Hadronic contributions to the muon $(g-2)$ from Lattice QCD

Hartmut Wittig

PRISMA Cluster of Excellence, Institute for Nuclear Physics and Helmholtz Institute Mainz

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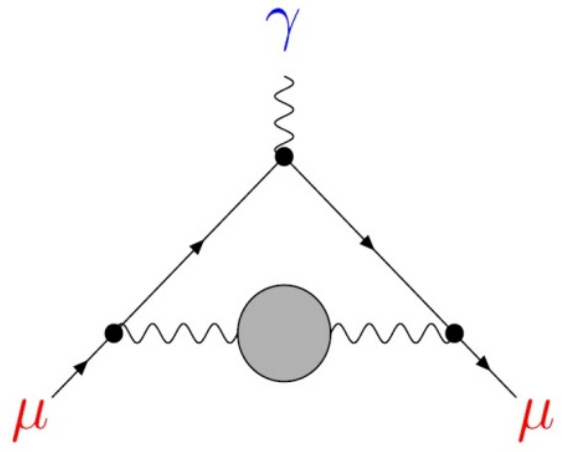
Anomalous magnetic moment of the muon

Current status:

$$a_\mu \equiv \frac{1}{2}(g-2)_\mu = \begin{cases} 116\,592\,089(54)(33)\cdot 10^{-11} & \text{Experiment} \\ 116\,591\,828(43)(26)(2)\cdot 10^{-11} & \text{SM prediction} \end{cases}$$

Hadronic vacuum polarisation:

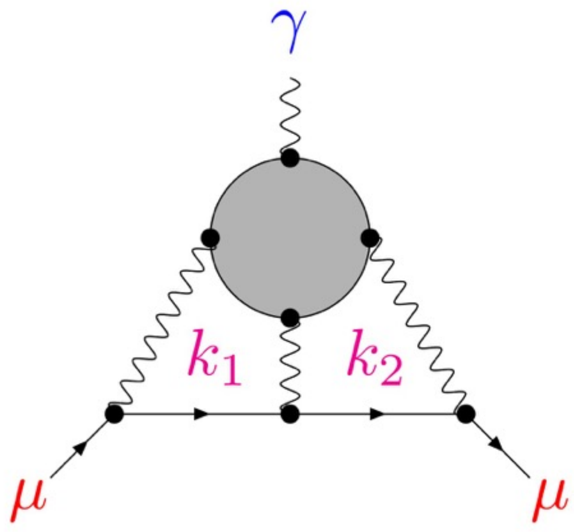


Dispersion theory:

$$a_\mu^{\text{hvp}} = (6949 \pm 43)\cdot 10^{-11}$$

(combined e^+e^- data)

Hadronic light-by-light scattering:



Model estimates:

$$a_\mu^{\text{hlbl}} = (105 \pm 26)\cdot 10^{-11}$$

"Glasgow consensus"

Anomalous magnetic moment of the muon

Hadronic Vacuum Polarisation and Dispersion Theory

$$a_\mu^{\rm hvp} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left\{ \int_{m_\pi^2}^{E_{\rm cut}^2} ds\, \frac{R_{\rm had}^{\rm data}(s)\hat{K}(s)}{s^2} + \int_{E_{\rm cut}^2}^{\infty} ds\, \frac{R_{\rm had}^{\rm pQCD}(s)\hat{K}(s)}{s^2} \right\}$$

* SM estimate subject to experimental uncertainties in $R_{\rm had}(e^+e^- \to \text{hadrons})$

Hadronic Light-by-Light Scattering

* Model uncertainties difficult to quantify
* Dispersive formalism much more complicated than HVP
  [Colangelo et al., JHEP 1409 (2014) 091, PLB 738 (2014) 6, JHEP 1509 (2015) 074]
  [Pauk & Vanderhaeghen, PRD 90 (2014) 113012]
* Identify dominant sub-processes, e.g. $\gamma^*\gamma^* \longrightarrow \pi^0, \eta, \eta'$

Anomalous magnetic moment of the muon

Motivation for first-principles approach:

* No reliance on experimental data
  — except for simple hadronic quantities to fix bare parameters
* No model dependence
  — except for chiral extrapolation and constraining the IR regime

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* Can lattice QCD deliver estimates with sufficient accuracy in the coming years?

$$\delta a_\mu^{\rm hvp}/a_\mu^{\rm hvp} < 0.5\%, \qquad \delta a_\mu^{\rm hlbl}/a_\mu^{\rm hlbl} \lesssim 10\%$$

$\Rightarrow$ Crucial for exploring the limits of the Standard Model

Outline

I. Hadronic Vacuum Polarisation

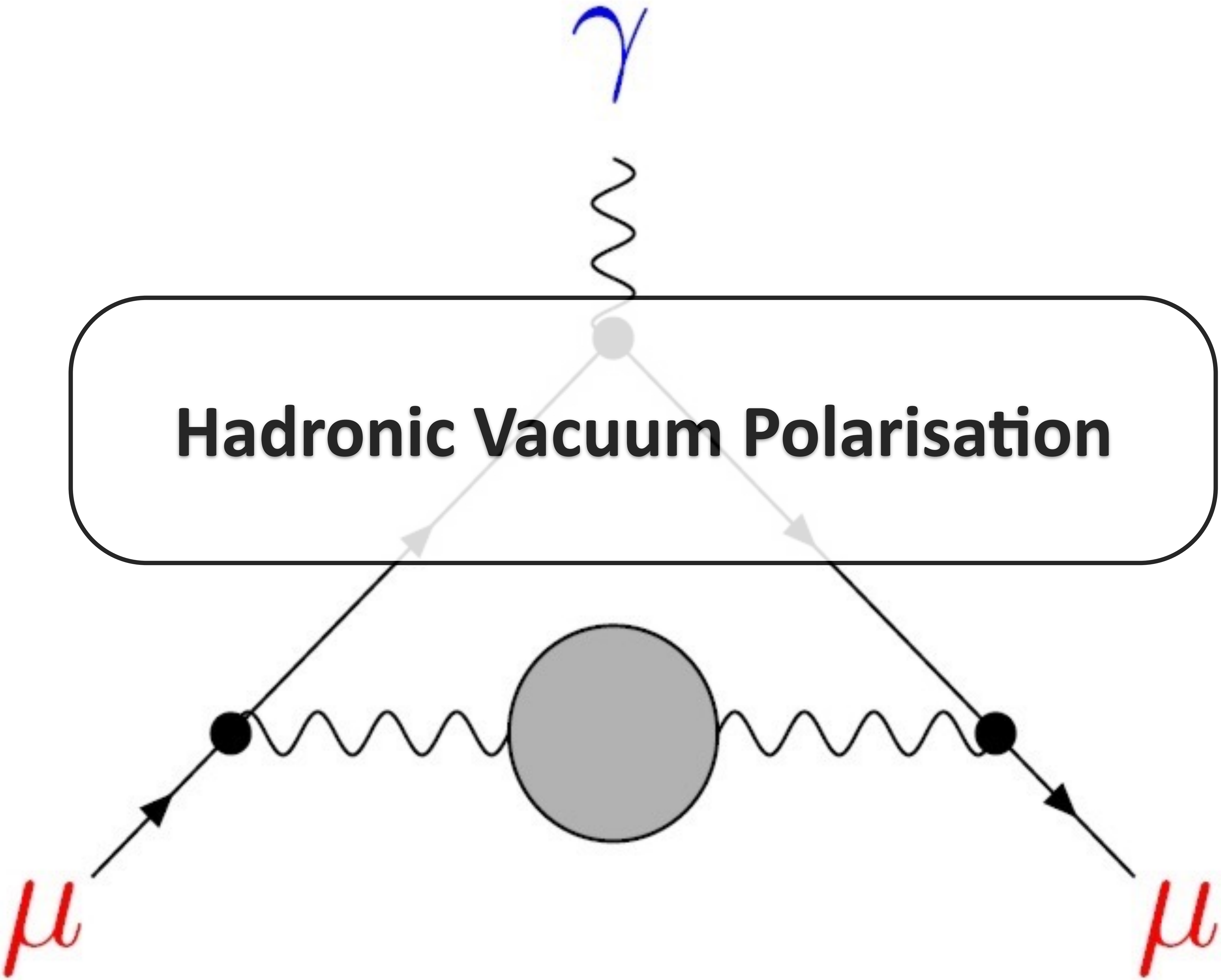
Constraining the infrared regime
Quark-disconnected diagrams
Finite-volume effects
Results overview

II. Hadronic Light-by-Light Scattering

Lattice QCD approaches to HLbL
Recent calculations

III. Summary

Hadronic Vacuum Polarisation



Lattice QCD approach to HVP

* Convolution integral over Euclidean momenta: *[Lautrup & de Rafael; Blum]*

$$a_\mu^{\rm hvp} = 4\alpha^2 \int_0^\infty dQ^2\, f(Q^2)\left\{\Pi(Q^2) - \Pi(0)\right\}$$

$$\Pi_{\mu\nu}(Q) = \int e^{iQ\cdot(x-y)}\left\langle J_\mu(x)J_\nu(y)\right\rangle \equiv \left(Q_\mu Q_\nu - \delta_{\mu\nu}Q^2\right)\Pi(Q^2)$$

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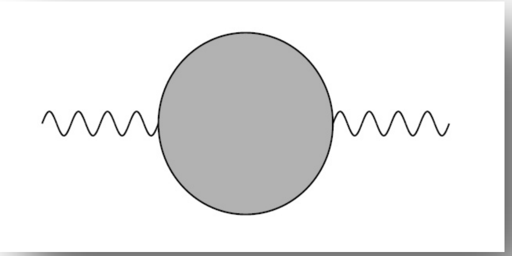

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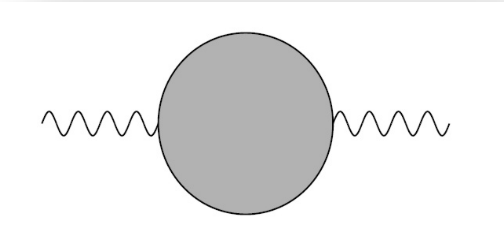

* Determine VPF $\Pi(Q^2)$ and additive renormalisation $\Pi(0)$
* Integrand peaked near $Q^2 \approx (\sqrt{5}-2)m_\mu^2$

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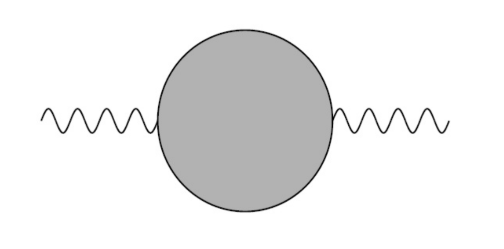
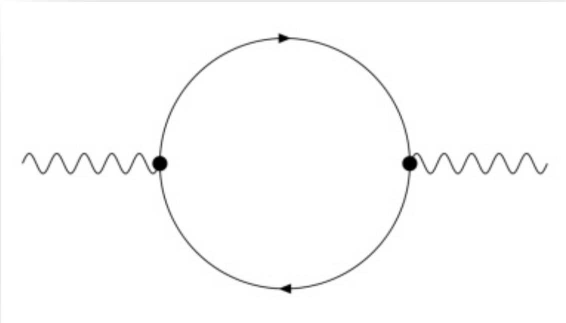


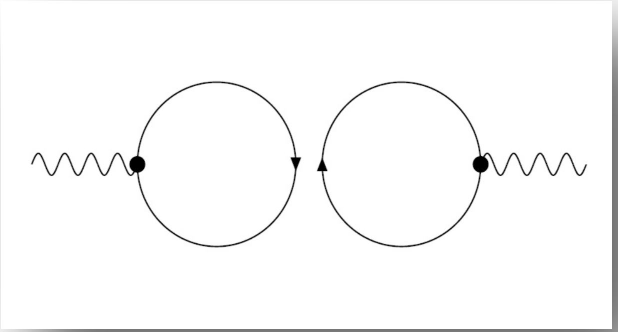
* Determine VPF $\Pi(Q^2)$ and additive renormalisation $\Pi(0)$
* Integrand peaked near $Q^2 \approx (\sqrt{5}-2)m_\mu^2$
* Lattice momenta are quantised: $Q_\mu = \dfrac{2\pi}{L_\mu}$
* Statistical accuracy of $\Pi(Q^2)$ deteriorates as $Q \longrightarrow 0$

Lattice QCD approach to HVP

Main issues:

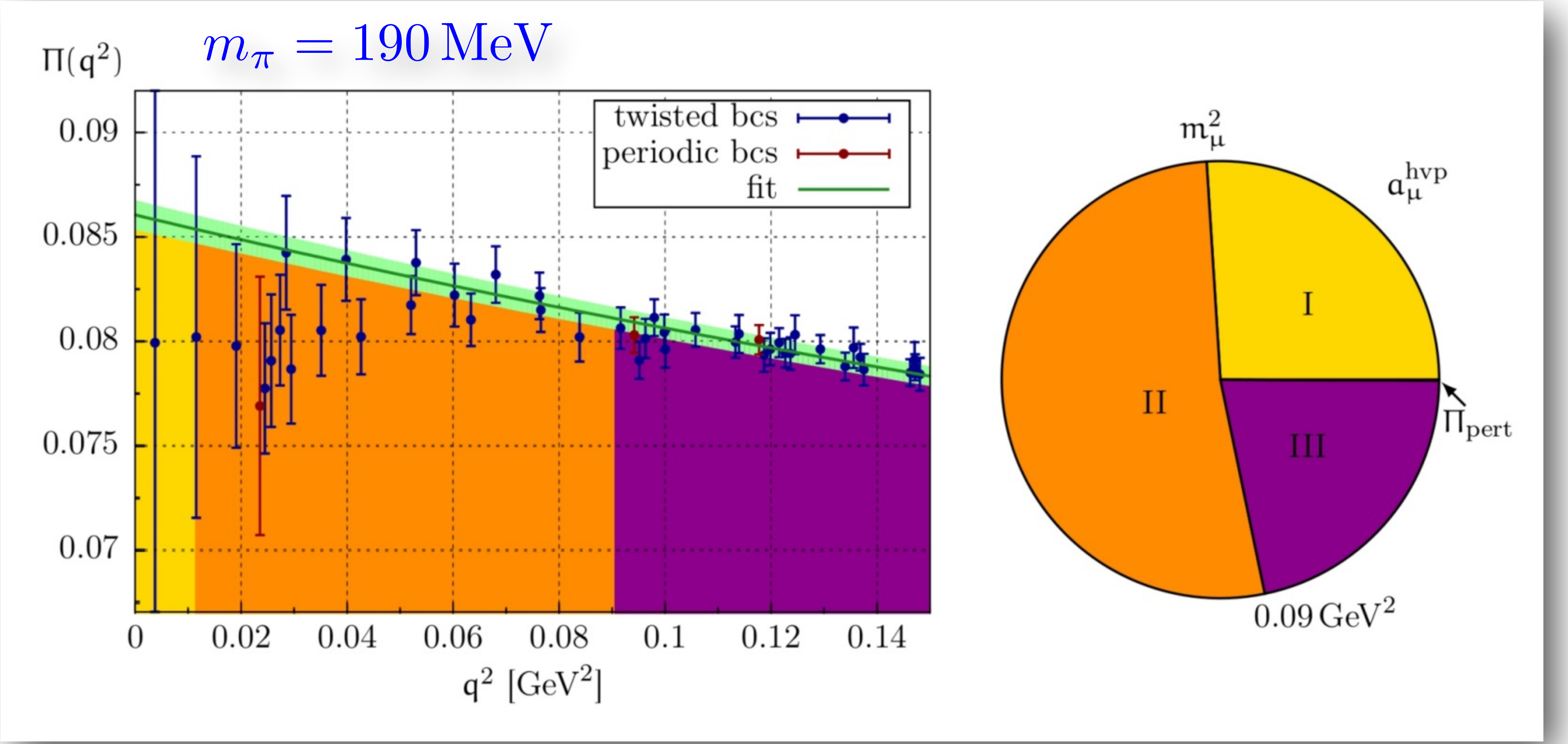
- Statistical accuracy at the sub-percent level required
- Reduce systematic uncertainty associated with region of small Q^2
  ⇔ accurate determination of $\Pi(0)$
- Perform comprehensive study of finite-volume effects
- Include quark-disconnected diagrams





- Include isospin breaking: $m_u \neq m_d$, QED corrections

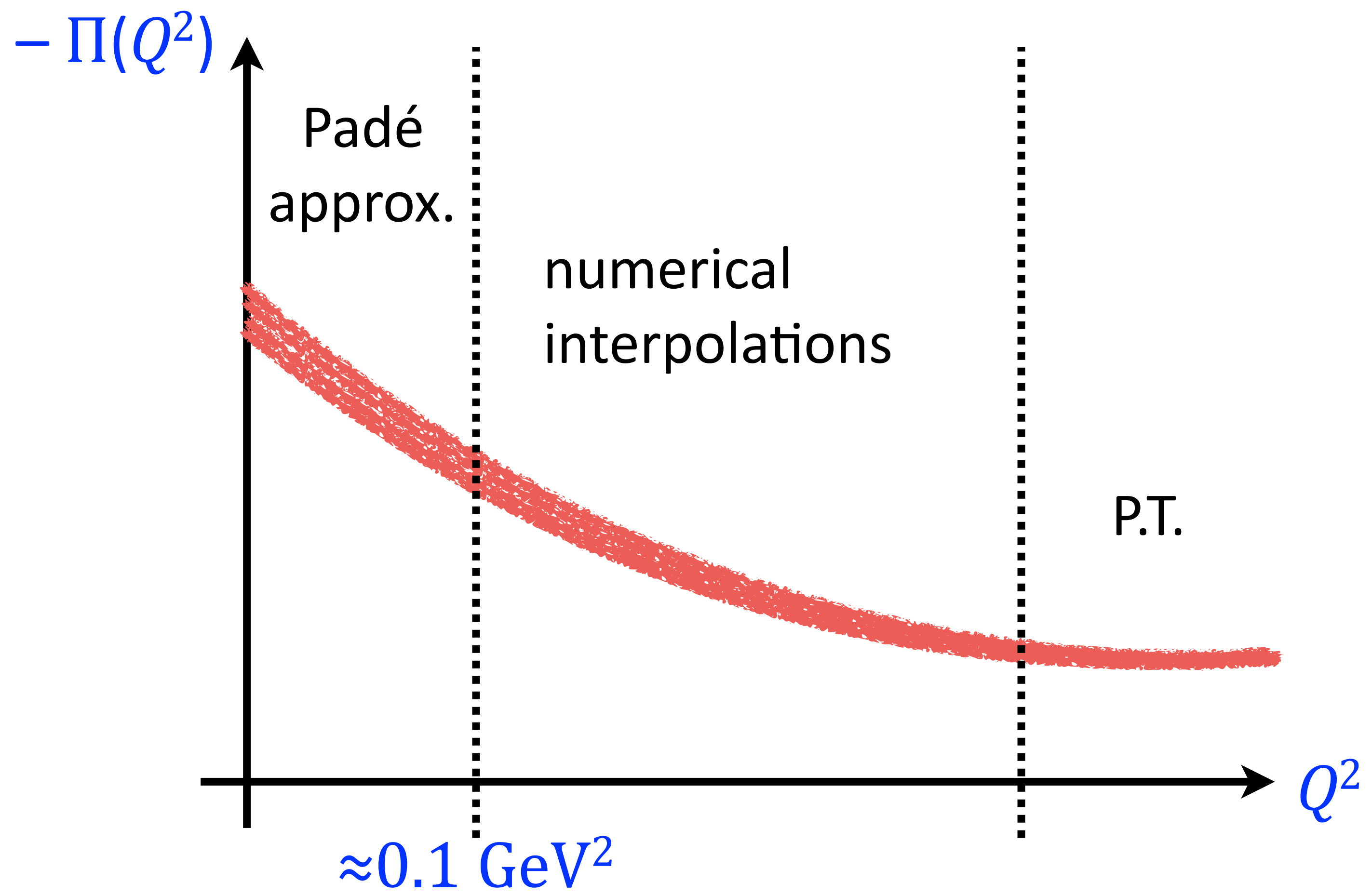
Low-momentum region: Twisted BCs



- Model-independent fits compromised when applied to $Q^2 \gg m_\mu{}^2$
- Determination of $\Pi(0)$ may be biased by more accurate data at large Q^2

Low-momentum region: "Hybrid method"

* Minimise model dependence: *[Golterman, Maltman & Peris, Phys Rev D90 (2014) 074508]*



* Determine $\Pi(0)$ and $\Pi(Q^2)$ from models in small-momentum region:
  Padé approximants, conformal polynomials, time moments

Low-momentum region: Time moments

* Expansion of VPF at low-Q^2: $\Pi(Q^2) = \Pi_0 + \sum_{j=1}^{\infty} Q^{2j}\Pi_j$

* Vacuum polarisation for $Q = (\omega, \vec{0})$:

$$\Pi_{kk}(\omega) = a^4 \sum_{x_0} \mathrm{e}^{i\omega x_0} \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle$$

* Spatially summed vector correlator: $G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle$

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* Time moments:

[Chakraborty et al., Phys Rev D89 (2014) 114501]

$$G_{2n} \equiv a \sum_{x_0} x_0^{2n} G(x_0) = (-1)^n \frac{\partial^{2n}}{\partial\omega^{2n}} \left\{ \omega^2 \hat{\Pi}(\omega^2) \right\}_{\omega^2=0}$$

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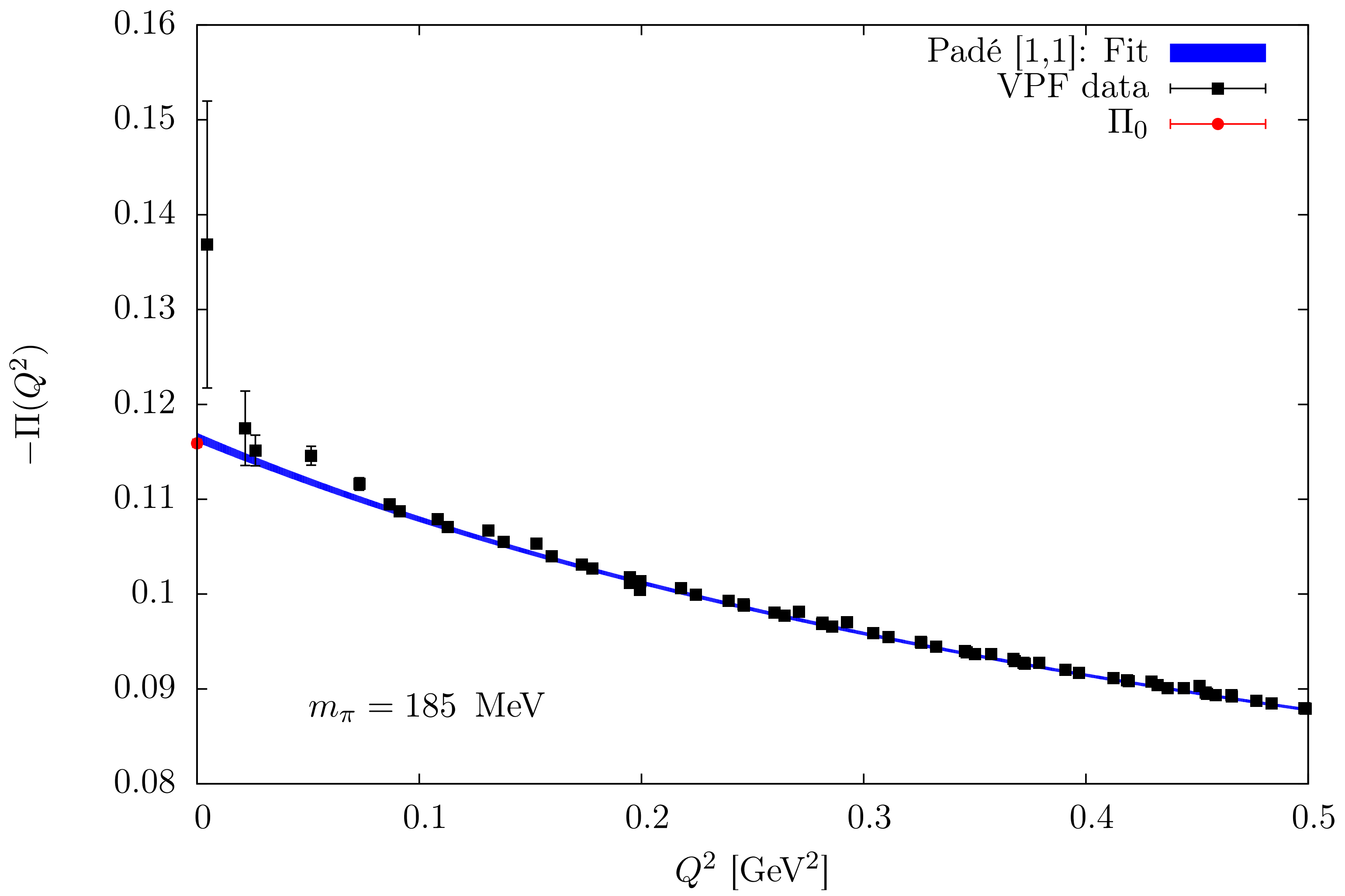
* Expansion coefficients: $$\Pi(0) \equiv \Pi_0 = \frac{1}{2} G_2, \quad \Pi_j = (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!}$$

Hybrid Method versus Time Moments

* Construct Padé approximants either from fits or time moments



Fit Padé [1,1] for

$$Q^2_{\rm low} \lesssim 0.5\,{\rm GeV}^2$$

[Hanno HORCH, TUE 14:40]

