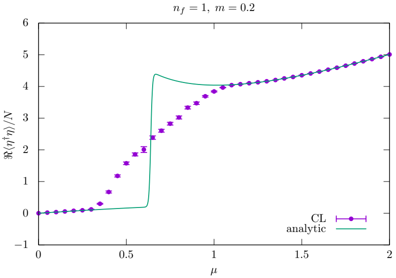

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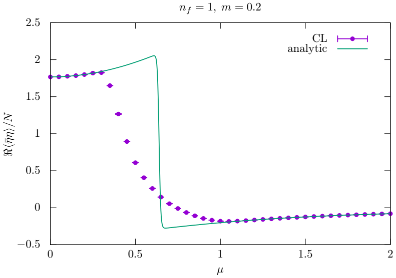

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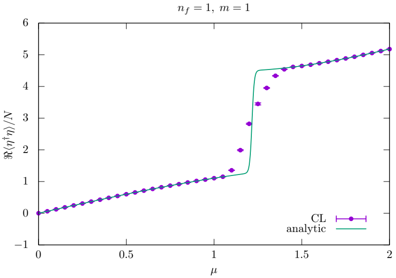

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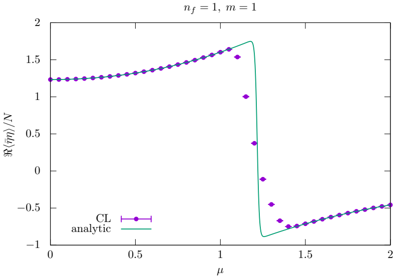

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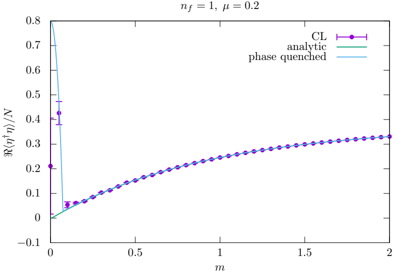

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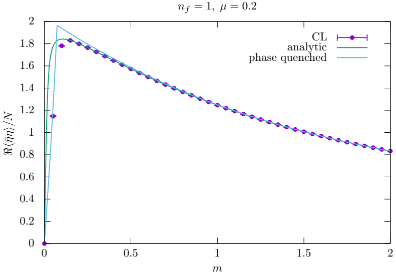

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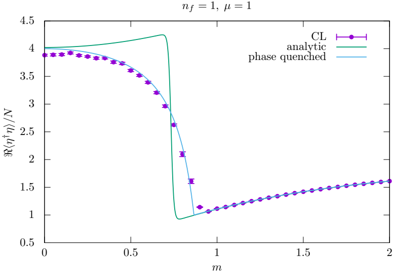

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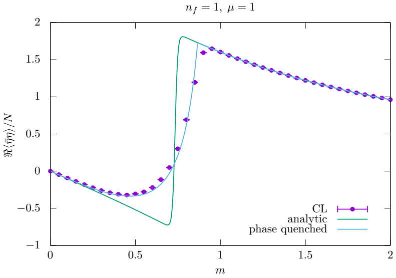

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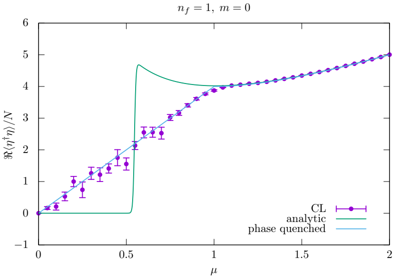

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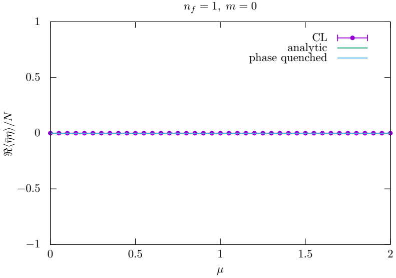

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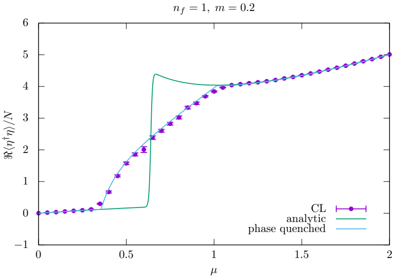

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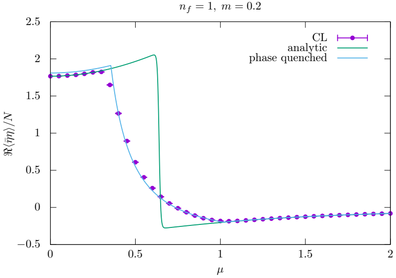

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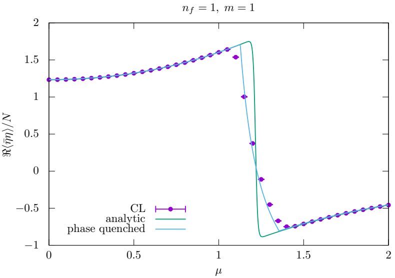

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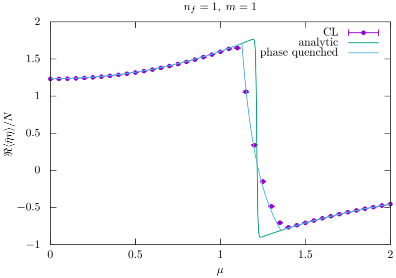

$\langle \bar{\eta} \eta \rangle$ for $m = 1$

Conclusions and outlook

- studied the Complex Langevin algorithm for an RMT model for QCD
- can compare with exact analytical results for all the range of parameters(m, μ)
- compared to previous similar studies this model posseses a phase transition to a phase with non-zero baryon density
- fails to converge to QCD and it converges to $|QCD|$
- standard ways to fix it $\rightarrow$ gauge cooling Seiler, Sexty and Stamatescu(2012), Nagata, Nishimura, Shimasaki (2015)
- work in progress...
- work in progress employing the Lefschetz thimbles Witten (2010), Christoforetti et al (2012), Eruzzi and Di Renzo(2015)
- *Thanks a lot for your attention!*

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for upcoming results ...