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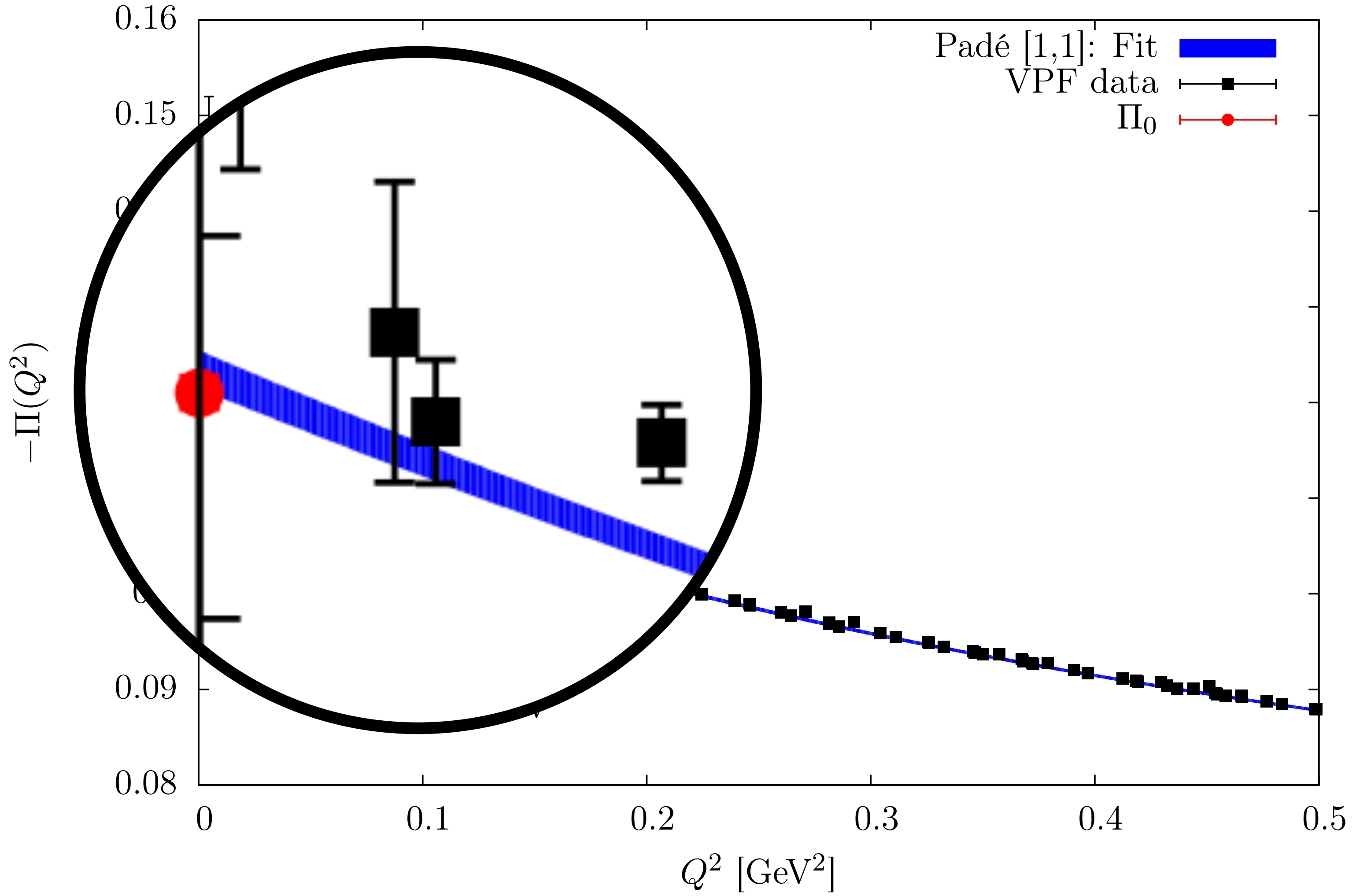

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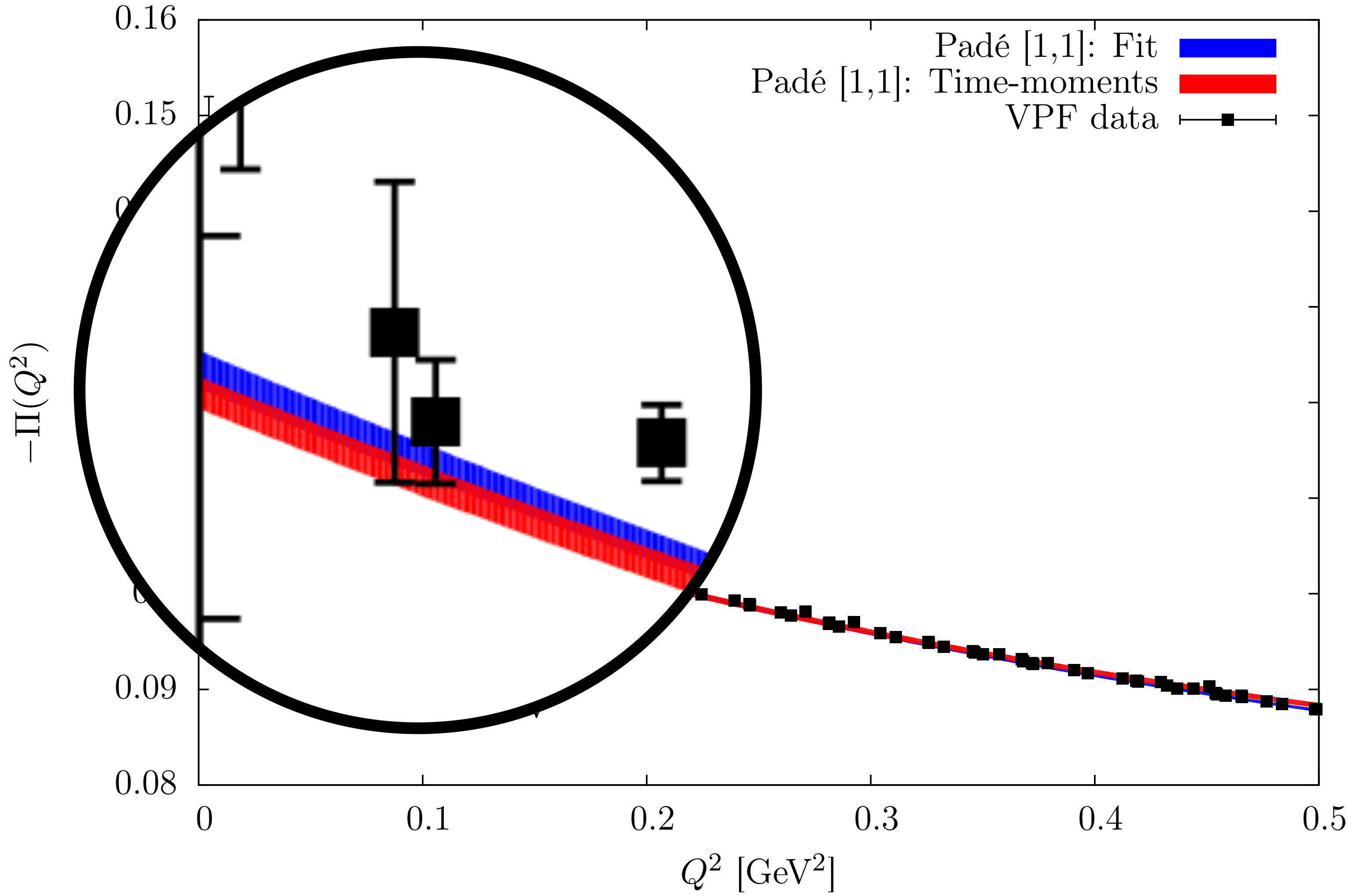

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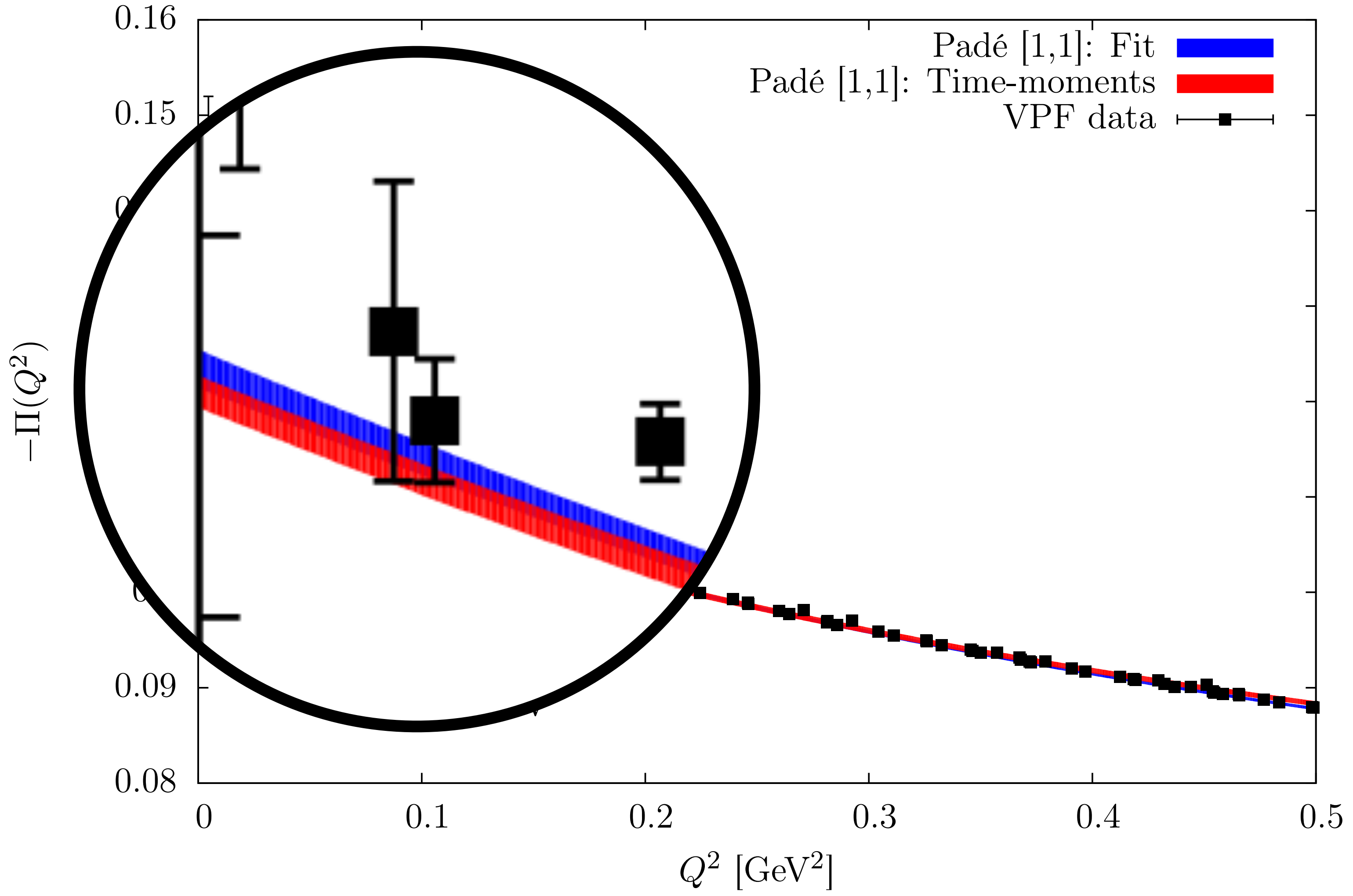

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* Low-order Padé approximants consistent for $Q^2 < 0.5\ \mathrm{GeV}^2$

Time-Momentum Representation

* Integral representation of subtracted VPF $\hat{\Pi}(Q^2) \equiv \Pi(Q^2) - \Pi(0)$

$$\Pi(Q^2) - \Pi(0) = \frac{1}{Q^2}\int_0^\infty dx_0\, G(x_0)\left[Q^2 x_0^2 - 4\sin^2\left(\tfrac{1}{2}Qx_0\right)\right]$$

$$G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0)\rangle$$

[Bernecker & Meyer, Eur Phys J A47 (2011) 148,
Francis et al. Phys Rev D88 (2013) 054502,
Feng et al., Phys Rev D88 (2013) 034505]

* Q^2 is a tuneable parameter

* No extrapolation to $Q^2 = 0$ required

* Must determine $I = 1$ vector correlator $G(x_0)$ at all distances

  $\longrightarrow$ $G(x_0)$ dominated by two-pion state for $x_0 \rightarrow \infty$

  $\longrightarrow$ Include multi-particle states to capture long-distance behaviour

Equivalence of time moments and TMR

* Subtracted VPF: $\hat{\Pi}(Q^2) \equiv \Pi(Q^2) - \Pi(0)$
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TMR and time moments: model dependence

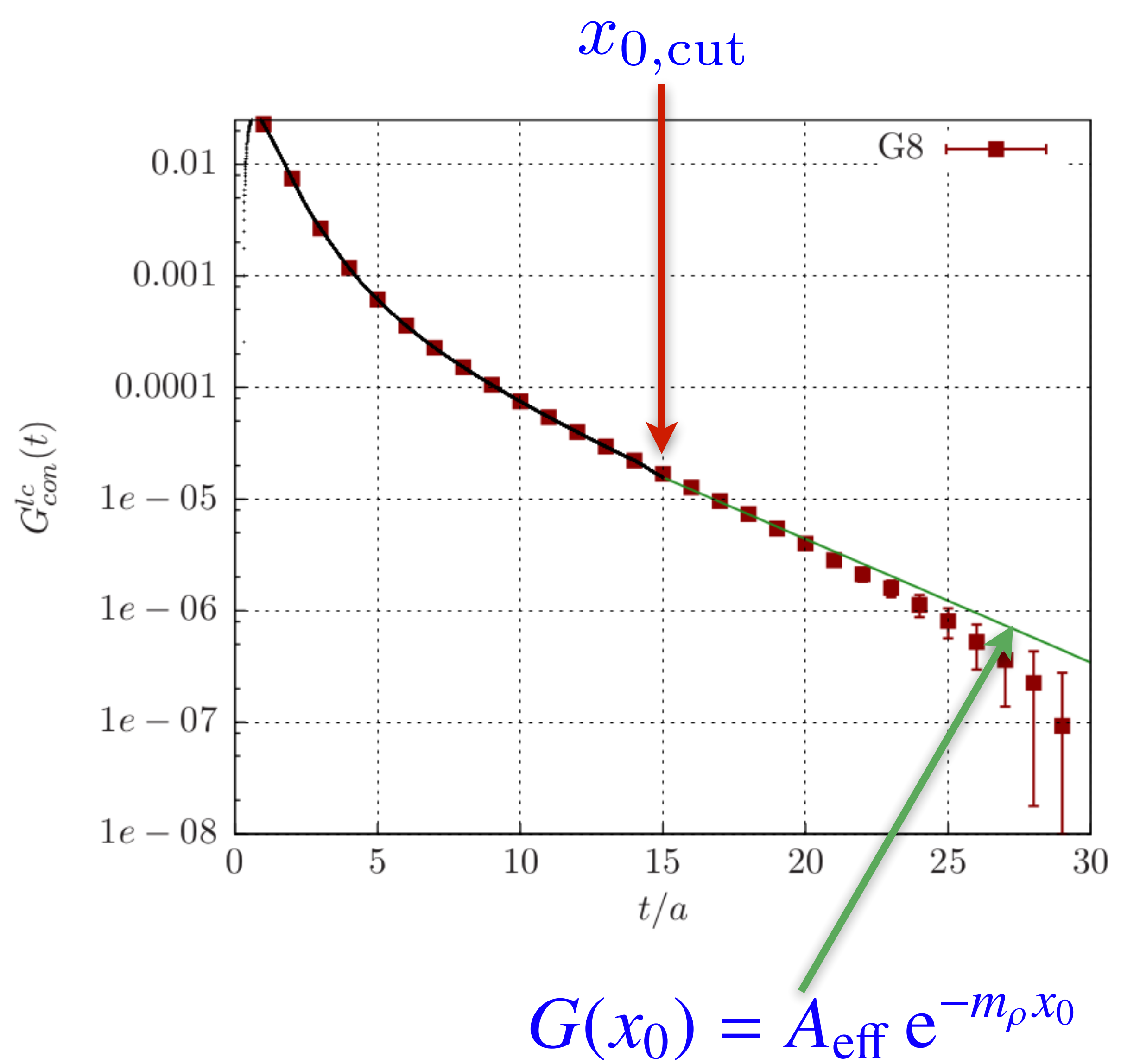
* Determine long-distance contribution to vector correlator:

$$G(x_0) = \begin{cases} G(x_0)^{\text{data}}, & x_0 \le x_{0,\text{cut}} \\ G(x_0)^{\text{fit}}, & x_0 > x_{0,\text{cut}} \end{cases}$$

* Single-exponential fit for $x > x_{0,\text{cut}}$

* $G(x_0)$ dominated by two-pion state at long distances

⇒ Include multi-particle states to eliminate model dependence when $x_0 \longrightarrow \infty$



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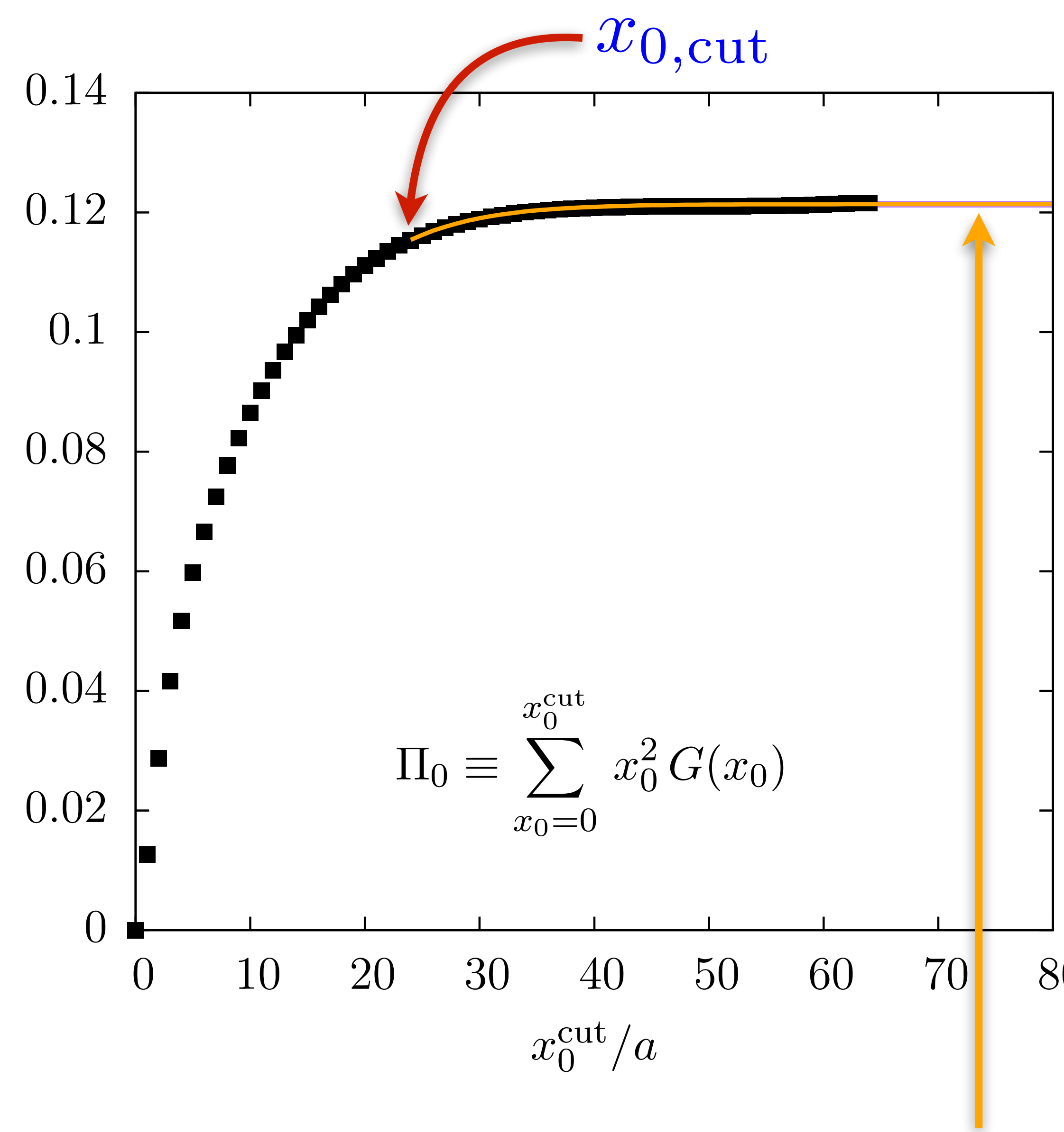

$$G(x_0) = A_{\text{eff}}\, e^{-m_\rho x_0}$$

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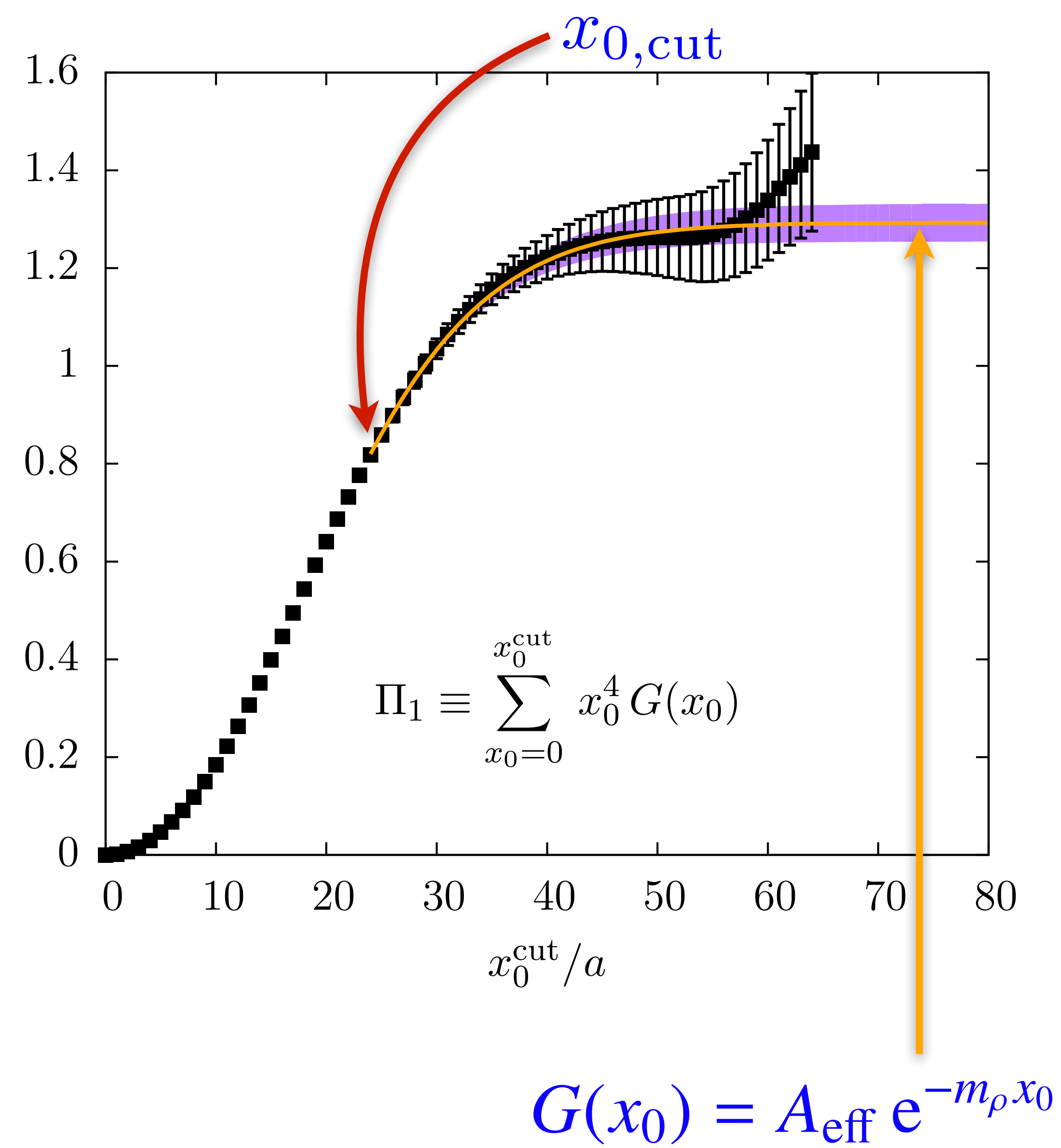

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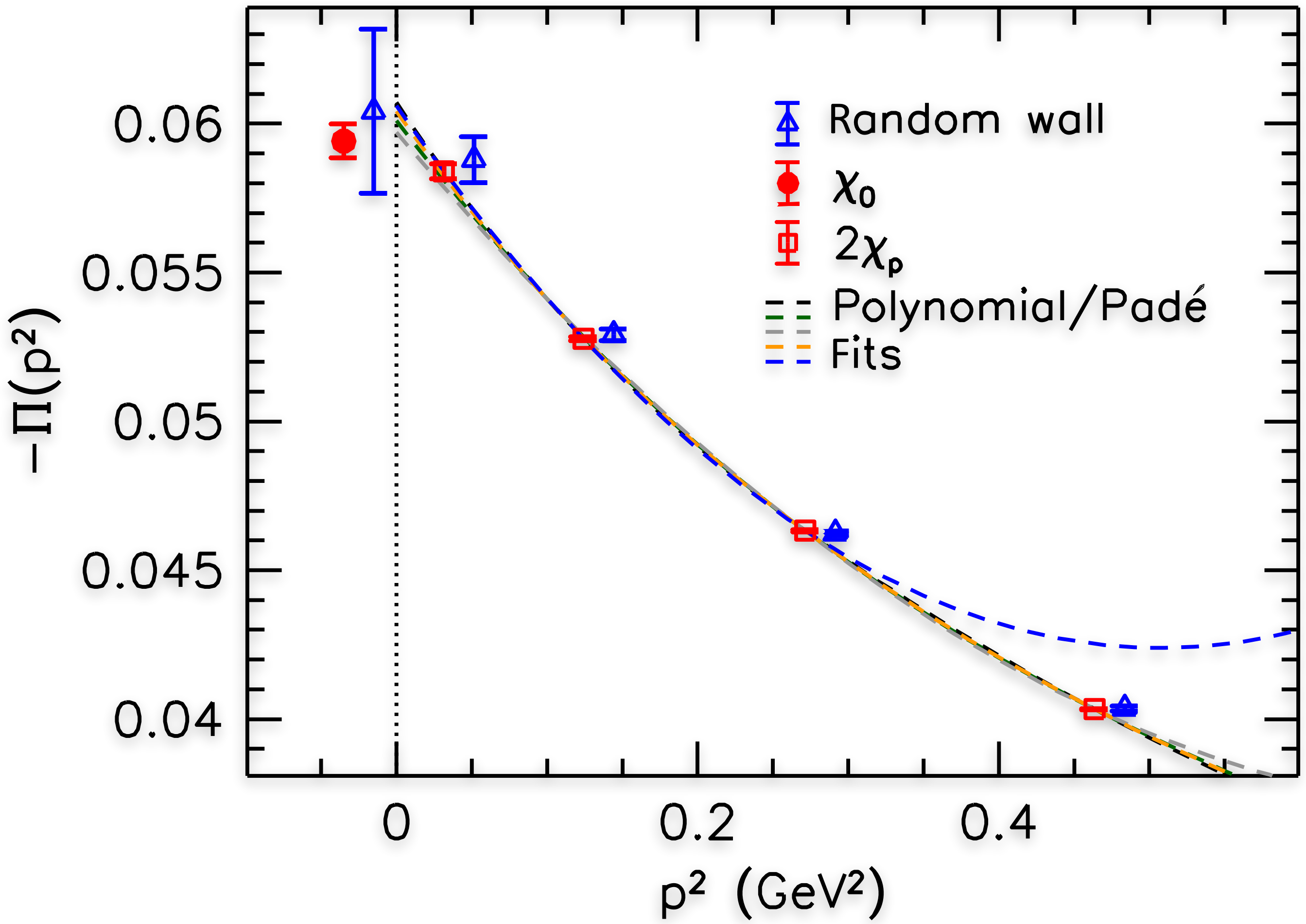
HVP from magnetic susceptibilities

* $\Pi(p^2)$ can be interpreted as a magnetic susceptibility
* $\Pi(0)$ obtained from homogeneous background field: $\chi_0 = \Pi(0)$
* Efficient evaluation employing 4D random noise sources at fixed momentum
* Precise determination of disconnected contribution

(rooted staggered quarks;
physical pion mass; $a = 0.29$ fm)

[Bali & Endrődi, Phys Rev D92 (2015) 054506]



Disconnected Contributions

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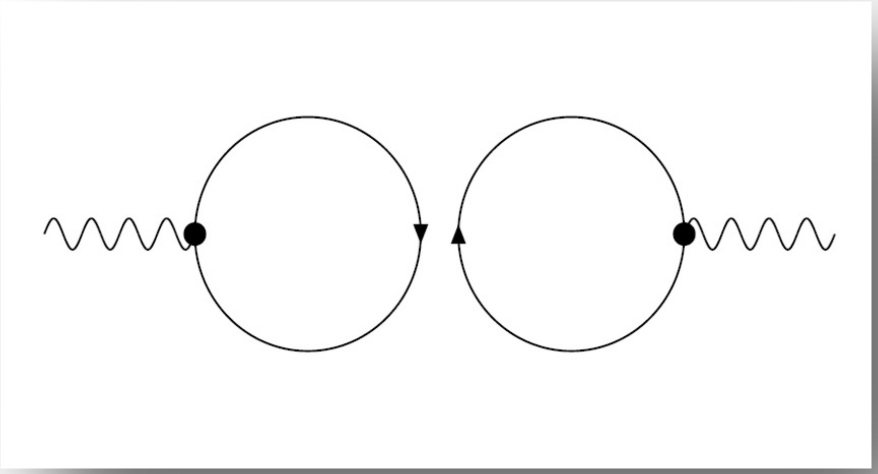
- Current-current correlator contains quark-disconnected contributions

$$\left\langle \mathrm{Tr}\left(\gamma_\mu S^{f}(x,x)\right) \mathrm{Tr}\left(\gamma_\nu S^{f'}(y,y)\right) \right\rangle$$



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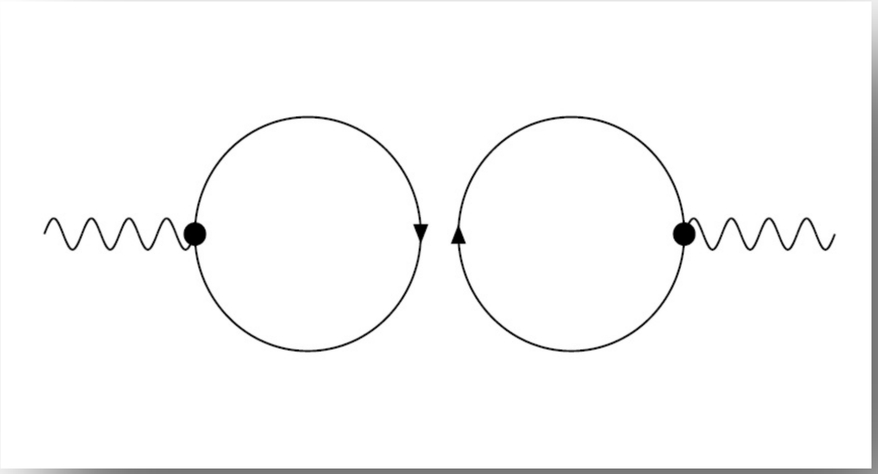
stochastic evaluation



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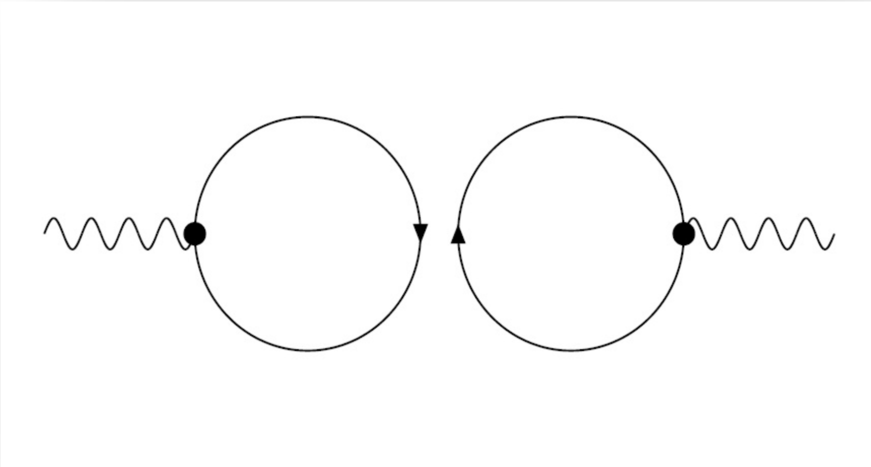


* Apply noise-reduction techniques

  - Stochastic noise cancellation *[Gülpers et al., arXiv:1411.7592; V. Gülpers, PhD Thesis 2015]*
  - Low-mode averaging *[Blum et al., PRL 116 (2016) 232002]*
  - Momentum sources *[Bali & Endrődi, PRD 92 (2015) 054506]*
  - All-mode-averaging, hopping parameter expansion, sparsening schemes,… *[Blum, Izubuchi & Shintani, PRD 88 (2013) 094503, Bali, Collins & Schäfer, CPC 181 (2010) 1570,…]*

Minimising stochastic noise

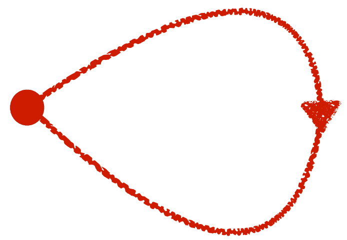
* Electromagnetic current correlator with *u, d, s* quarks:

$$G(x_0) := -a \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle, \quad J_k = \tfrac{2}{3}\bar{u}\gamma_k u - \tfrac{1}{3}\bar{d}\gamma_k d - \tfrac{1}{3}\bar{s}\gamma_k s$$

* $G(x_0)$ splits into connected and disconnected parts:

$$G(x_0) = \tfrac{5}{9}C^{\ell\ell}(x_0) + \tfrac{1}{9}C^{ss}(x_0) - \tfrac{1}{9}D^{\ell s}(x_0), \qquad (m_u = m_d = m_\ell)$$

$$D^{\ell s}(x_0) = \Big(\Delta^\ell(x_0) - \Delta^s(x_0)\Big)\Big(\Delta^\ell(0) - \Delta^s(0)\Big), \qquad \Delta^f(x_0) \sim \gamma_k$$



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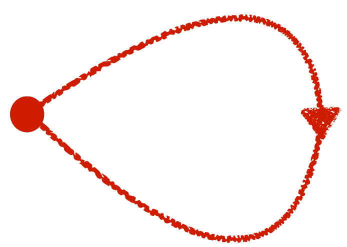
$$D^{\ell s}(x_0) = \underbrace{\left(\Delta^\ell(x_0) - \Delta^s(x_0)\right)}_{\text{noise cancellation}} \left(\Delta^\ell(0) - \Delta^s(0)\right), \qquad \Delta^f(x_0) \sim \gamma_k$$



* Preserve correlation in stochastic evaluation of $\Delta^\ell(x_0) - \Delta^s(x_0)$ to achieve noise cancellation *[Gülpers et al., arXiv:1411.7592; V. Gülpers, PhD Thesis 2015]*

Minimising stochastic noise

* Electromagnetic current correlator with u, d, s quarks:

$$G(x_0) := -a \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle, \quad J_k = \tfrac{2}{3}\bar{u}\gamma_k u - \tfrac{1}{3}\bar{d}\gamma_k d - \tfrac{1}{3}\bar{s}\gamma_k s$$

* $G(x_0)$ splits into connected and disconnected parts:

$$G(x_0) = \tfrac{5}{9} C^{\ell\ell}(x_0) + \tfrac{1}{9} C^{ss}(x_0) - \tfrac{1}{9} D^{\ell s}(x_0), \qquad (m_u = m_d = m_\ell)$$

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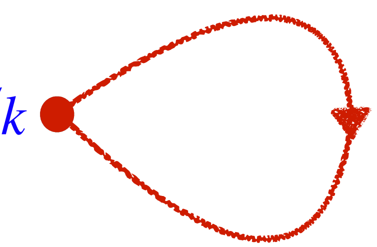

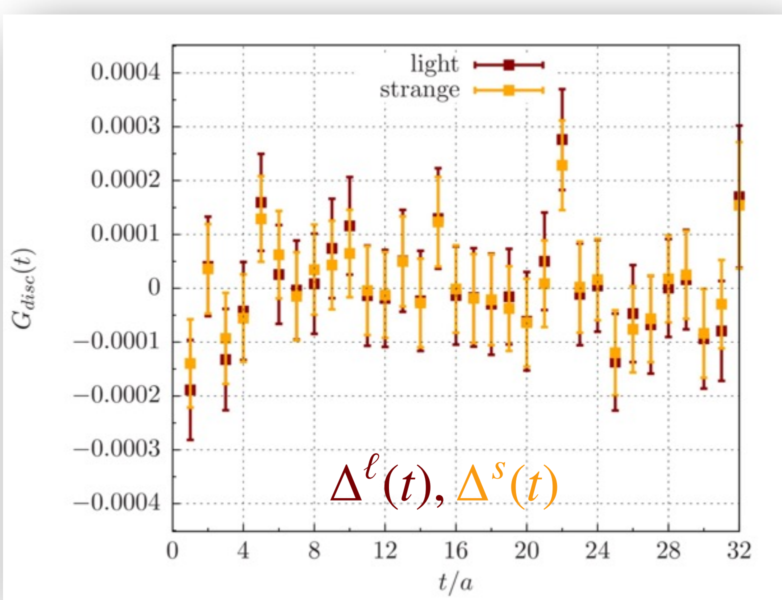

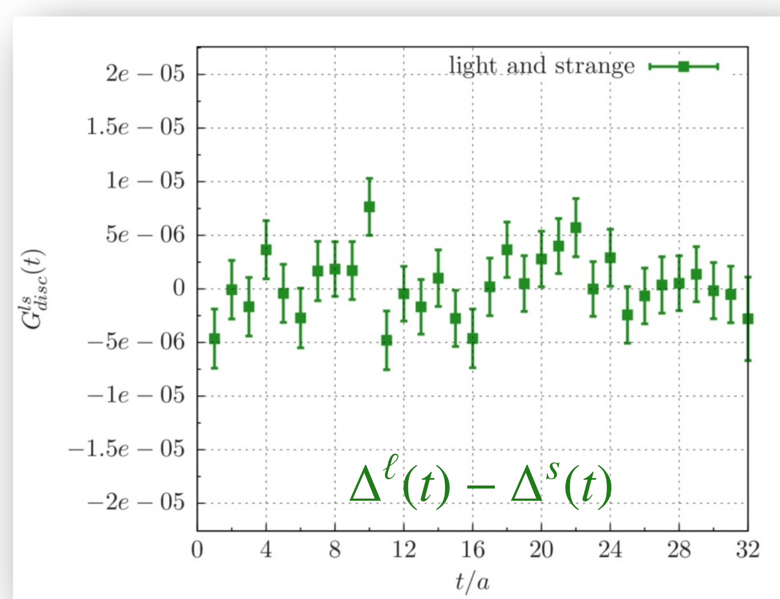

[Gülpers et al., arXiv:1411.7592]

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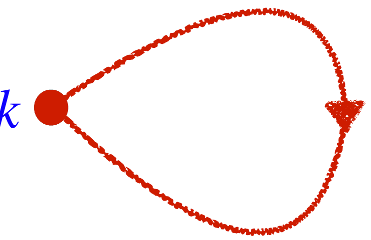

- Refinements: exact treatment of low modes [T. Blum et al., PRL 116 (2016) 232002]

$$S^f(x,y) = \sum_{k=1}^{N_{\rm ev}} \frac{v_k(x) \otimes v_k(y)^\dagger}{\lambda_k} + S_\perp^f(x,y), \qquad \lambda_{N_{\rm ev}} \simeq m_s$$

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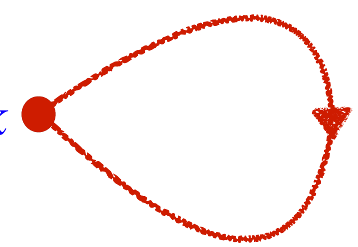

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stochastic evaluation

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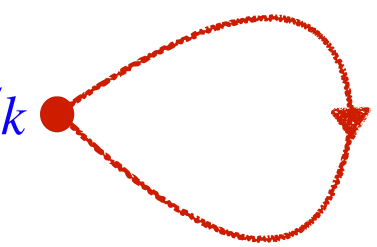

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stochastic evaluation (of $S_\perp^{f}(x,y)$)

- AMA, time translations, sparsening schemes

Random noise sources

* Random noise source at fixed momentum: *[Bali & Endrődi, PRD 92 (2015) 054506]*

$$\phi^{(r)}(x|p) = \sum_z D(x,z)^{-1}\eta^{(r)}(z)\,\mathrm{e}^{-ip\cdot z}, \qquad p_\mu = n_\mu \frac{2\pi}{L_\mu}, \qquad r = 1,\dots N_\mathrm{r}$$

$$\lim_{N_\mathrm{r}\to\infty} \mathrm{Tr}\left[\eta(x)^\dagger\,\gamma_k\,\phi(x|p)\right] = \sum_x \mathrm{e}^{-ip\cdot x}\,\mathrm{Tr}\left(\gamma_k S\,(x,x)\right)$$ (stochastic average)

⇒ disconnected contribution to $-p^2\,\Pi(p^2)$ at fixed momentum

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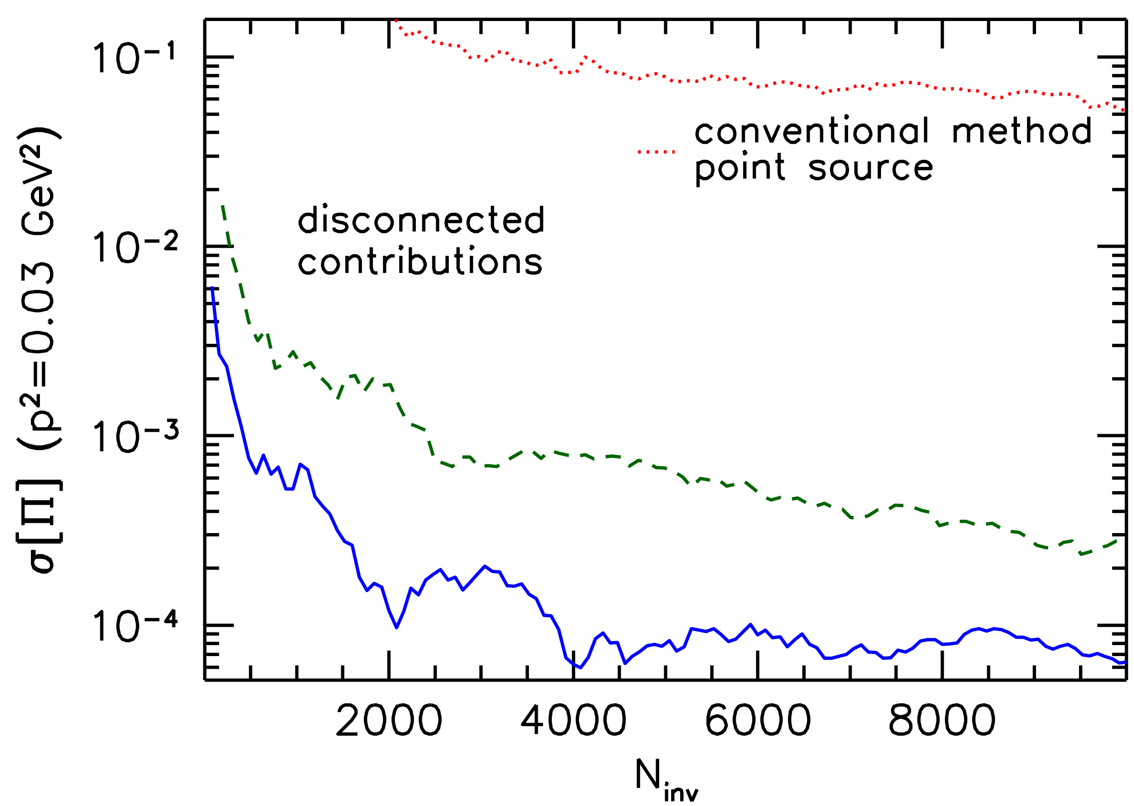

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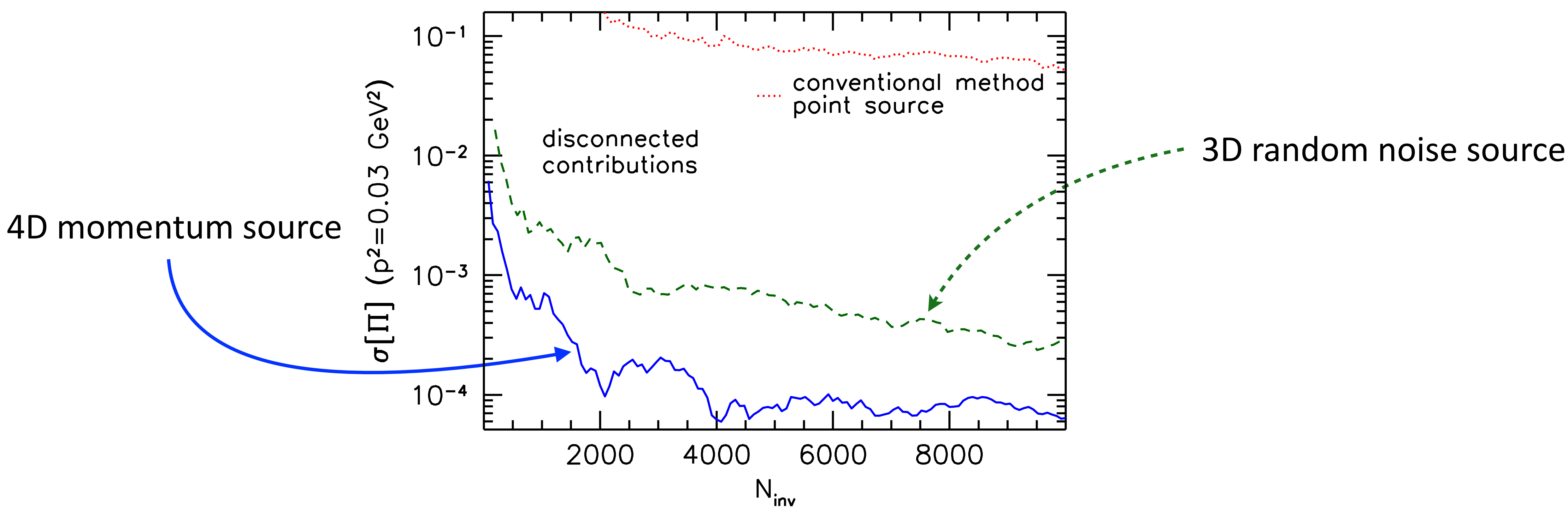

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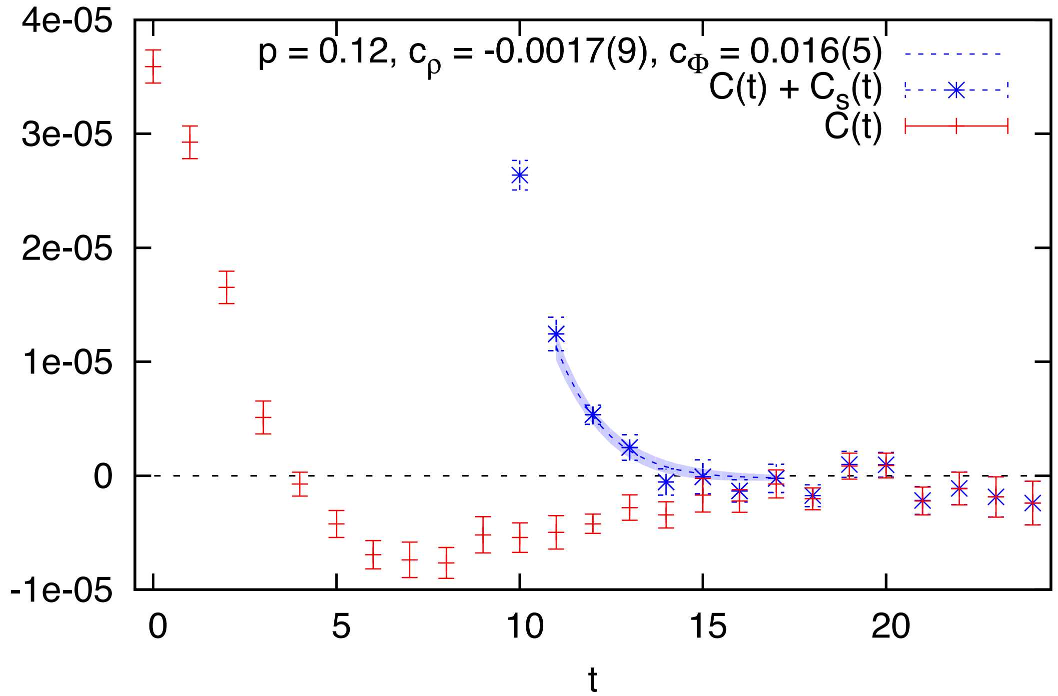

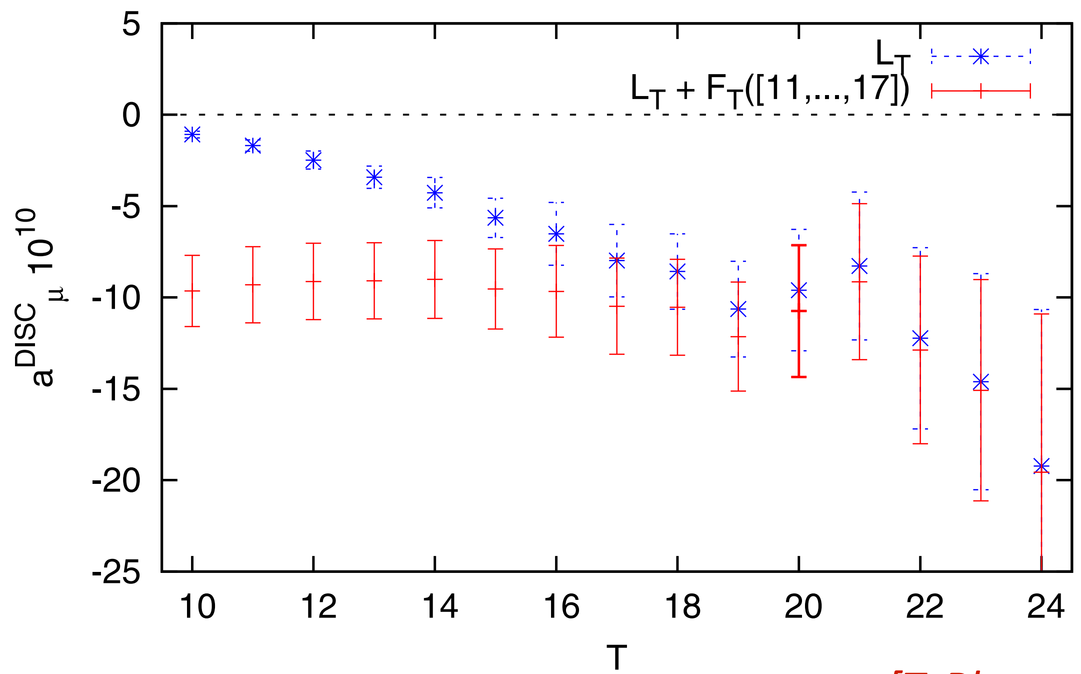

RBC/UKQCD Collaboration:

- Domain wall fermions; physical pion mass
- $a \approx 0.11$ fm, $m_\pi L \approx 3.9$
- Low-mode averaging, AMA

Monitor saturation of

$$L_T = \sum_{x_0=0}^{T} w(x_0)\, D^{\ell s}(x_0) \xrightarrow{T\to\infty} (a_\mu^{\rm hvp})_{\rm disc}$$

$$w(x_0) = 4\alpha^2 \int_0^\infty \frac{dQ^2}{Q^2}\, f(Q^2)\left\{Q^2 x_0^2 - 4\sin^2\left(\tfrac{1}{2}Qx_0\right)\right\}$$

[T. Blum et al., PRL 116 (2016) 232002]

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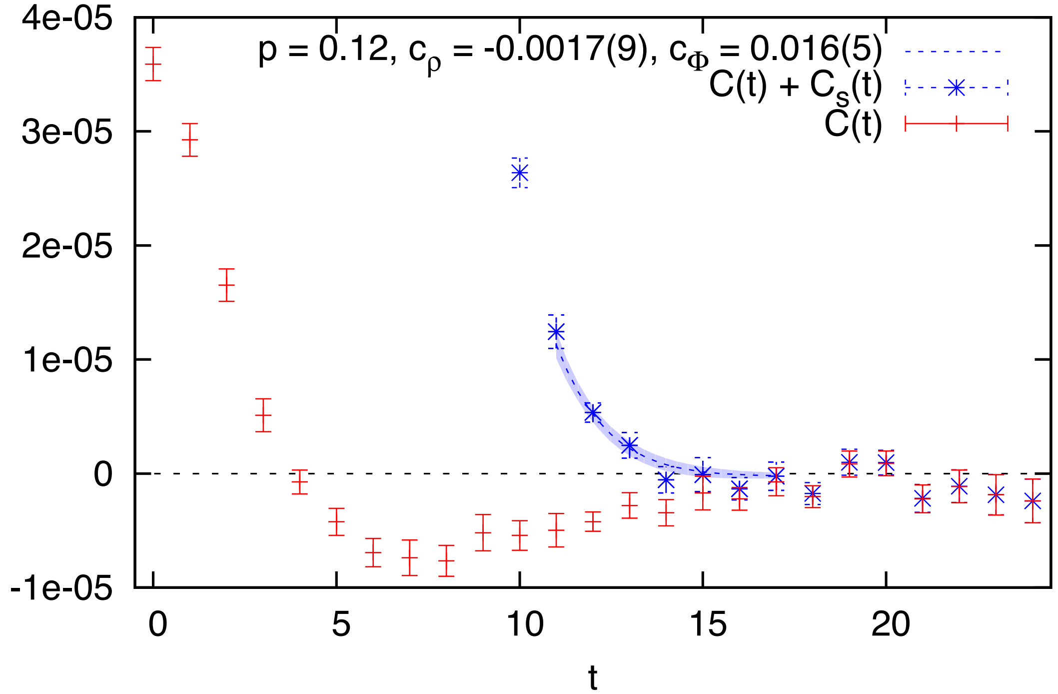

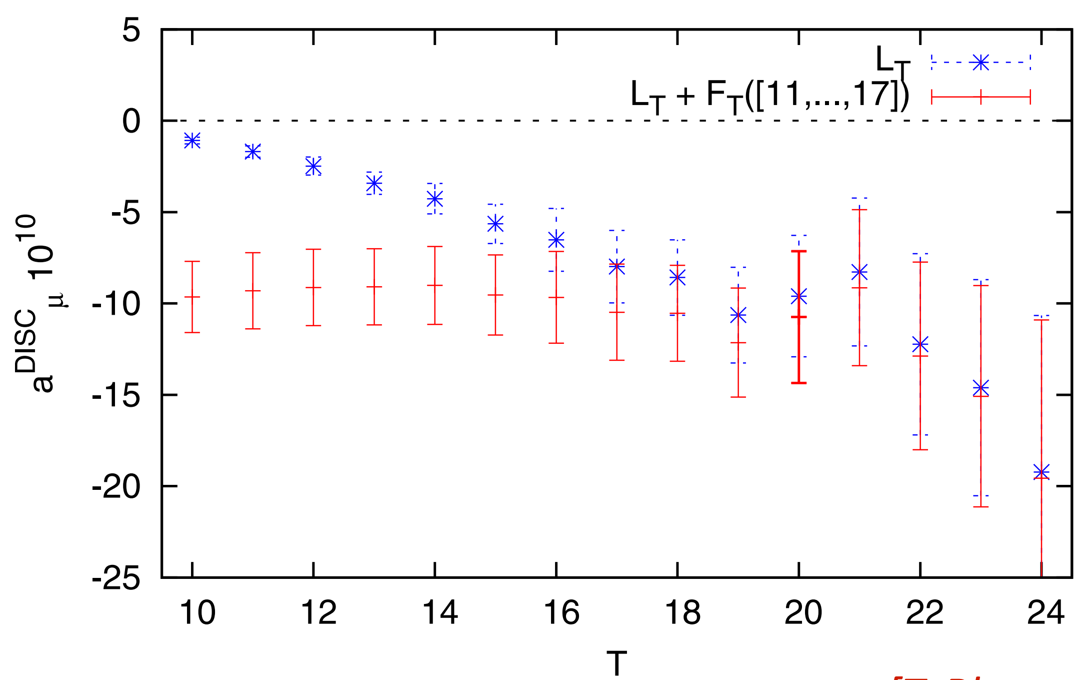

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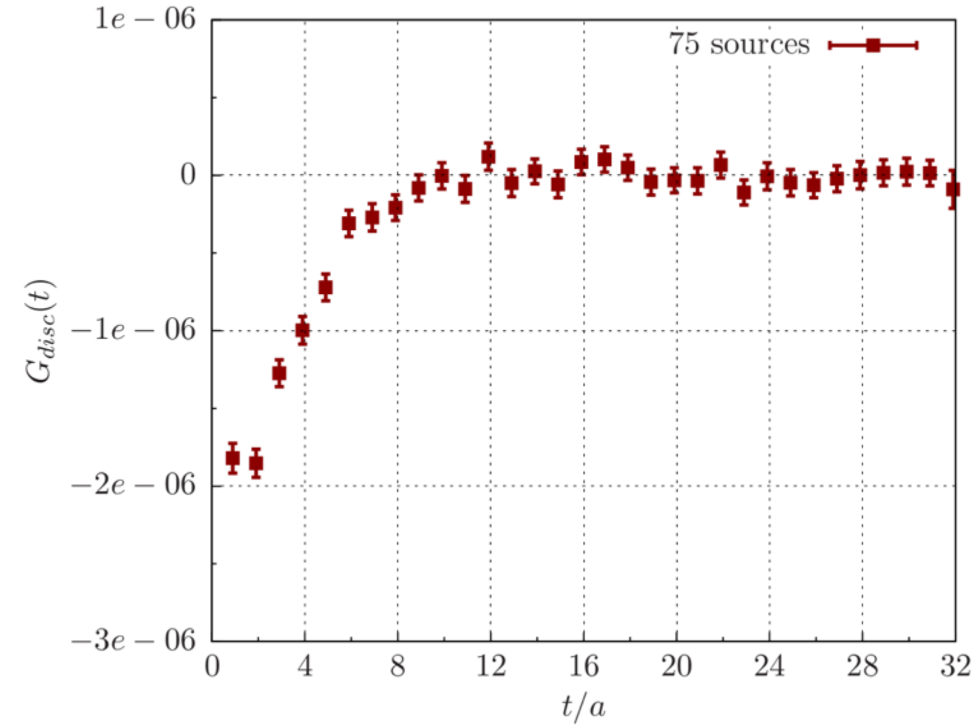
$$\Rightarrow \quad (a_\mu^{\rm hvp})_{\rm disc} = -(9.6 \pm 3.3 \pm 2.3)\cdot 10^{-10}$$

[T. Blum et al., PRL 116 (2016) 232002]

Disconnected Contributions

Mainz/CLS: *[Gülpers et al., in preparation]*

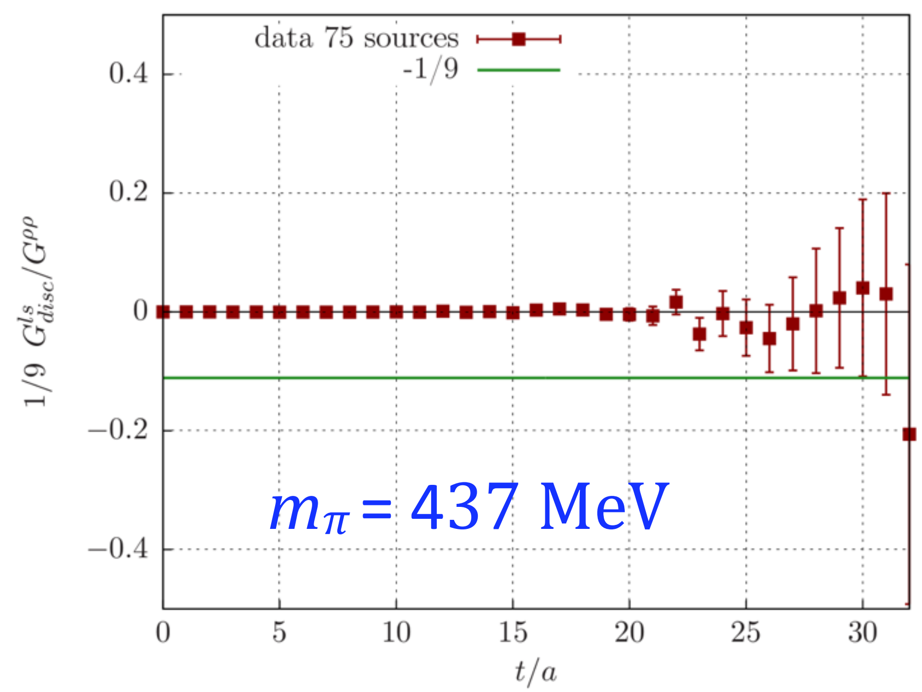
- $N_f = 2$ Clover fermions; $m_\pi = 311, 437$ MeV
- $a \approx 0.063$ fm, $m_\pi L > 4.0$
- HPE, stochastic noise cancellation
- Statistics: 4800 k measurements



$G^{\ell s}_{\rm disc}$ dominates uncertainty for $x_0 > 1.6$ fm

Disconnected contribution for $x_0 \longrightarrow \infty$:

$$-\frac{1}{9}\frac{G^{\ell s}_{\rm disc}}{G^{\rho\rho}} \xrightarrow{x_0\to\infty} -\frac{1}{9}$$



Upper bound on disconnected contribution:

$$G^{\ell s}_{\rm disc} = \begin{cases} 0, & x_0 \leq 1.6\,\text{fm} \\ -1/9, & x_0 > 1.6\,\text{fm} \end{cases}$$

$\Rightarrow$ sub-percent effect

Disconnected Contributions: Results Summary

* Non-zero disconnected contribution can be resolved

HPQCD: Anisotropic Clover action; $m_\pi = 391$ MeV; $a_s \approx 0.12$ fm; Distillation

$$(a_\mu^{\rm hvp})_{\rm disc}/(a_\mu^{\rm hvp})_{\rm con}^{(\ell\ell)} = -0.14(5)\%, \quad (a_\mu^{\rm hvp})_{\rm disc} \approx -0.84\cdot 10^{-10}$$

RBC/UKQCD: Domain wall fermions; physical pion mass; $a \approx 0.11$ fm, $m_\pi L \approx 3.9$;

$$(a_\mu^{\rm hvp})_{\rm disc}/(a_\mu^{\rm hvp})_{\rm con}^{(\ell\ell)} = -1.6(7)\%, \quad (a_\mu^{\rm hvp})_{\rm disc} = -(9.6\pm 3.3\pm 2.3)\cdot 10^{-10}$$

CLS/Mainz: $N_{\rm f} = 2$ Clover fermions; $m_\pi = 311, 437$ MeV; $a = 0.063$ fm;

$$(a_\mu^{\rm hvp})_{\rm disc}/(a_\mu^{\rm hvp})_{\rm con}^{(\ell\ell)} < -1\%$$

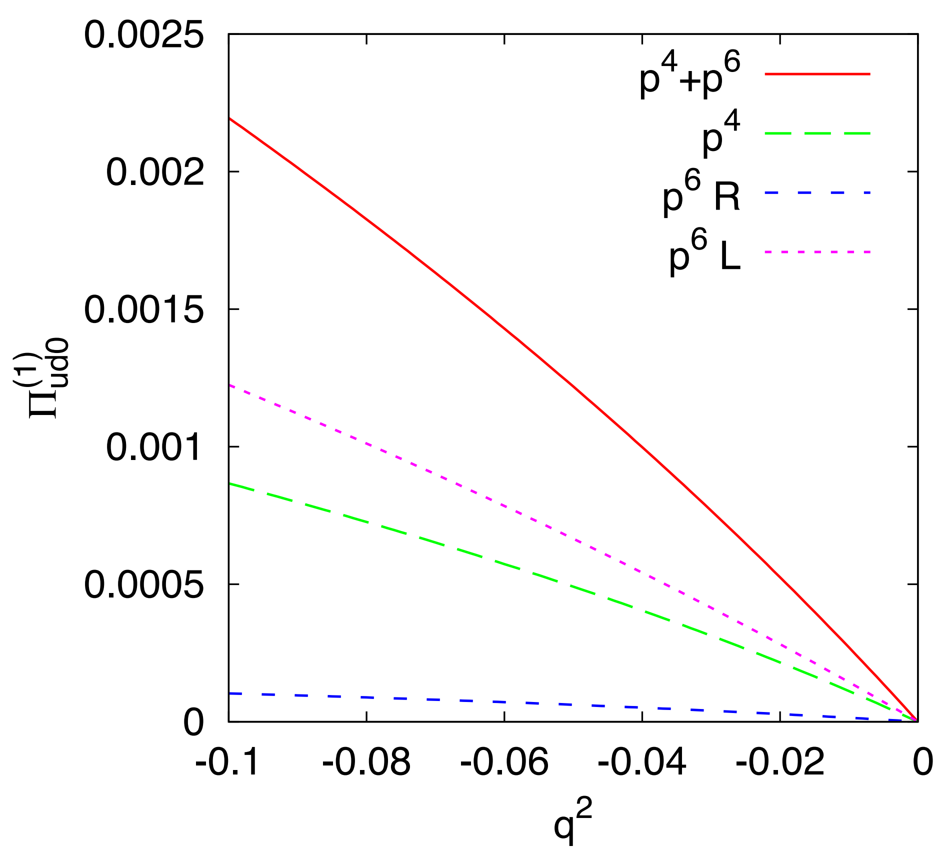
Bali & Endrődi: Rooted staggered fermions; physical pion mass; $a = 0.1 - 0.29$ fm;

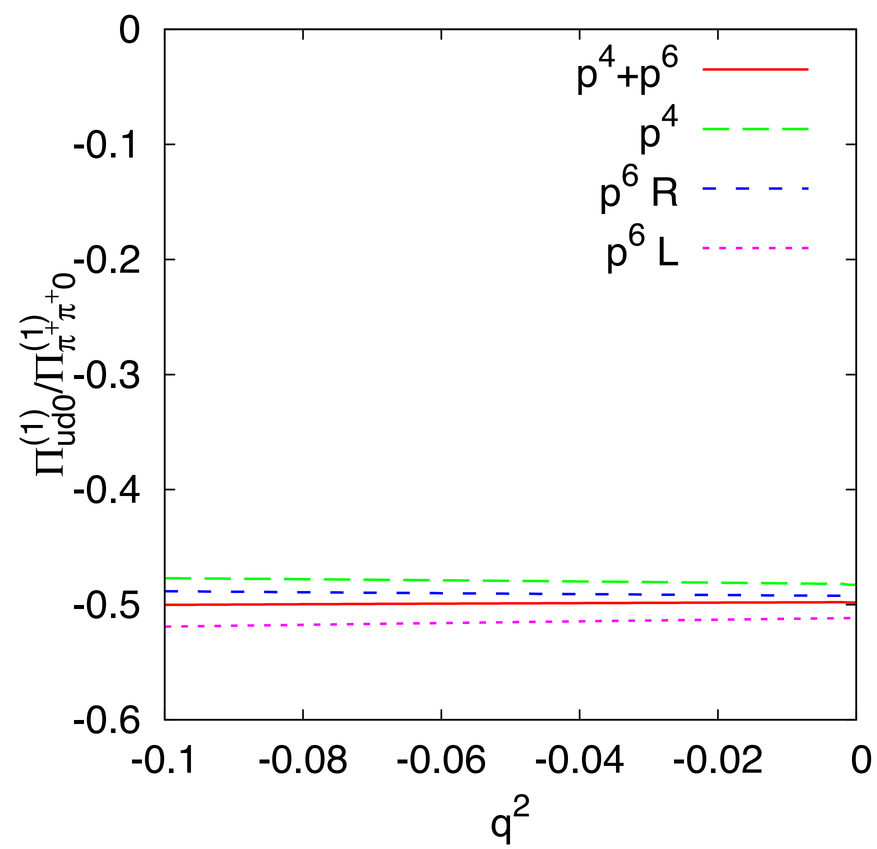
$$\Pi^{\rm disc}/\Pi^{\rm con} = -(3.6\pm 4.5)\cdot 10^{-4} \quad \text{at } Q^2 = 0.03\ {\rm GeV}^2$$

Chiral Perturbation Theory

* Two-loop calculations of connected vs. disconnected and effects of twisted boundary conditions in $\Pi_{\mu\nu}(Q)$ *[Hans BIJNENS, THU 14:00]*

* NLO ChPT estimate: $\Pi^{\rm disc}/\Pi^{\rm con} = -1/10$ *[Jüttner & Della Morte, JHEP 1011 (2010) 154]*





⇒ Corrections are large, but not in the ratio $\Pi^{\rm disc}/\Pi^{\rm con}$

Finite-volume effects

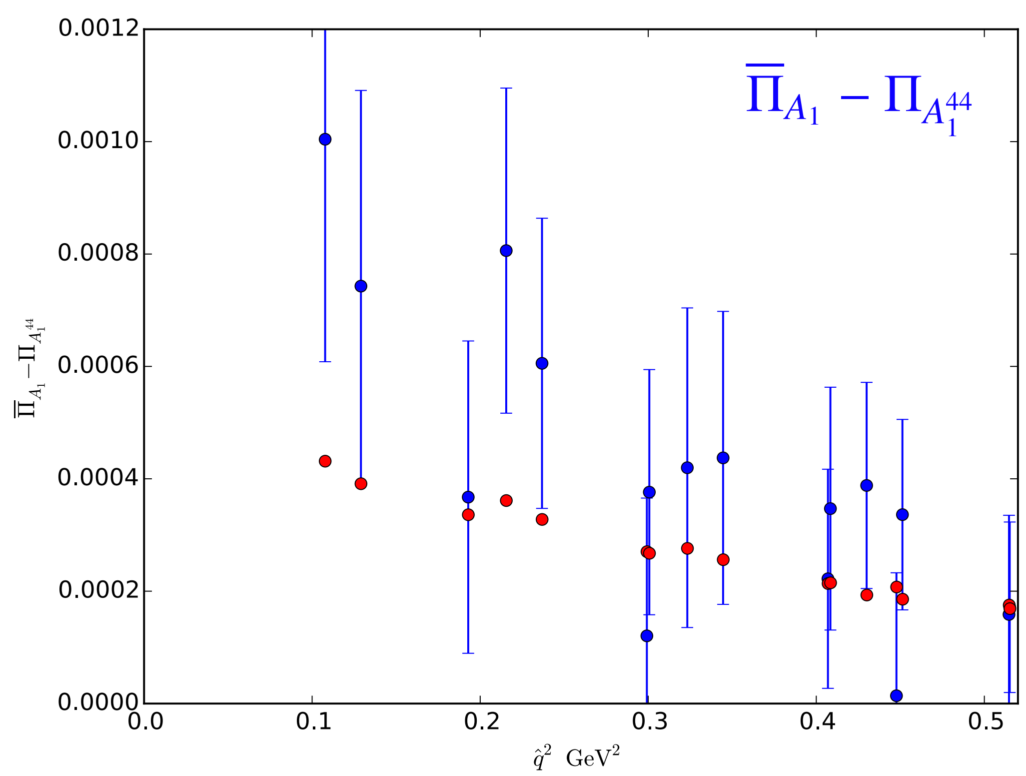
Finite-volume effects: Anisotropy studies

* Consider subtracted VP tensor in a finite volume of $L^3 \cdot T$:

$$\overline{\Pi}_{\mu\nu}(p) = \sum_{\kappa,\lambda} P^T_{\mu\kappa}\Big(\Pi_{\kappa\lambda}(p) - \Pi_{\kappa\lambda}(0)\Big) P^T_{\lambda\nu}$$ $\Rightarrow$ satisfies Ward Identities

$\Rightarrow$ contains five irreducible substructures: $A_1,\ A_1^{44},\ T_1,\ T_2,\ E$

* Study deviation between different irreps. for $m_\pi = 220\ \mathrm{MeV},\quad L = 3.8\ \mathrm{fm}$



- Finite-volume effects in $\Pi_{\mu\nu}$ well described by SChPT@LO:
- Impact on HVP contribution:

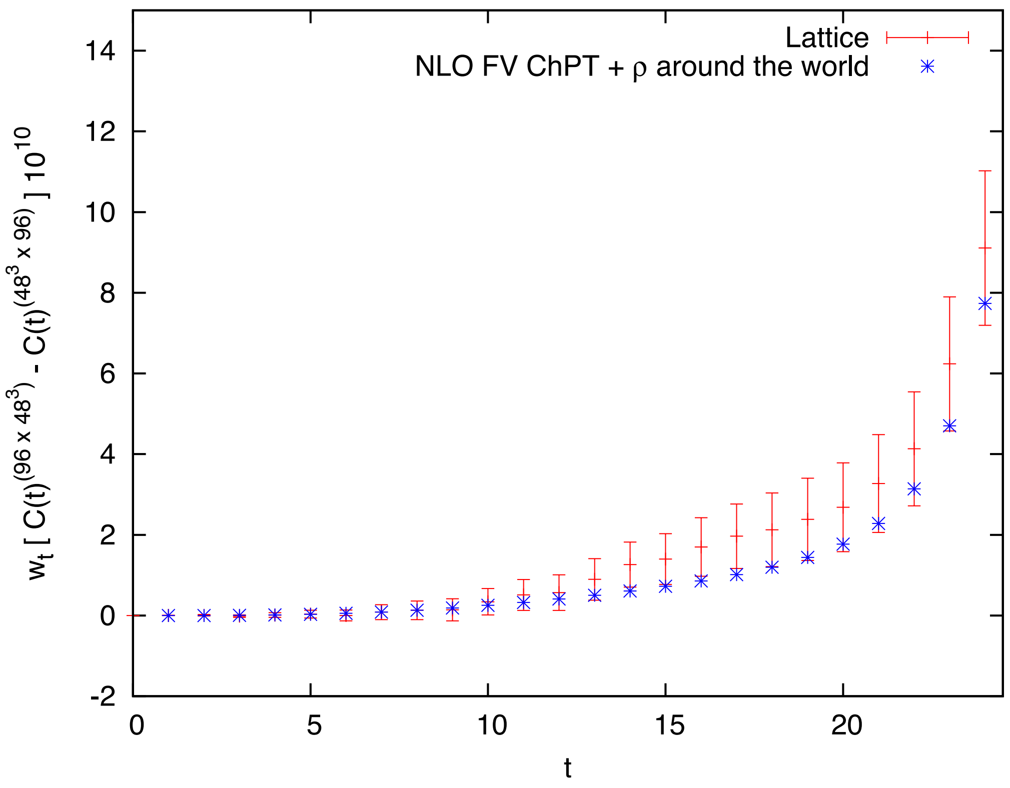
$$a^{\mathrm{hvp}}_{\mu,A_1} - a^{\mathrm{hvp}}_{\mu,A_1^{44}} = 10 - 15\%$$

- Simulations with $m_\pi = 220\ \mathrm{MeV},\ m_\pi L = 4.2$ not sufficient for percent-level accuracy

[Aubin et al., PRD 93 (2016) 054508]

Finite-volume effects: Anisotropy studies

* Compare vector correlator along spatial and temporal directions



[Christoph LEHNER, TUE 14:00]

$$w(x_1)\,G^{(L)}(x_1) - w(x_0)\,G^{(T)}(x_0)$$

(domain wall fermions)

* Anisotropy well described by FV ChPT after removing backward propagating ρ-meson

* FV correction for $m_\pi = 140$ MeV, $L = 5.3$ fm: $a_\mu^{\rm hvp}(\infty) - a_\mu^{\rm hvp}(L) \approx 3\%$

Finite-volume effects: taste breaking

* Finite volume effects for calculations using staggered (HISQ) quarks

* Consider effective theory of photons, pions and rho-mesons; compute hadronic contributions to photon propagator:

$$\Pi(Q^2) - \Pi(0) = -\frac{4Q^2}{3}\int \frac{d^3k}{(2\pi)^3}\, F(E_a, E_b, \boldsymbol{k}) + \cdots$$

* Taylor expansion for $m_{a,b} = m_\pi$ yields coefficients $\Pi_j^{(\pi\pi)}$ (similarly for $\Pi_j^{(\rho)}$)

* Replace integral by a finite sum over discrete momenta $\boldsymbol{k}$

[Chakraborty et al., arXiv:1601.03071, Christine DAVIES, poster session]

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* Replace integral by a finite sum over discrete momenta $\boldsymbol{k}$

* Average over taste multiplets and determine shift in $\Pi_j^{(\pi\pi)}$

  $m_\pi = 140$ MeV, $L = 4.5$ fm:

$$\delta\Pi_1/\Pi_1 \approx 10\% \quad \Rightarrow \quad a_\mu^{\rm hvp}(\infty) - a_\mu^{\rm hvp}(L) \approx 7\%$$

[Chakraborty et al., arXiv:1601.03071, Christine DAVIES, poster session]

Finite-volume effects: TMR analysis

* Starting point: $a_\mu^{\rm hvp}(L) = \int_0^\infty dx_0\, G(x_0, L)\, w(x_0)$

* Small x_0: Compute $G(x_0, \infty) - G(x_0, L)$ using Poisson-resummation

* Large x_0: Relate $G(x_0, L)$ to low-lying energy eigenstates on a torus

$$G(x_0, \infty) = \int_0^\infty d\omega\, \omega^2 \rho(\omega^2) {\rm e}^{-\omega|x_0|} = \frac{1}{48\pi^2} \int_0^\infty d\omega\, \omega^2 (1 - 4m_\pi^2/\omega^2)^{3/2} |F_\pi(\omega)|^2 {\rm e}^{-\omega x_0}$$

* Finite volume: $G(x_0, L) = \sum_n |A_n|^2\, {\rm e}^{-\omega_n x_0}, \quad \omega_n = 2\sqrt{m_\pi^2 + k_n^2}$

$$\delta_{11}(k) + \phi\Big(\frac{kL}{2\pi}\Big) = n\pi, \quad n = 1, 2, \ldots$$

[M. Lüscher 1991]

$$|F_\pi(\omega)|^2 = \left\{ (z\phi'(z))_{z=kL/2\pi} + k\frac{\partial\delta_1(k)}{\partial k} \right\} \frac{3\pi\omega^2}{2k^2}\, |A|^2$$

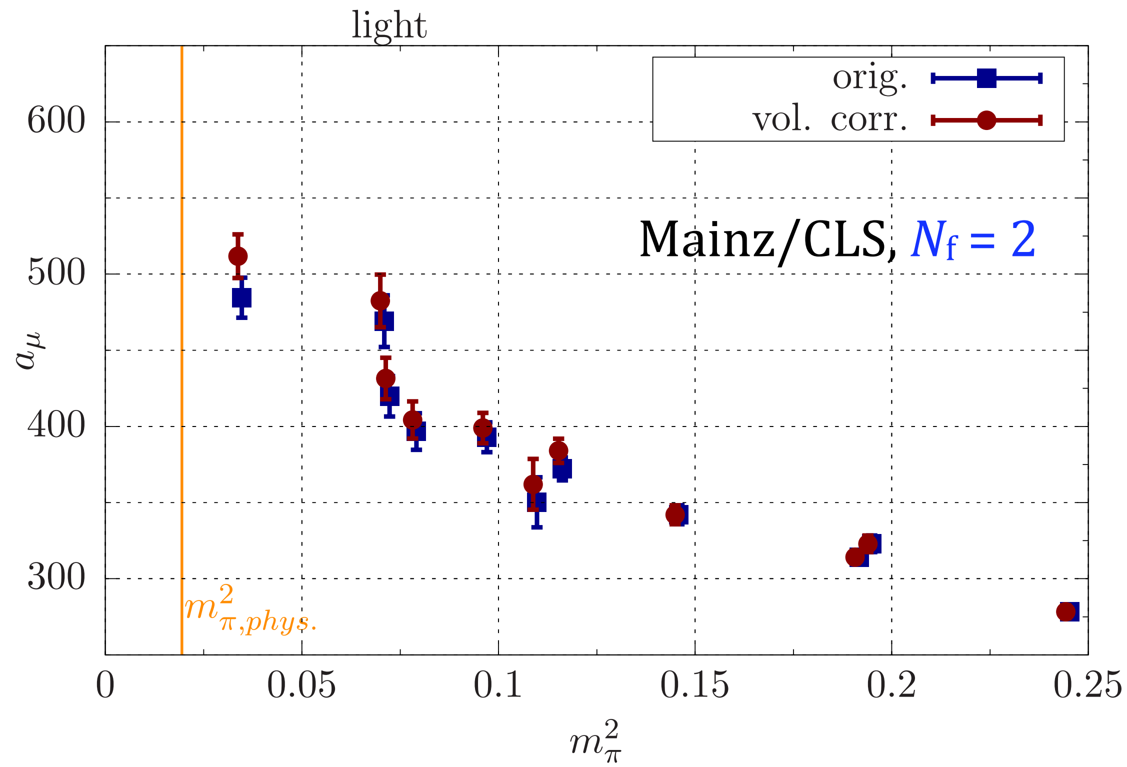
[H.B. Meyer, PRL 107 (2011) 072002]

[A. Francis et al., PRD 88 (2013) 054502]

Finite-volume effects: TMR analysis

- Input quantity: timelike pion form factor $F_\pi(\omega) = |F_\pi(\omega)|\,\mathrm{e}^{i\delta_{11}(k)}$
- Use Gounaris-Sakurai parameterisation and evaluate $|F_\pi(\omega)|$, $\delta_{11}(k)$ for given (m_π, m_ρ) of a given gauge ensemble
- Finite-volume effects in HVP dominated by long-distance contribution
- For $m_\pi = 190$ MeV, $L = 4.0$ fm, $m_\pi L = 4.0$:

$$a_\mu^{\rm hvp}(\infty) - a_\mu^{\rm hvp}(L) = 5.2\%$$



[H.B. Meyer et al., in preparation]

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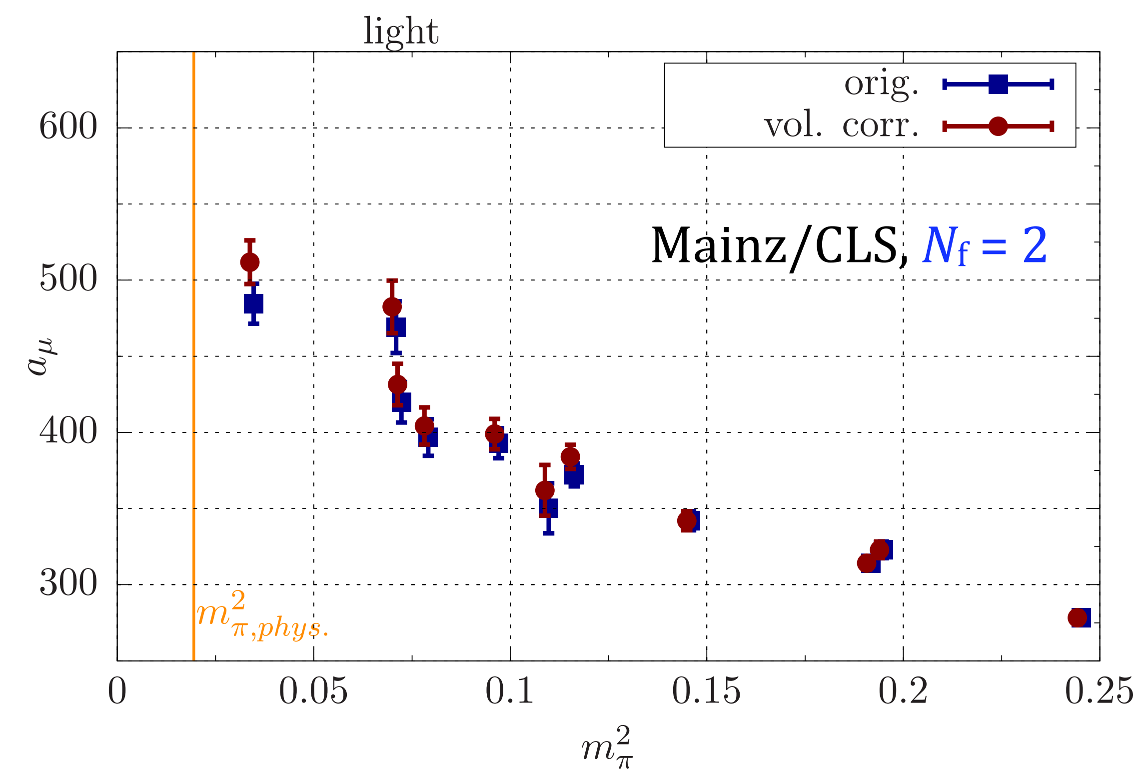
* Procedural variations:
  assign uncertainty of $\approx 10\%$

⇒ Dynamical theory of finite-volume effects in terms of m_ρ/m_π and $m_\pi L$



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Recent results: HPQCD

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  $N_f = 2+1+1$ flavours of staggered quarks (HISQ)

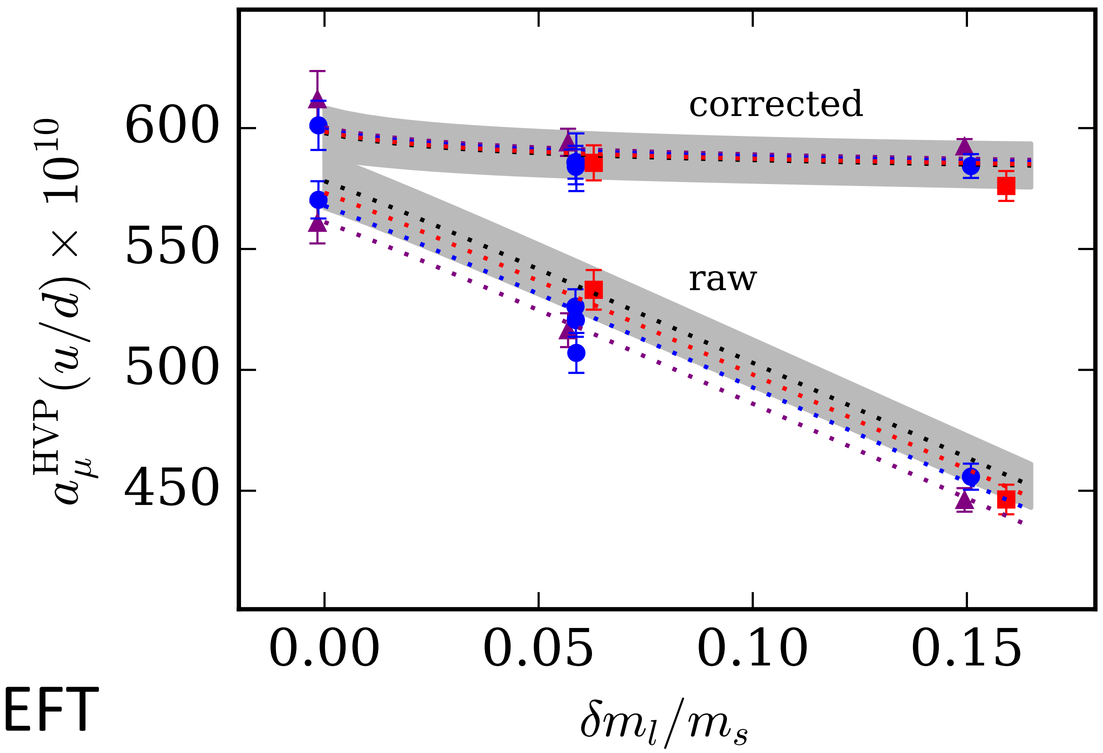
  10 ensembles; three lattice spacings: $a = 0.09, 0.12, 0.15$ fm

  Physical pion mass: $m_\pi^{\rm min} L = 3.9$

  Statistics: ≈ 16000 per ensemble

- $\Pi(Q^2) - \Pi(0)$ determined from time moments

- Reduce $m_{u,d}$-dependence of $a_\mu^{\rm hvp}$:
  - Rescale $\pi^+\pi^-$ contribution in continuum EFT
  - Rescale time moment Π_j by $(m_\rho^{\rm lat}/m_\rho^{\rm phys})^{2j}$



- Combined chiral and continuum extrapolation using Bayesian priors

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$$(a_\mu^{\rm hvp})_{\rm con} \cdot 10^{10} = \begin{cases} 598 \pm 11 & (u,d) \\ 53.4 \pm 0.6 & (s) \\ 14.4 \pm 0.4 & (c) \end{cases}$$

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contains finite-volume and isospin corrections

Recent results: HPQCD

* Simulation details: [Chakraborty et al., arXiv:1601.03071, Christine DAVIES, poster session]

  $N_f = 2+1+1$ flavours of staggered quarks (HISQ)

  10 ensembles; three lattice spacings: $a = 0.09, 0.12, 0.15$ fm

  Physical pion mass: $m_\pi^{\rm min} L = 3.9$

  Statistics: ≈ 16000 per ensemble

* Results:

$$(a_\mu^{\rm hvp})_{\rm con} \cdot 10^{10} = \begin{cases} 598 \pm 11 & (u,d) \\ 53.4 \pm 0.6 & (s) \\ 14.4 \pm 0.4 & (c) \end{cases}$$

contains finite-volume and isospin corrections

* Disconnected contribution: $(a_\mu^{\rm hvp})_{\rm disc} = (0 \pm 9) \cdot 10^{-10}$

* Final estimate: $a_\mu^{\rm hvp} = (666 \pm 6 \pm 12) \cdot 10^{-10}$

Recent results: RBC/UKQCD

* Simulation details: [Blum et al., JHEP 04 (2016) 063]

  $N_f = 2+1$ flavours; Möbius domain wall fermions

  Two lattice spacings: $a = 0.11,\ 0.084\ \mathrm{fm}$

  Physical pion mass: $m_\pi^{\min} L = 3.9$

  Noise reduction: AMA, deflation

Recent results: RBC/UKQCD

* Simulation details: *[Blum et al., JHEP 04 (2016) 063]*

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* Employ "Hybrid Method": *[Matthew SPRAGGS, TUE 17:10]*

  Padé fits, conformal polynomials in low-Q^2 regime, time moments

  Numerical integration techniques

* Lattice and experimental data; Finite-volume study *[Christoph LEHNER, TUE 14:00]*

* Isospin breaking *[James HARRISON, TUE 15:00, Vera GÜLPERS, TUE 15:20,]*

Recent results: RBC/UKQCD

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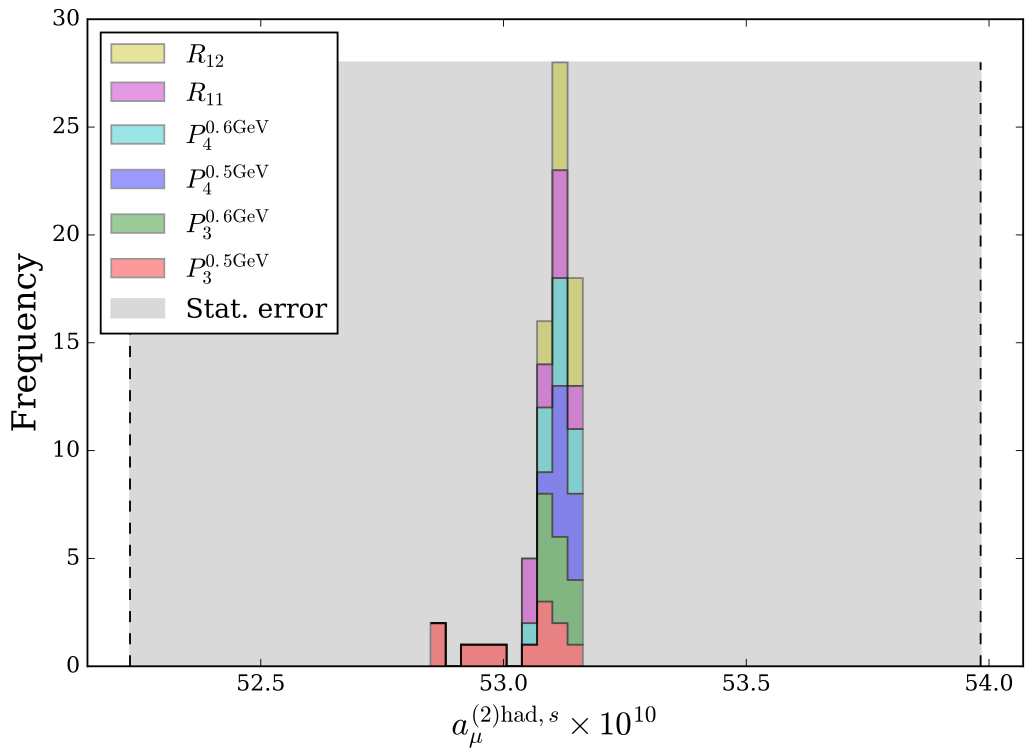
* "Hybrid method": systematic effects via procedural variations

* Strange quark contribution:

  [Blum et al., JHEP 04 (2016) 063]

$$a_\mu^{(s)\,\rm hvp} = 53.1(9)\binom{+1}{-3}\cdot 10^{-10}$$



[Matthew SPRAGGS, TUE 17:10]

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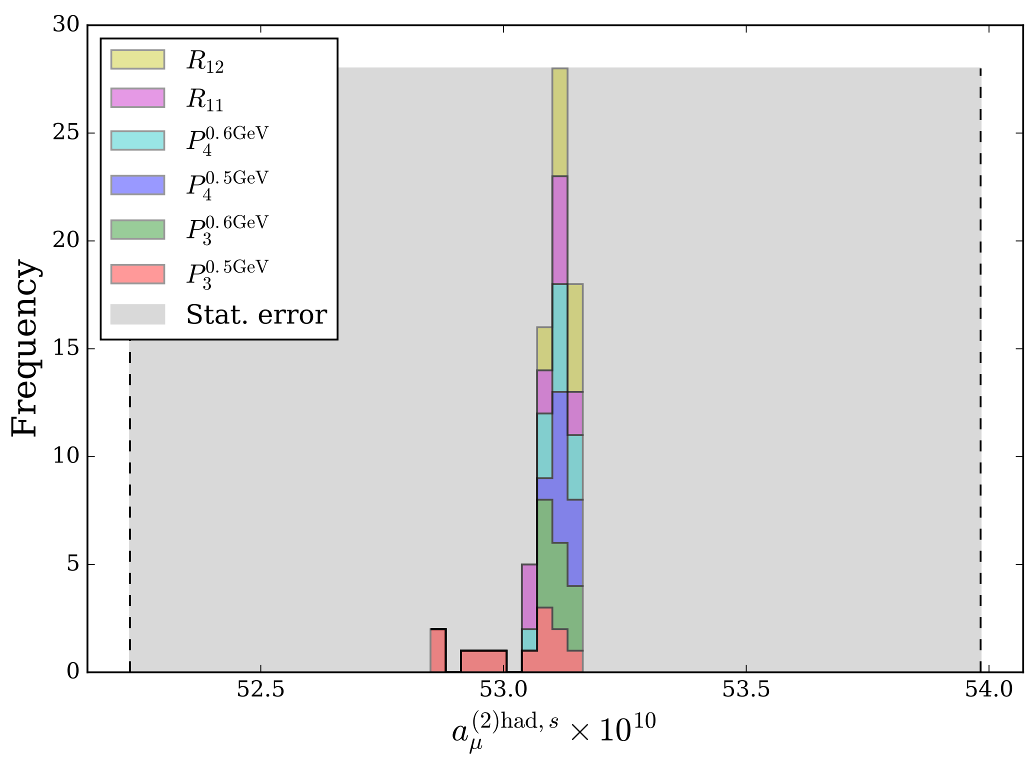
⇒ dominated by statistical error

[Matthew SPRAGGS, TUE 17:10]



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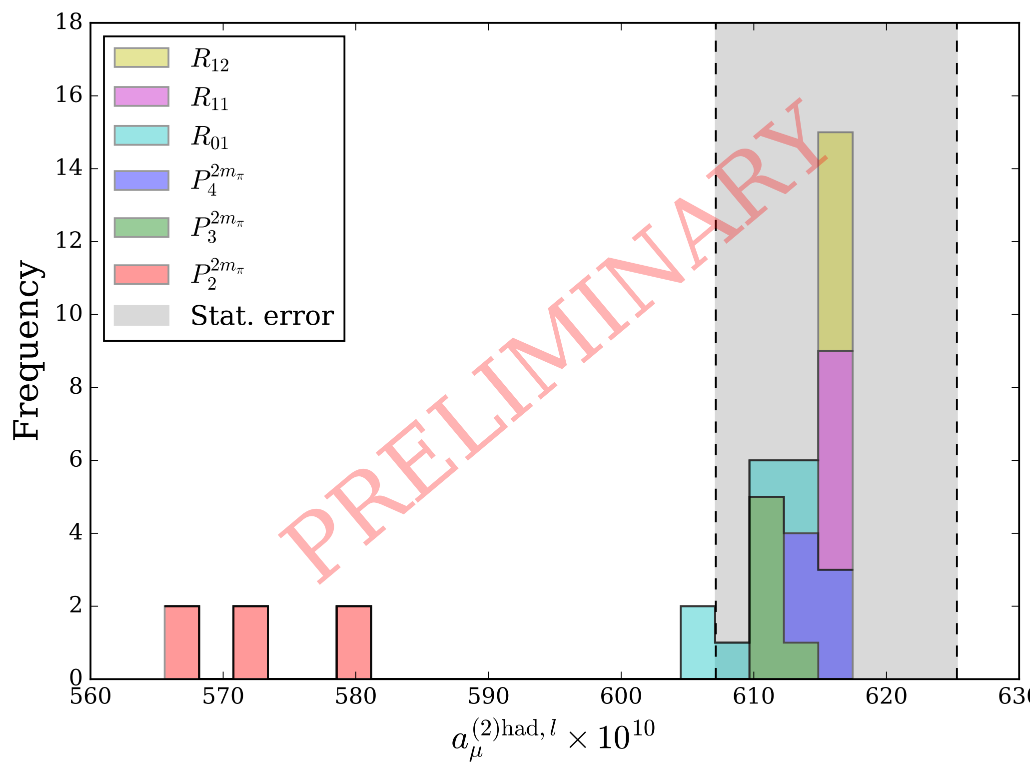
* Light quark contribution:



[Matthew SPRAGGS, TUE 17:10]

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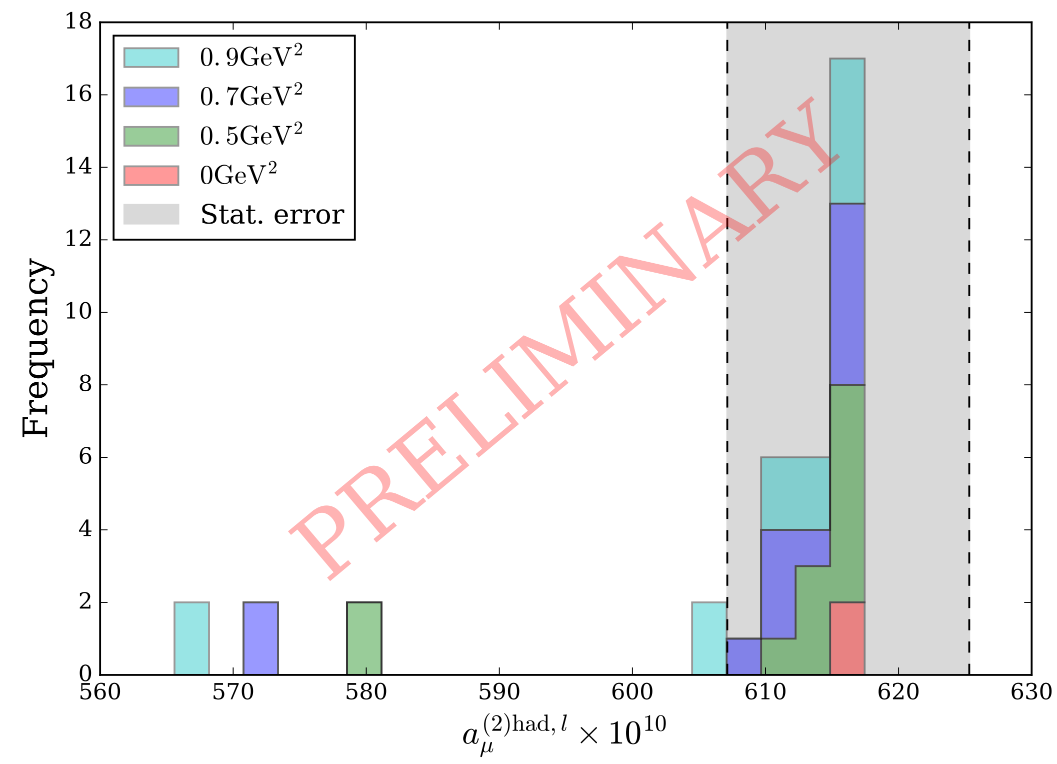

[Matthew SPRAGGS, TUE 17:10]

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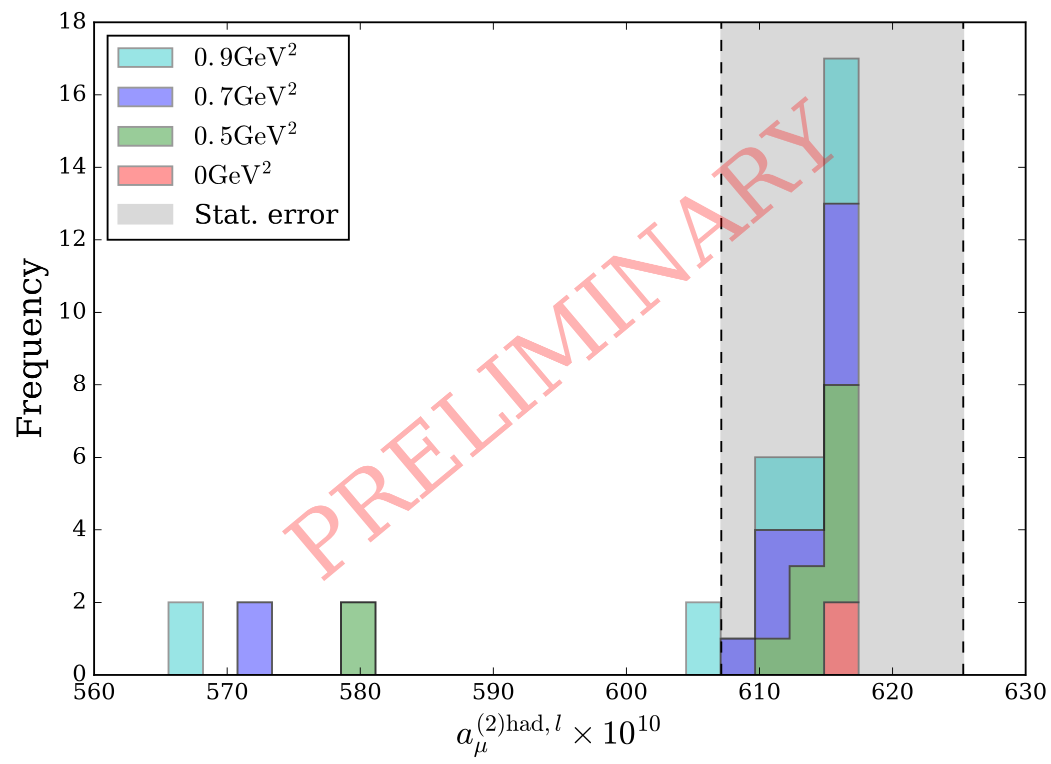
  $$a_\mu^{(s)\,\mathrm{hvp}} = 53.1(9)\left(^{+1}_{-3}\right)\cdot 10^{-10}$$

* Light quark contribution:

  $$\delta a_\mu^{(u,d)\,\mathrm{hvp}} / a_\mu^{(u,d)\,\mathrm{hvp}} \approx 3\%$$

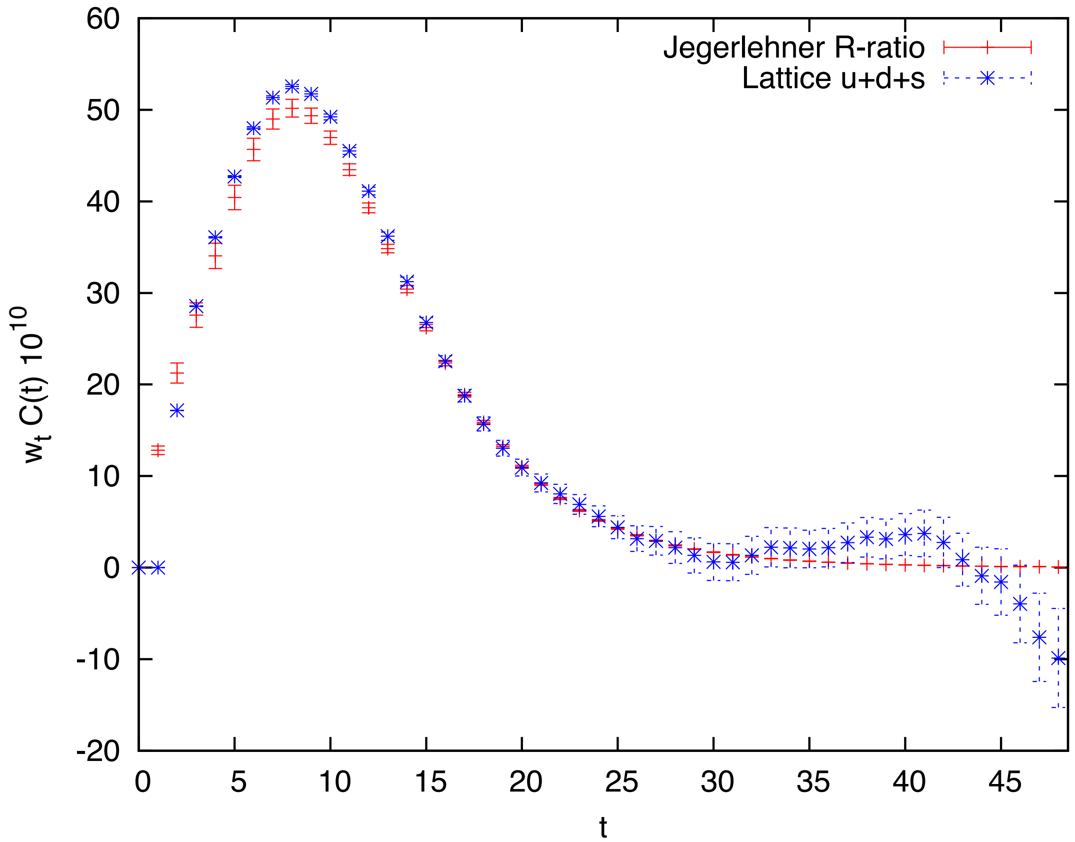
  [Matthew SPRAGGS, TUE 17:10]



Recent results: RBC/UKQCD

* Combining lattice and experimental data:

[Christoph LEHNER, TUE 14:00]



$$a_\mu^{\rm hvp} = \sum_{x_0=0}^{\infty} w(x_0)\, G(x_0),$$

- Experimental data more precise in long-distance regime

* Compute: $$a_\mu^{\rm hvp} = \sum_{x_0=0}^{x_0^{\rm lat/exp}} w(x_0)\, G^{\rm lat}(x_0)\Big|_{\rm con} + \sum_{x_0^{\rm lat/exp}}^{\infty} w(x_0)\, G^{\rm exp}(x_0)$$

$\Rightarrow\quad \delta a_\mu^{\rm hvp}/a_\mu^{\rm hvp} = 0.7\%$ at $x_0^{\rm lat/exp} = 1.7\,{\rm fm}$

Recent results: BMW

- Simulation details: *[Taichi KAWANAI, TUE 16:30, Kohtaroh MIURA, TUE 16:50]*

  $N_\mathrm{f} = 2+1+1$ flavours of stout-smeared staggered quarks; tree-level Symanzik

  17 ensembles; six lattice spacings: $a = 0.063 - 0.133\ \mathrm{fm}$

  Physical pion mass: $m_\pi^{\mathrm{min}} L \approx 4.2, \quad L \approx 6\,\mathrm{fm}$

  Statistics: $\approx 1.15\ \mathrm{M}$ for (u,d), $\approx 96\ \mathrm{k}$ for s, c

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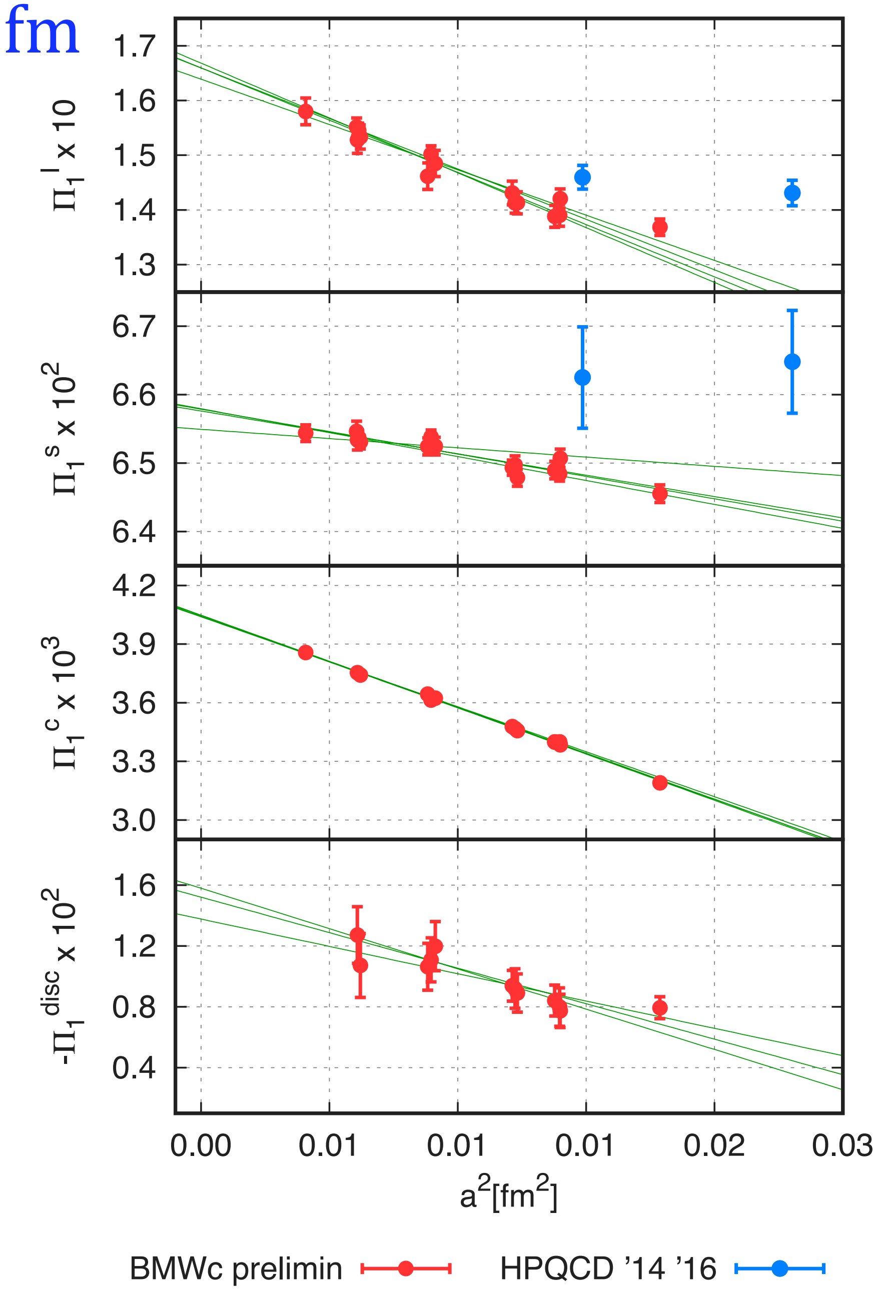

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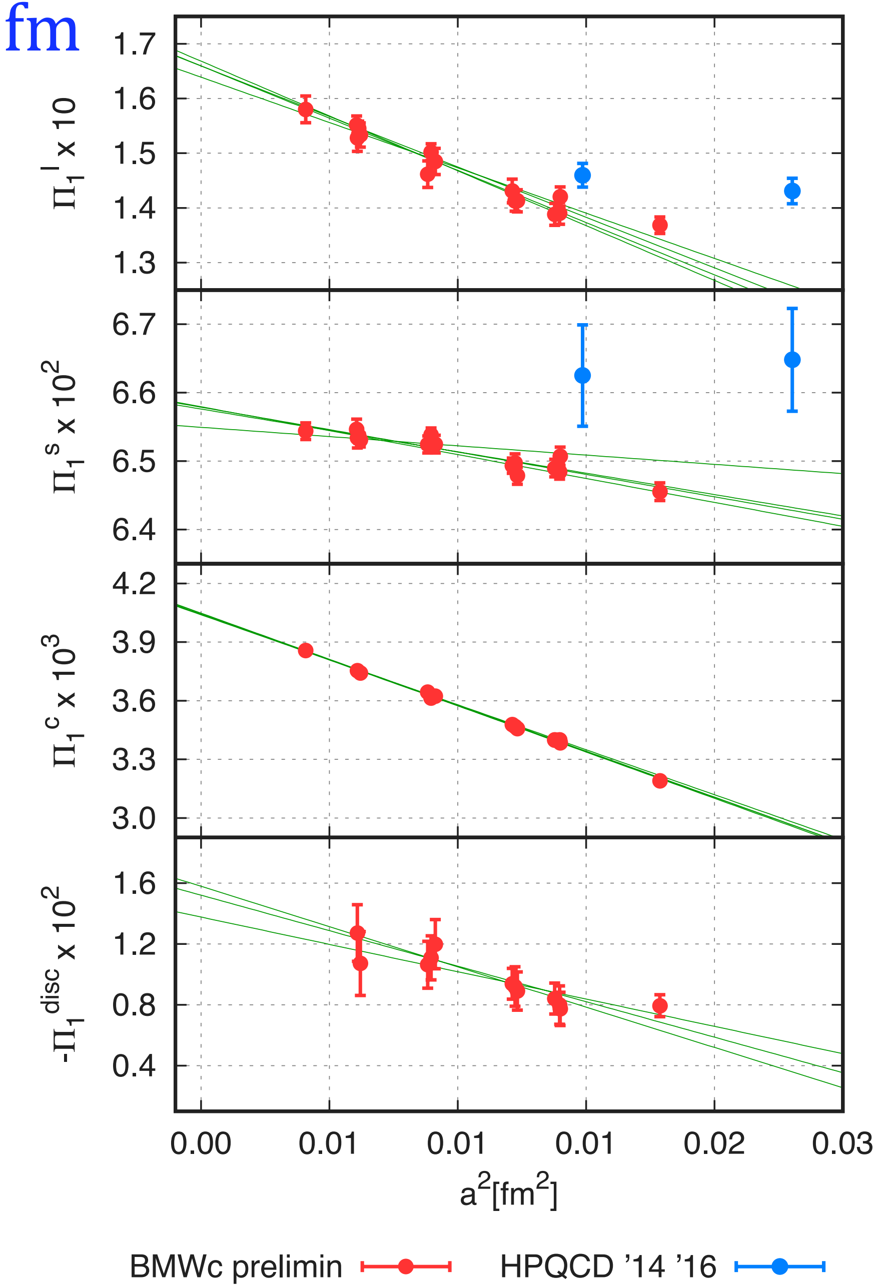

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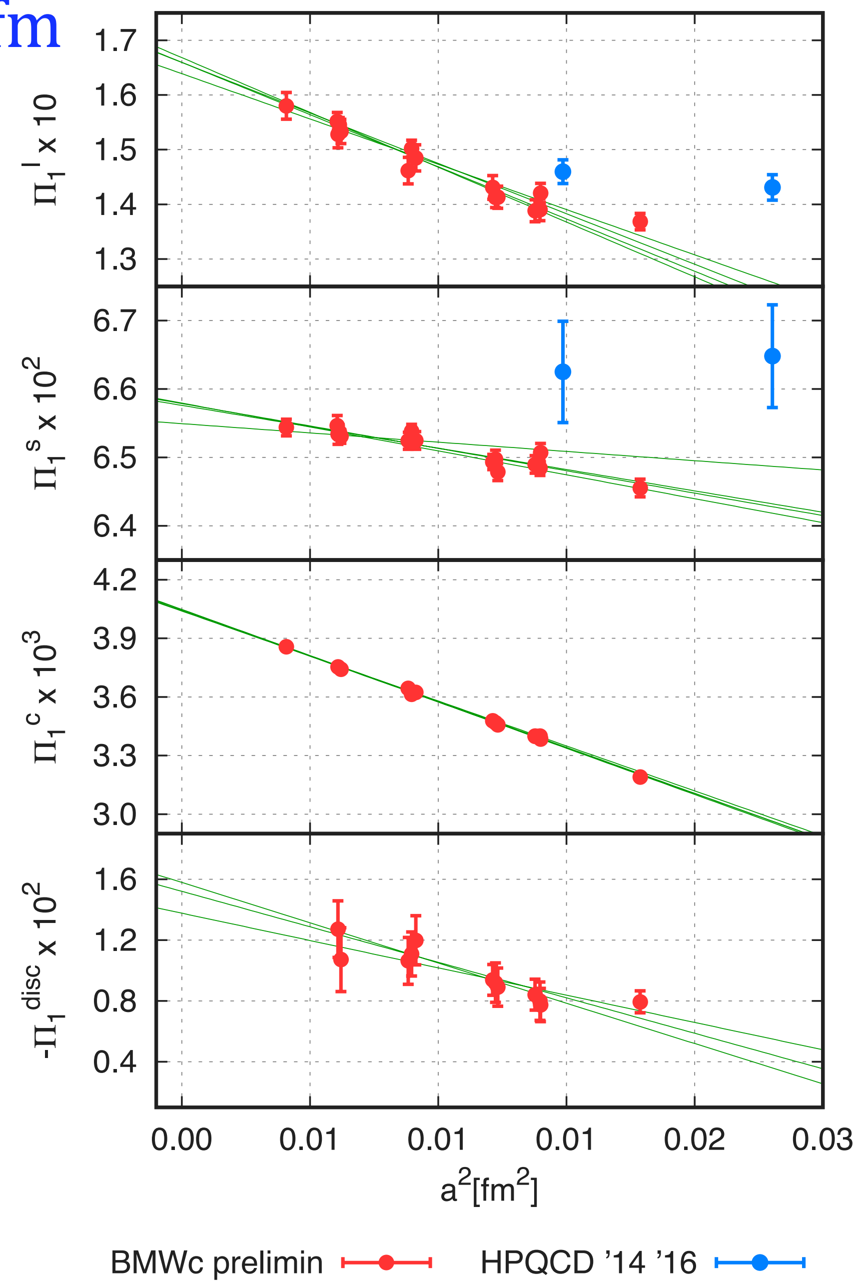
* Moments corrected for finite-volume effects in ChPT @ LO



Recent results: Mainz/CLS

[Hanno HORCH, TUE 14:40]

- Simulation details:

  $N_f = 2$ flavours; $O(a)$ improved Wilson fermions

  11 ensembles; three lattice spacings: $a = 0.049, 0.066, 0.076\,\text{fm}$

  Minimum pion mass: $m_\pi^{\text{min}} = 190\,\text{MeV}, \quad m_\pi^{\text{min}} L = 4.0$

  Statistics: $2000 - 4000$ per ensemble

- Determine $\Pi(Q^2) - \Pi(0)$ using Padé fits, time moments and TMR

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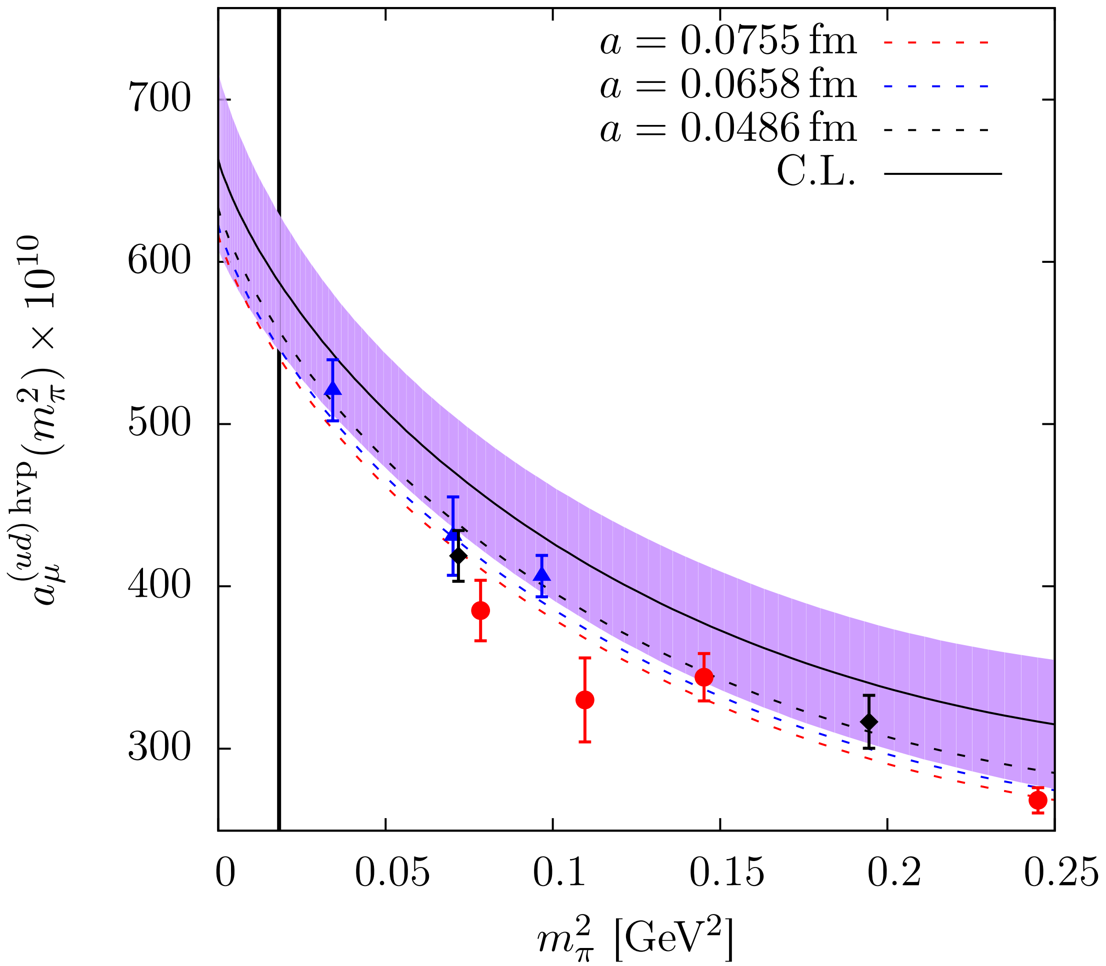
- Error estimates: "Extended Frequentist Method"



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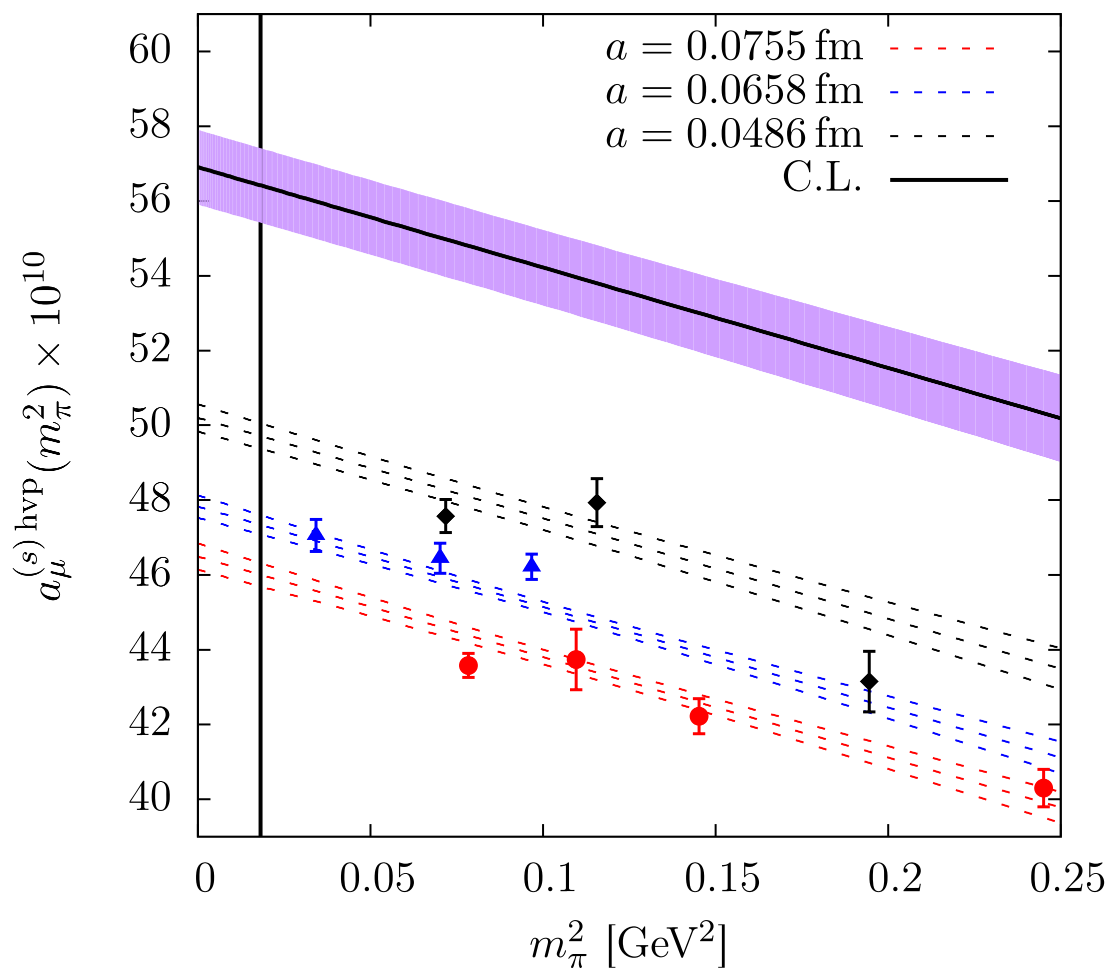

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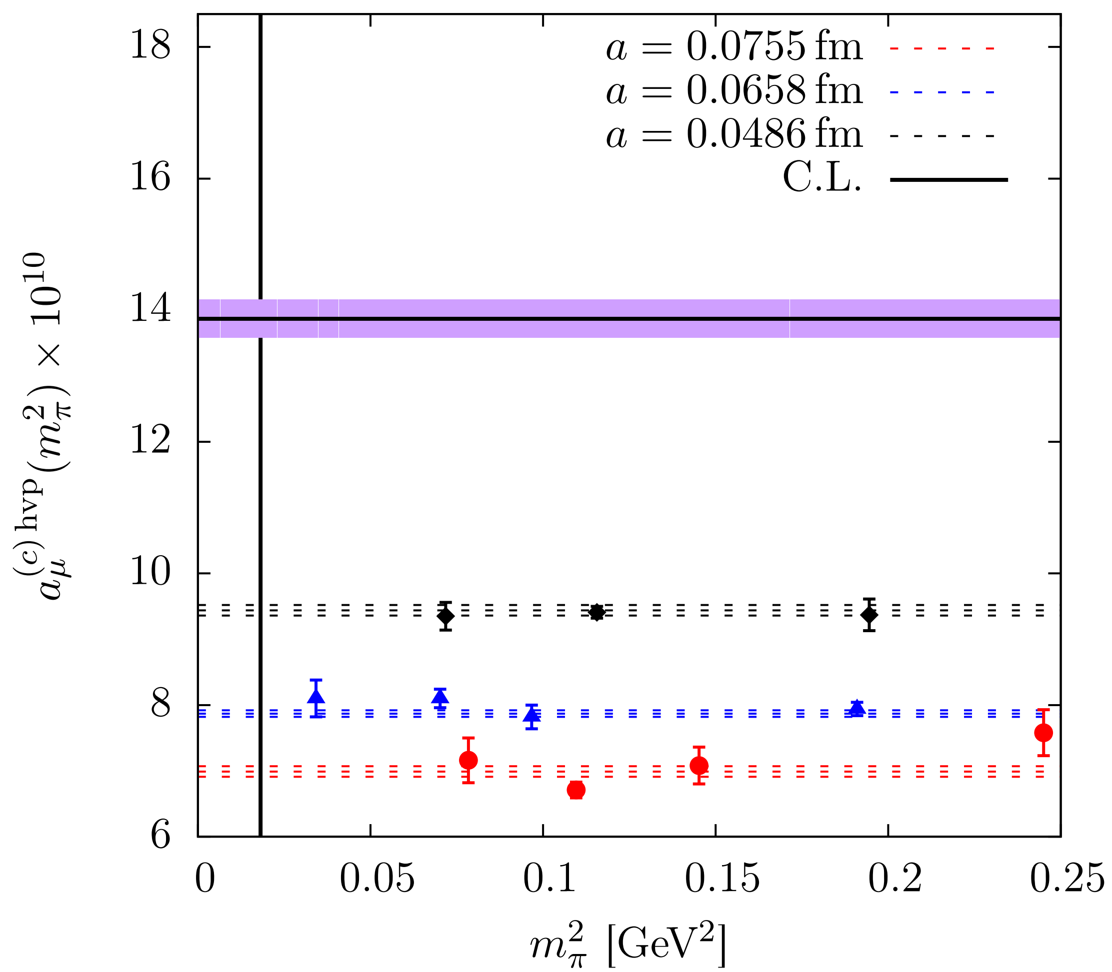
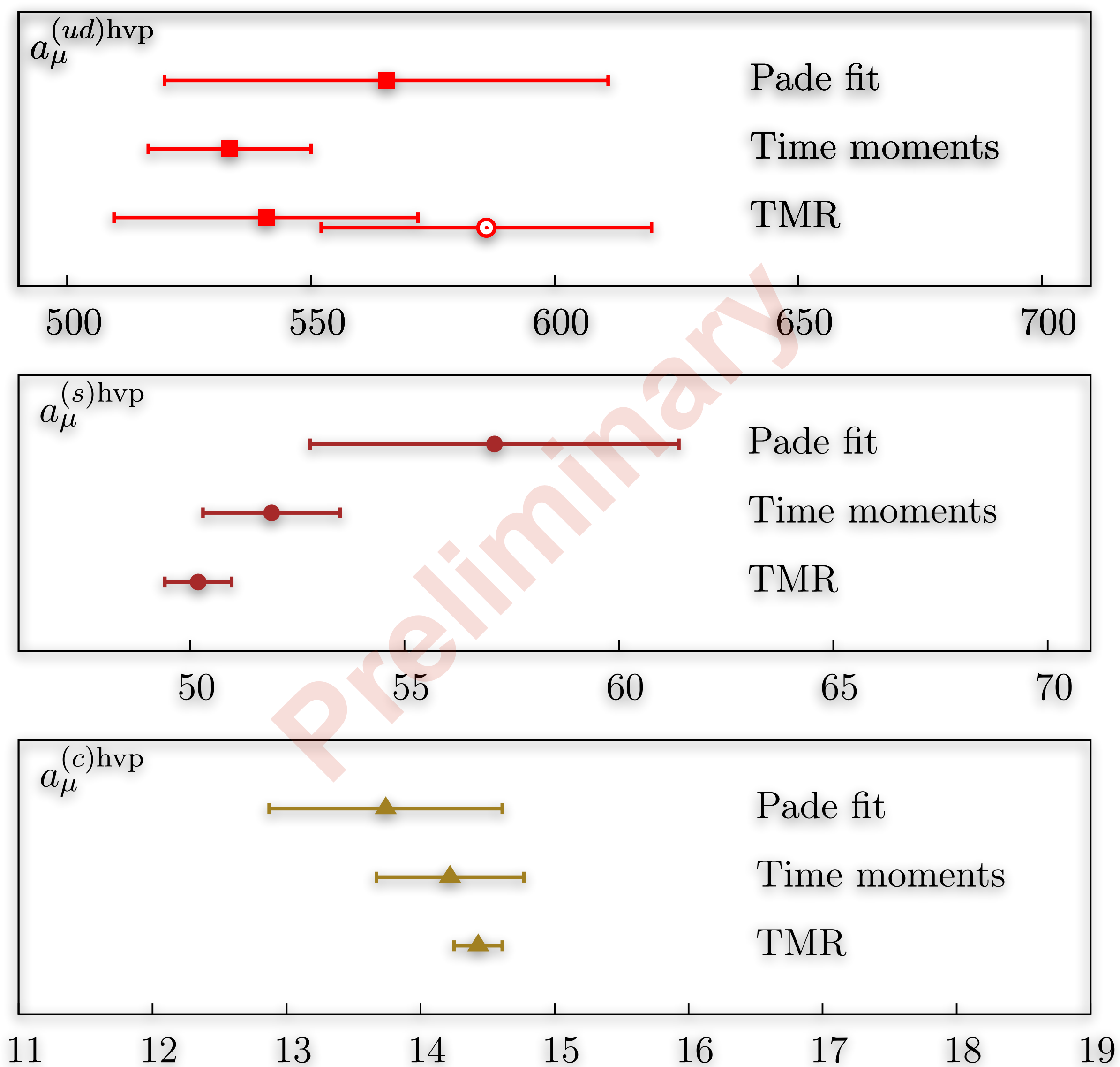


Recent results: Mainz/CLS

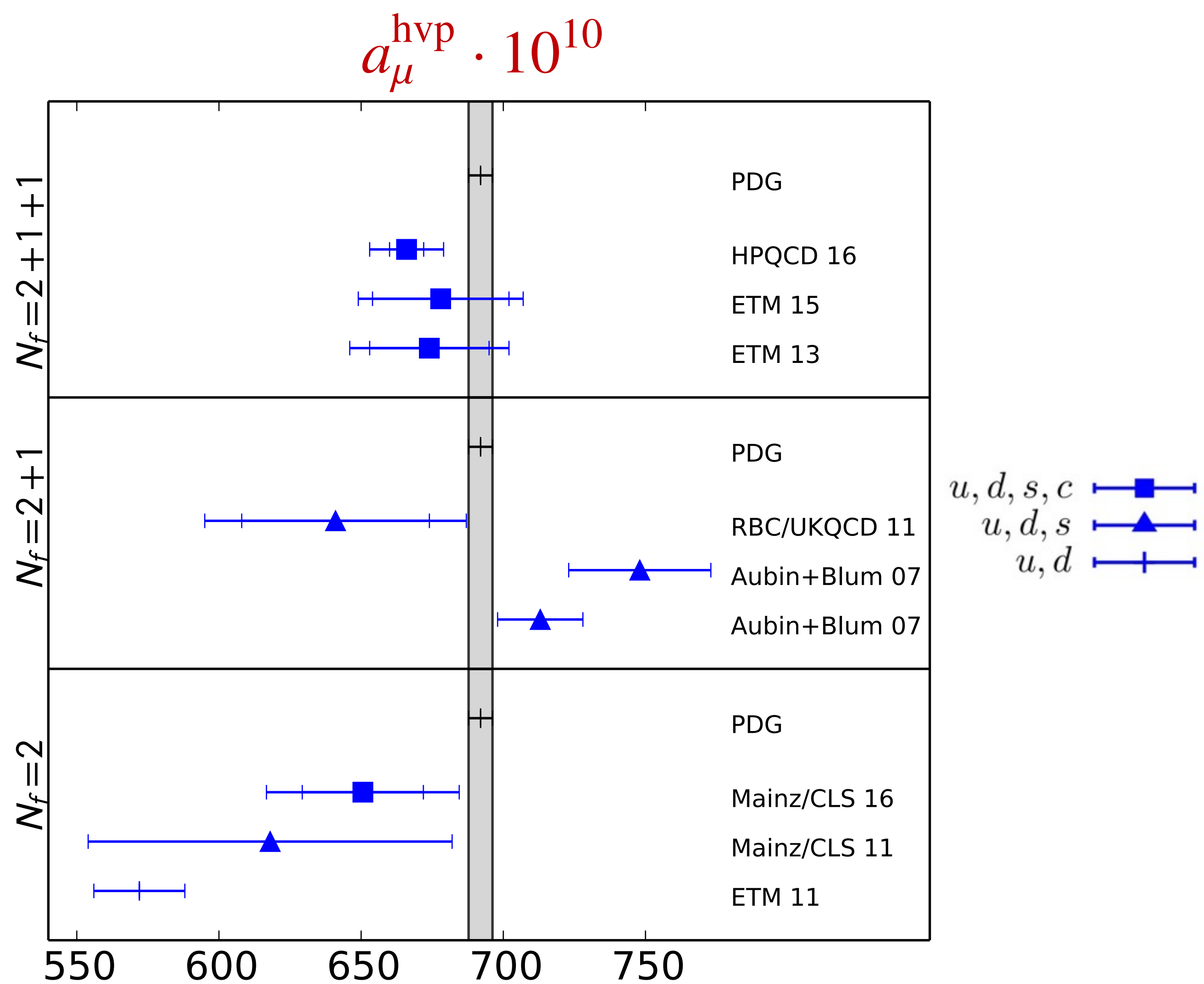
* Determine contributions from individual quark flavours: *(u,d), s, c*

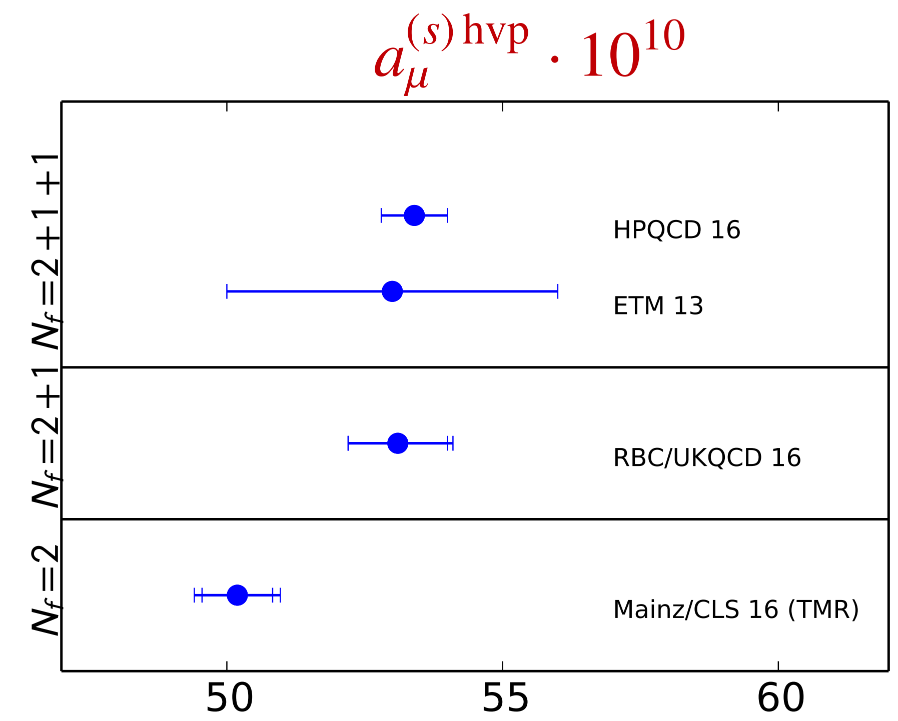


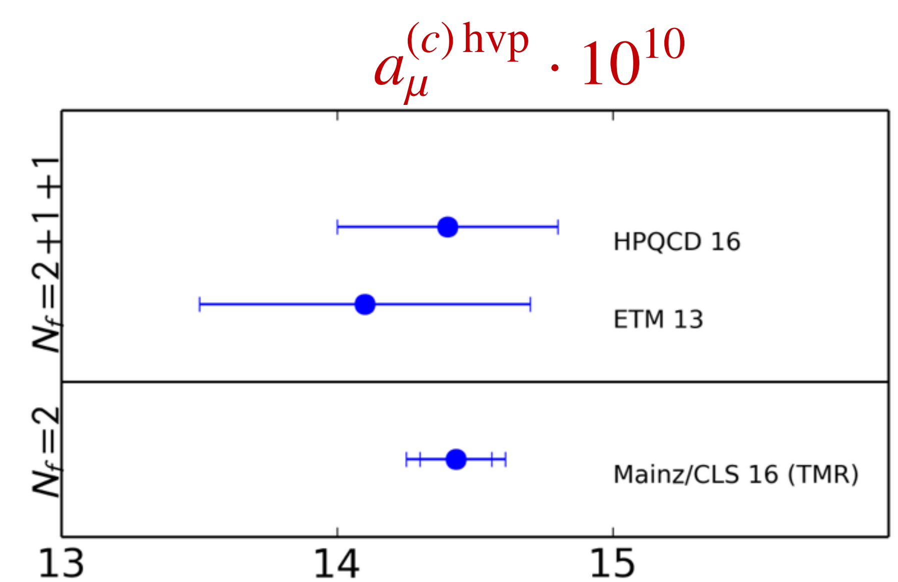
* Different methods consistent at the level of 1σ
* Significant shift due to finite-volume effects
* Overall accuracy dominated by *u, d* contribution
* Contributions from disconnected diagrams below 1%

[Hanno HORCH, TUE 14:40; Della Morte et al., in preparation]

Summary on $a_\mu^{\rm hvp}$



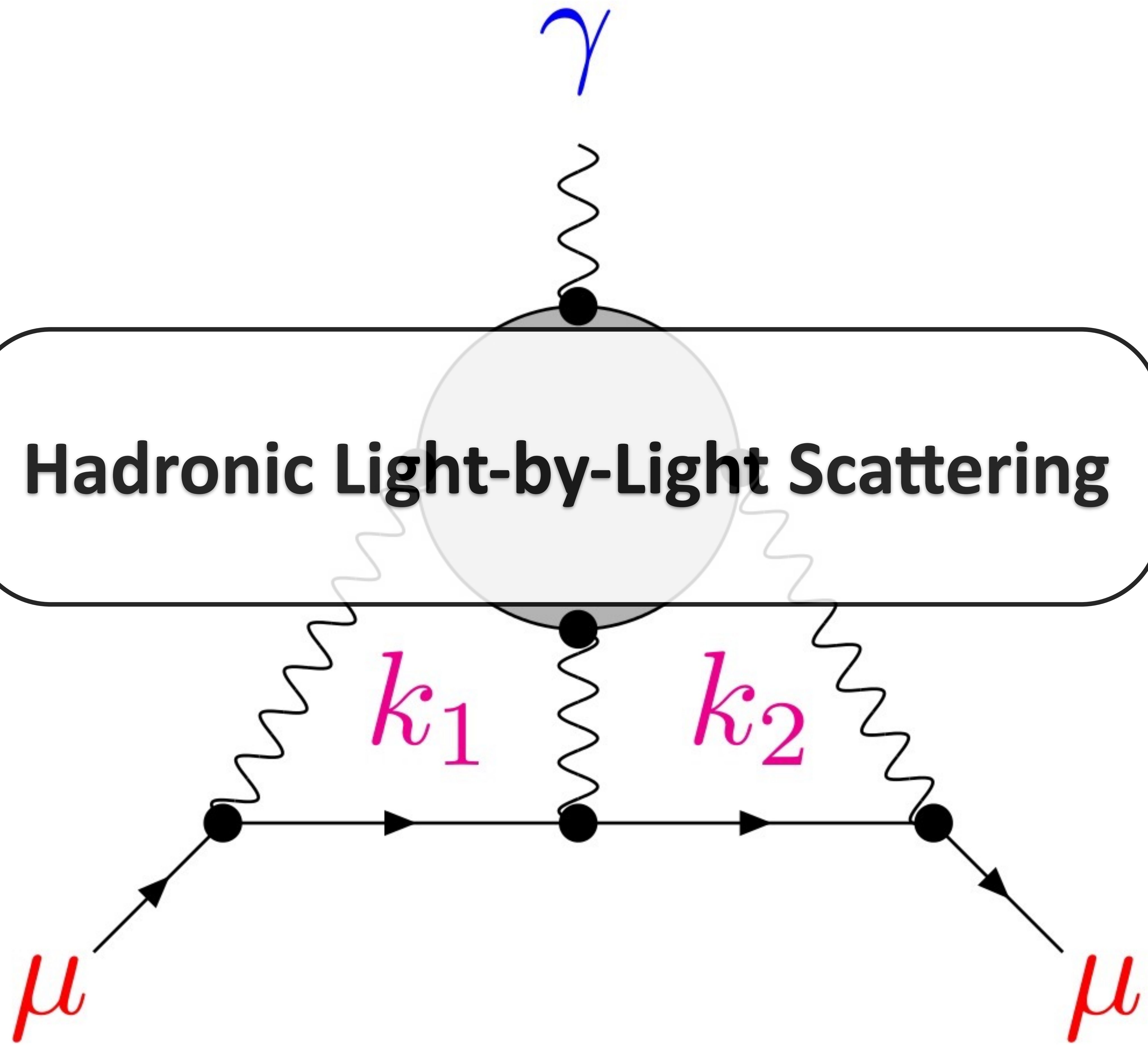




* Individual flavour contributions:

light (u,d)	≈ 90%
strange (s)	≈ 8%
charm (c)	≈ 2%

Hadronic Light-by-Light Scattering

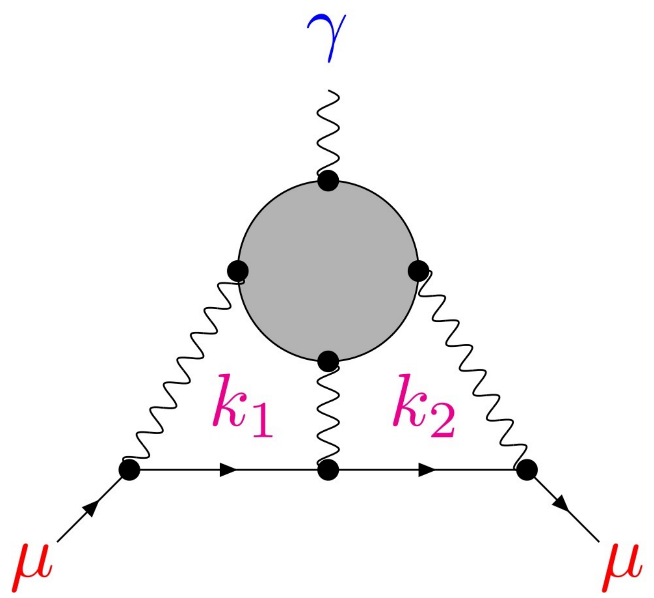


Lattice QCD approaches to HLbL scattering

* Numerically very demanding:
  - Compute 4pt correlation function for two independent momenta, k_1, k_2

    $\Rightarrow$ Cost scales proportional to $(\text{volume})^2$

  - Must take external momentum to zero: $q^2 \longrightarrow 0$



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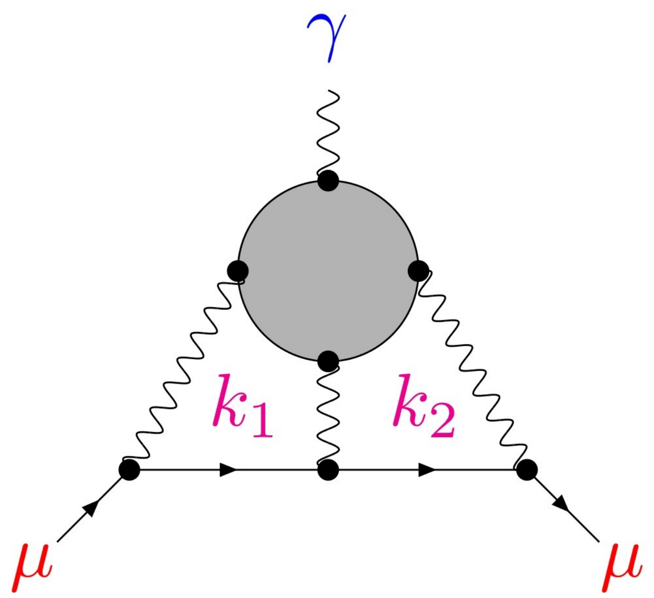

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Proposed techniques:

* QCD + QED simulations
  [Hayakawa et al., PoS LAT2005 (2006) 353, Blum et al., Phys Rev Lett 114 (2015) 012001]

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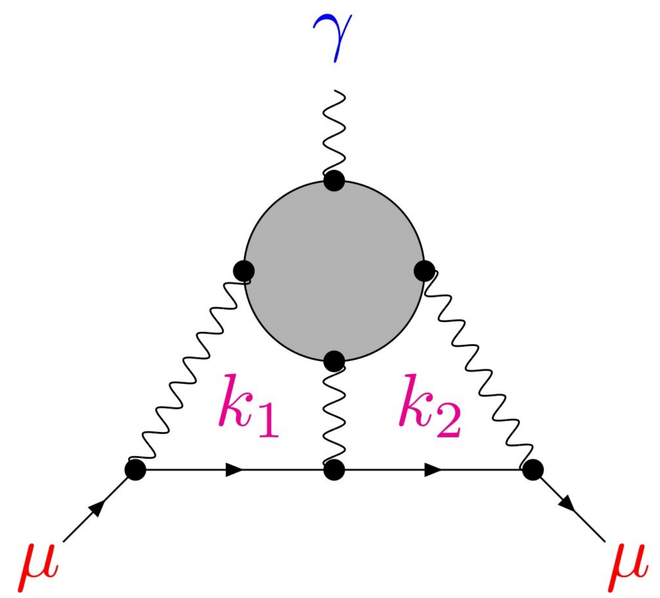

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* QCD + stochastic QED

  [Blum et al., Phys Rev D93 (2016) 014503, Luchang JIN, TUE 13:20]

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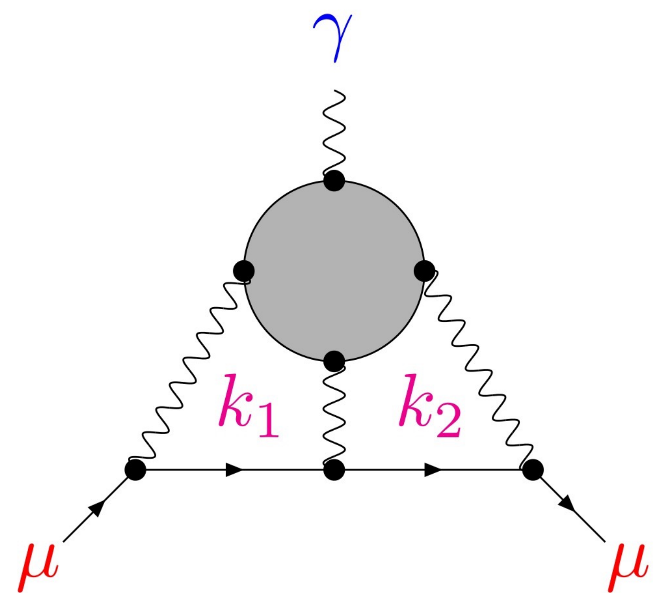

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* Light-by-light four-point function + exact QED kernel
  [Green, Gryniuk, von Hippel, Meyer, Pascalutsa, Phys Rev Lett 115 (2015) 222003, Nils ASMUSSEN, TUE 15:40]

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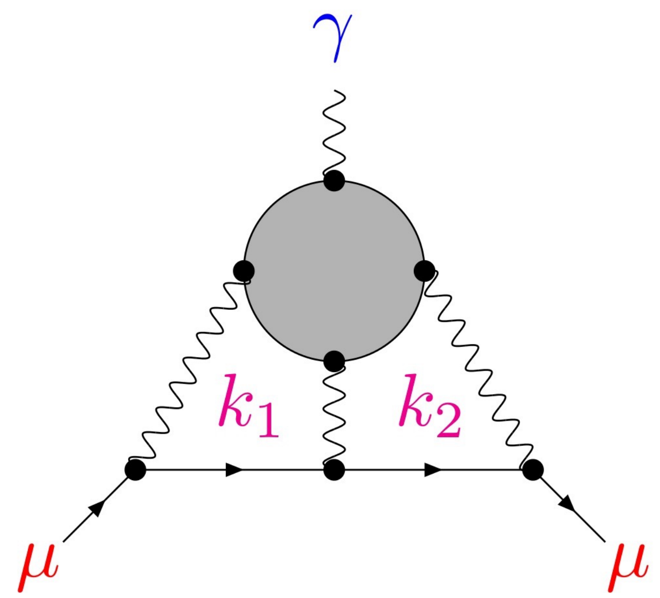

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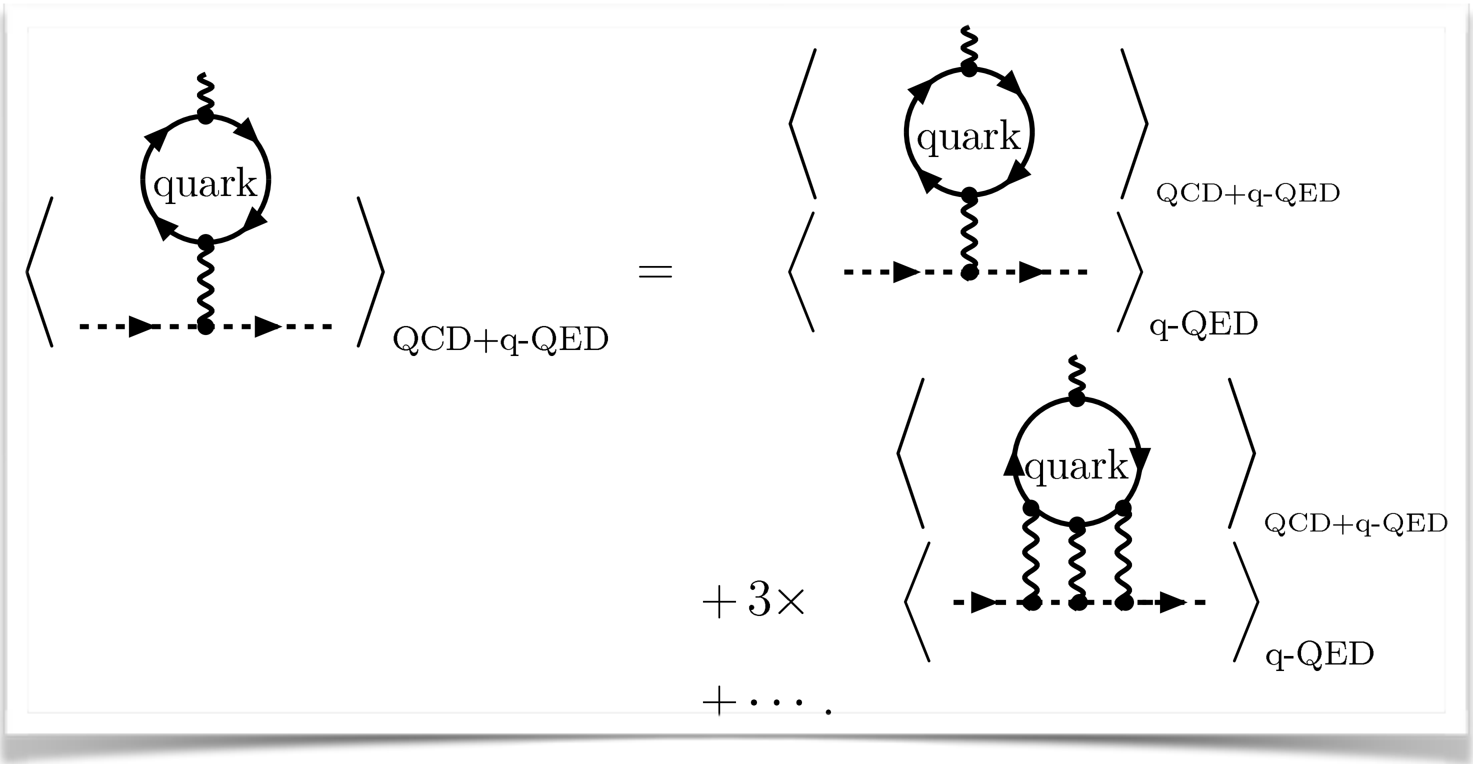
- Lattice calculations of dominant sub-processes
  [Feng et al., Phys Rev Lett 109 (2012) 182001, Antoine GÉRARDIN, FRI 18:10]

QCD + QED Simulations

- Compute matrix element of e.m. current between muon initial and final states:

$$\langle \mu(\boldsymbol{p}', s') \left| J_\mu(0) \right| \mu(\boldsymbol{p}, s) \rangle = -e\,\bar{u}(\boldsymbol{p}', s') \left(F_1(Q^2)\gamma_\mu + \frac{F_2(Q^2)}{2m}\sigma_{\mu\nu}Q_\nu \right) u(\boldsymbol{p}, s)$$

$$a_\mu^{\text{hlbl}} = F_2(0)$$



- Large statistical errors; subtract contributions of $\mathrm{O}(\alpha^4)$

[Blum et al., Phys Rev Lett 114 (2015) 012001]

QCD + Stochastic QED

* Abandon non-perturbative treatment of QED contribution:
  ⇒ insertion of three exact Feynman gauge photon propagators

$$G_{\mu\nu}(x,y) = \frac{1}{VT}\delta_{\mu\nu} \sum_{k,\,|\vec{k}|\neq 0} \frac{\mathrm{e}^{ik\cdot(x-y)}}{\hat{k}^2}$$

[Blum et al., Phys Rev D93 (2016) 014503]

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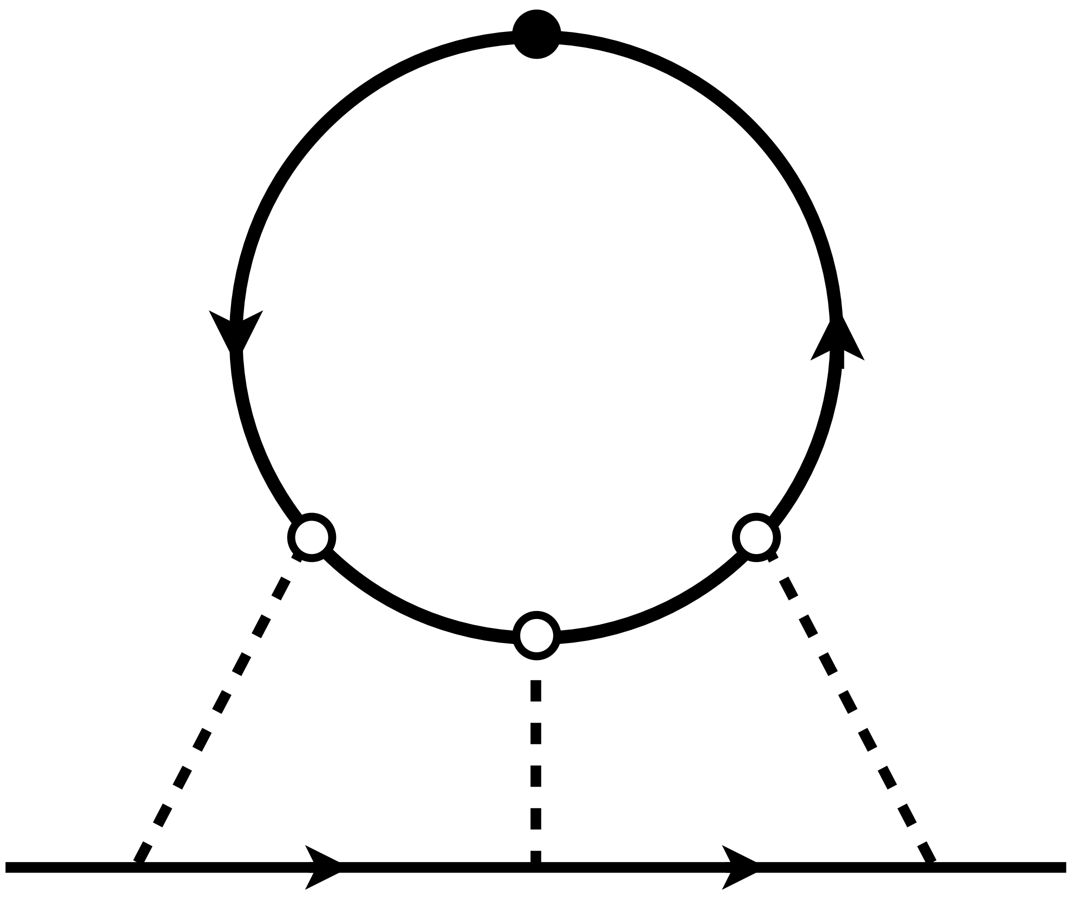

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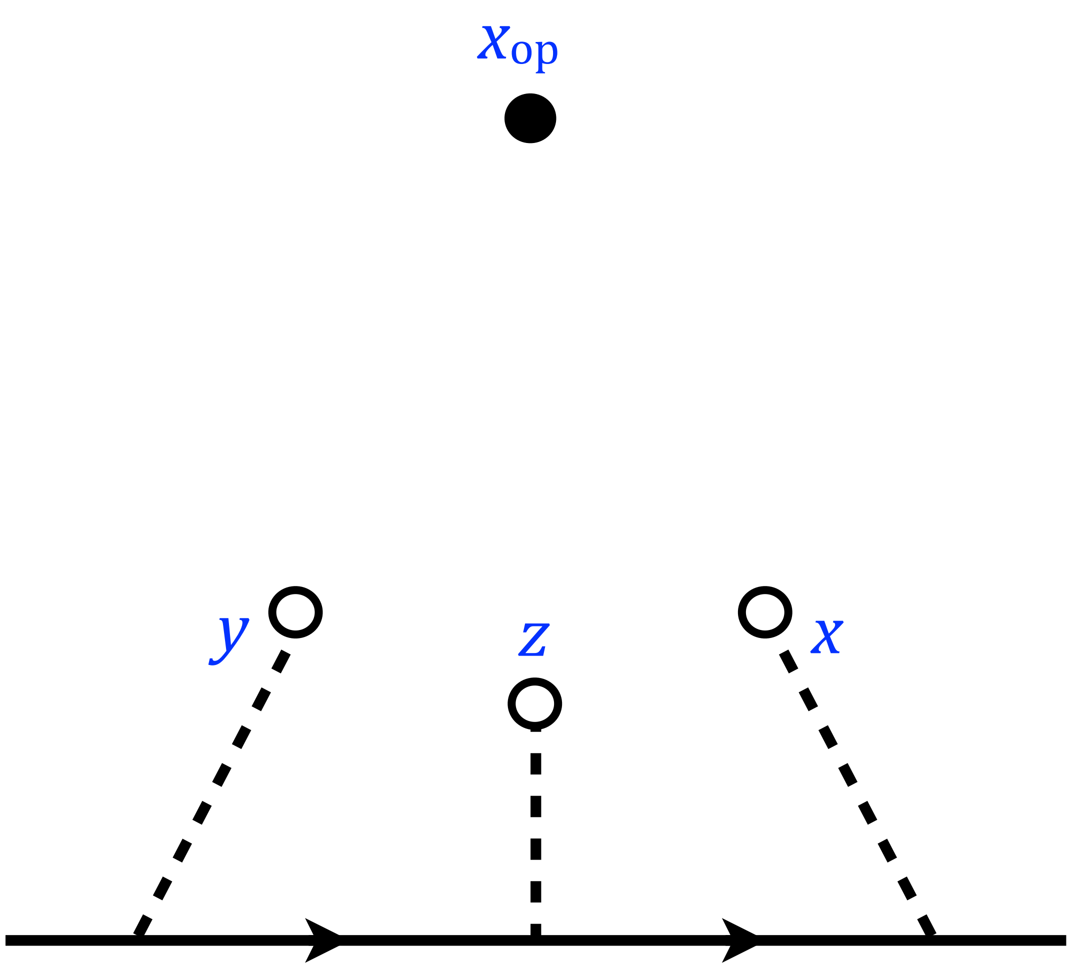

1. Stochastic selection of two internal photon vertices at x, y

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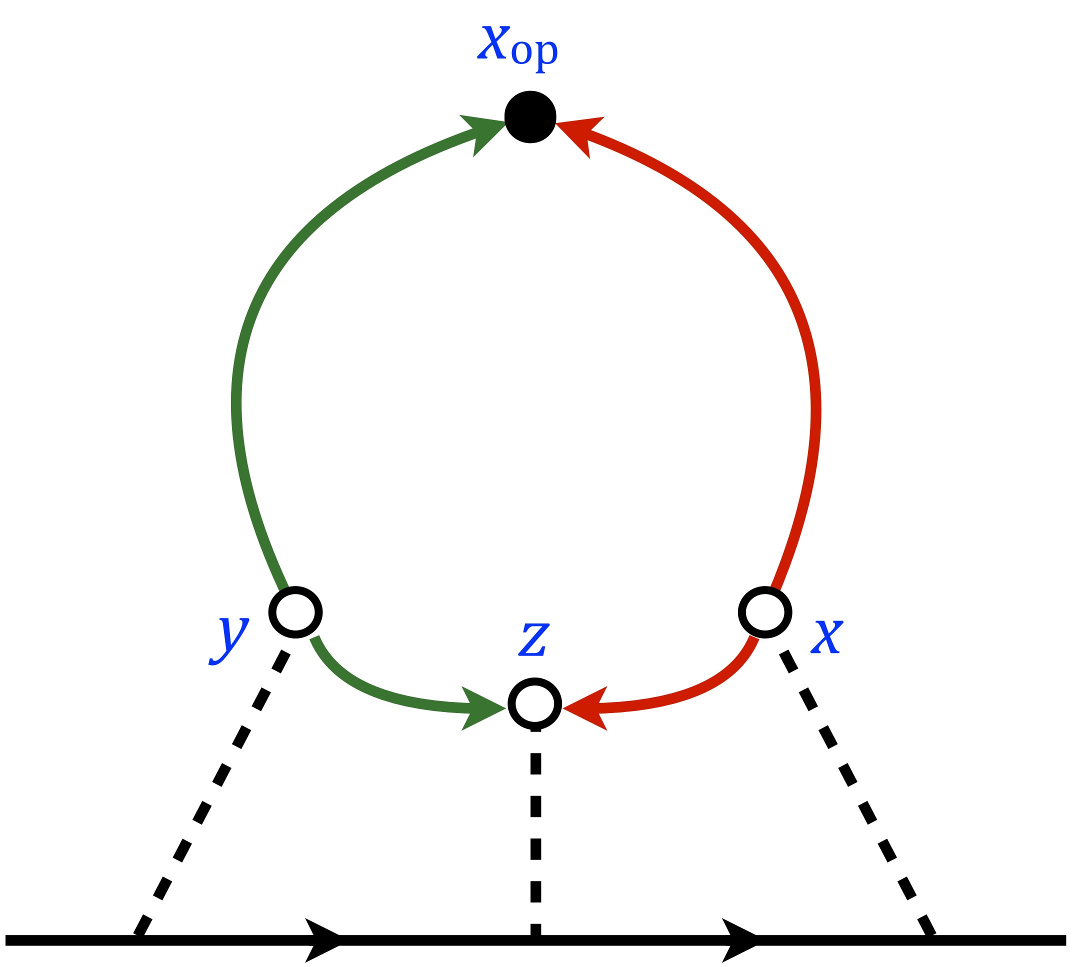
1. Stochastic selection of two internal photon vertices at x, y
2. Compute quark propagators from point sources at x and y



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[Blum et al., Phys Rev D93 (2016) 014503]

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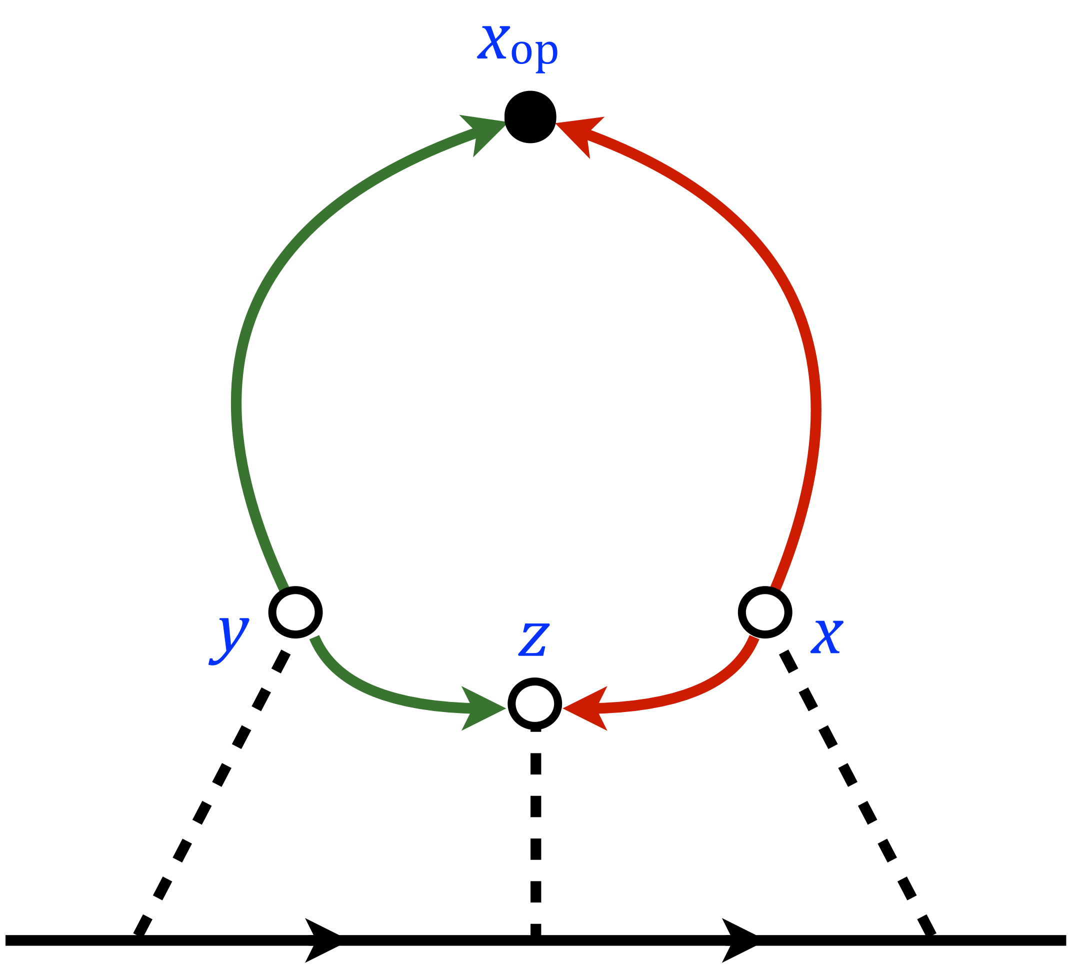
3. Current insertion x_{op} and third photon vertex z explicitly summed over



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1. Stochastic selection of two internal photon vertices at x, y
2. Compute quark propagators from point sources at x and y
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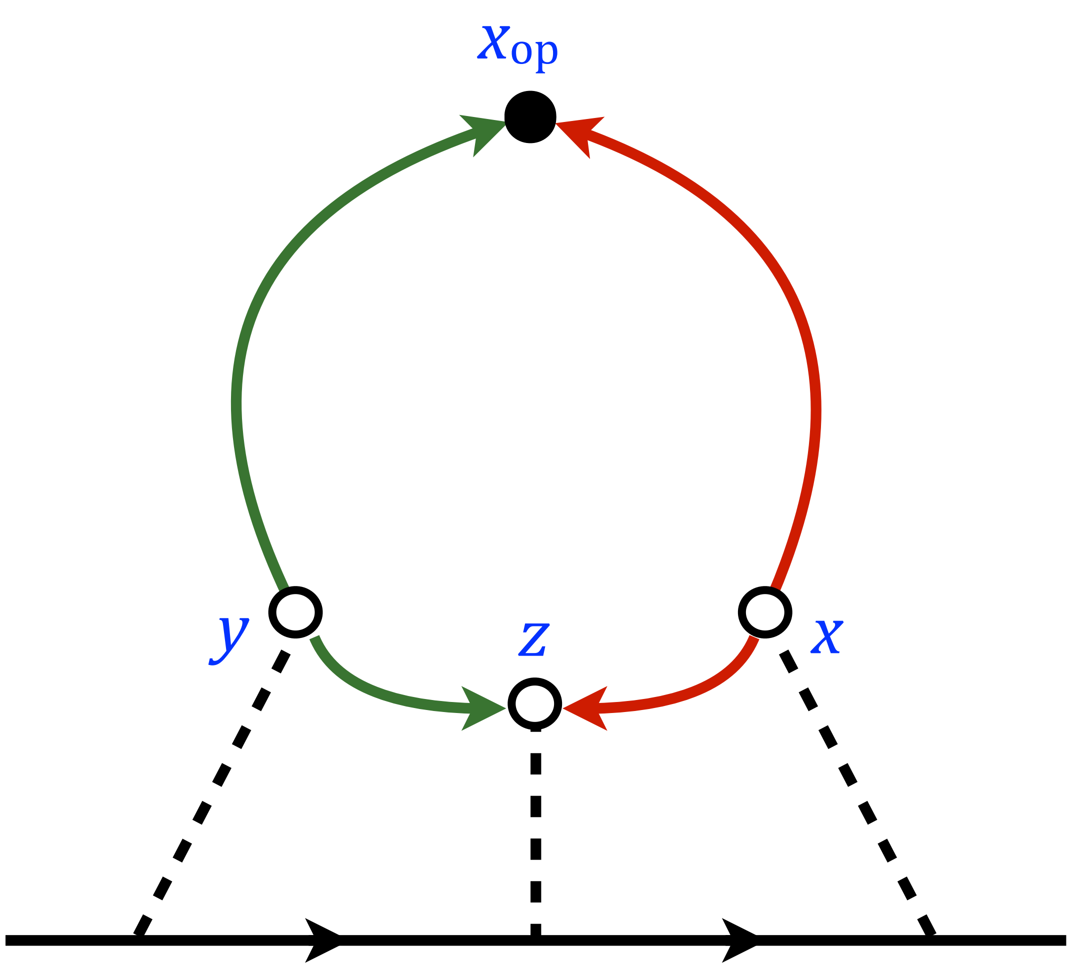
4. Avoid extrapolation to $q^2 = 0$ by computing first moment w.r.t. $x_{\rm op}$



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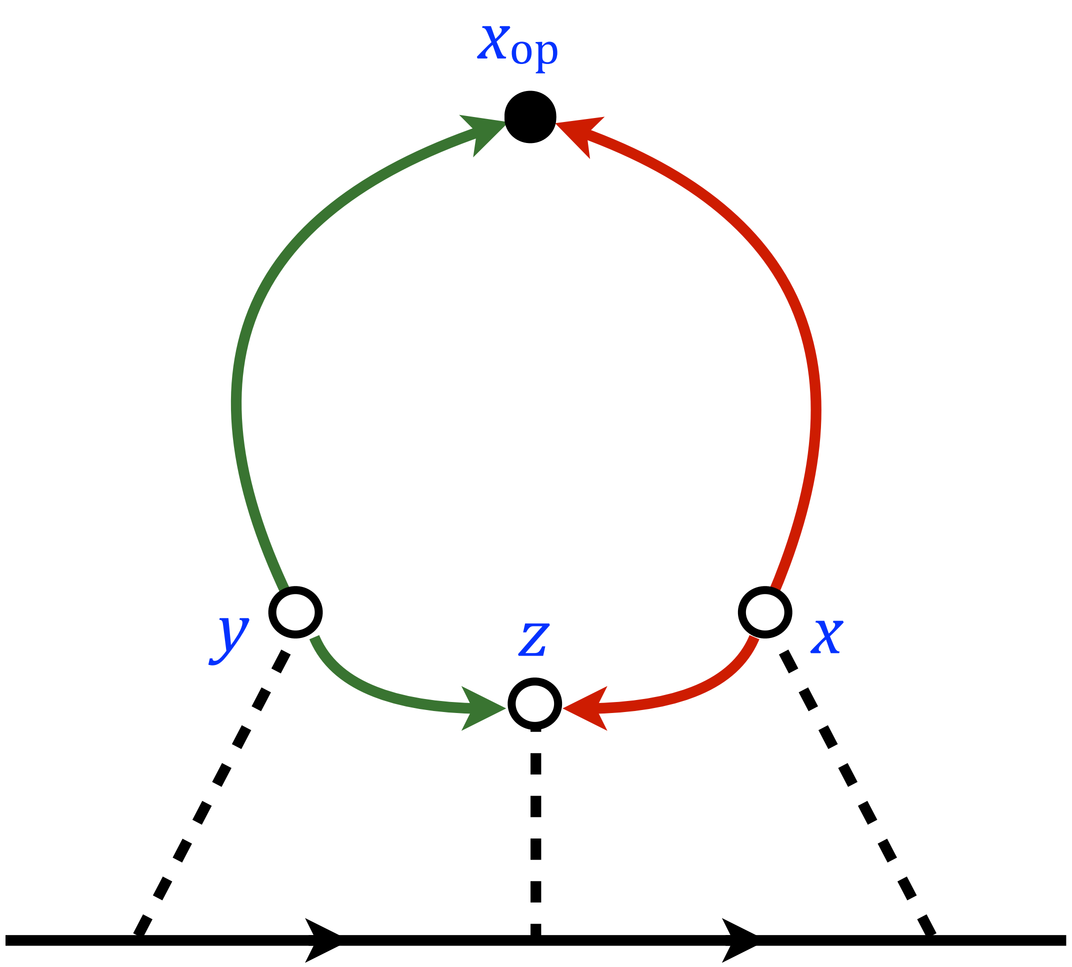
$$G_{\mu\nu}(x,y) = \frac{1}{VT}\delta_{\mu\nu} \sum_{k,\,|\vec{k}|\neq 0} \frac{\mathrm{e}^{ik\cdot(x-y)}}{\hat{k}^2}$$

[Blum et al., Phys Rev D93 (2016) 014503]

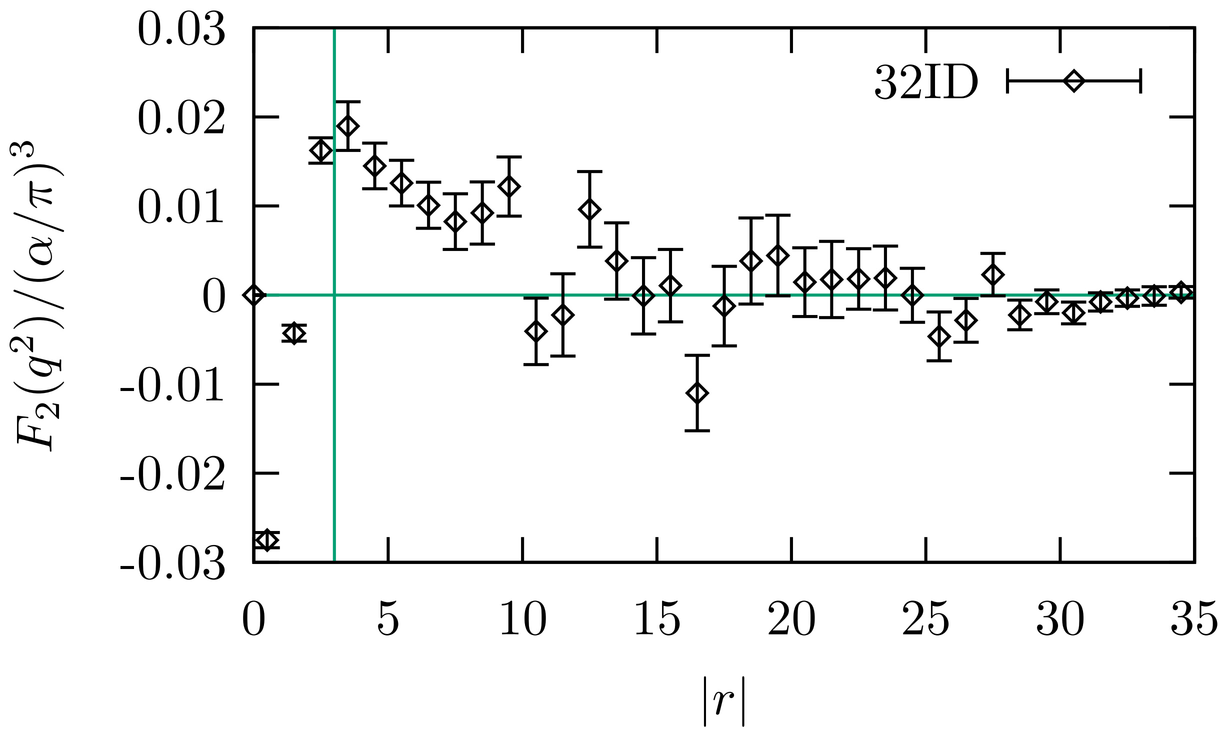
1. Stochastic selection of two internal photon vertices at x, y
2. Compute quark propagators from point sources at x and y
3. Current insertion $x_{\rm op}$ and third photon vertex z explicitly summed over
4. Avoid extrapolation to $q^2 = 0$ by computing first moment w.r.t. $x_{\rm op}$



* Efficiency gain: two orders of magnitude

QCD + Stochastic QED

* Final result: sum over relative coordinates $|r| \equiv |(x-y)_\mu|$



$N_f = 2+1$ flavours; DWFs

171 MeV pion mass; $a = 0.14$ fm

[Blum et al., Phys Rev D93 (2016) 014503]

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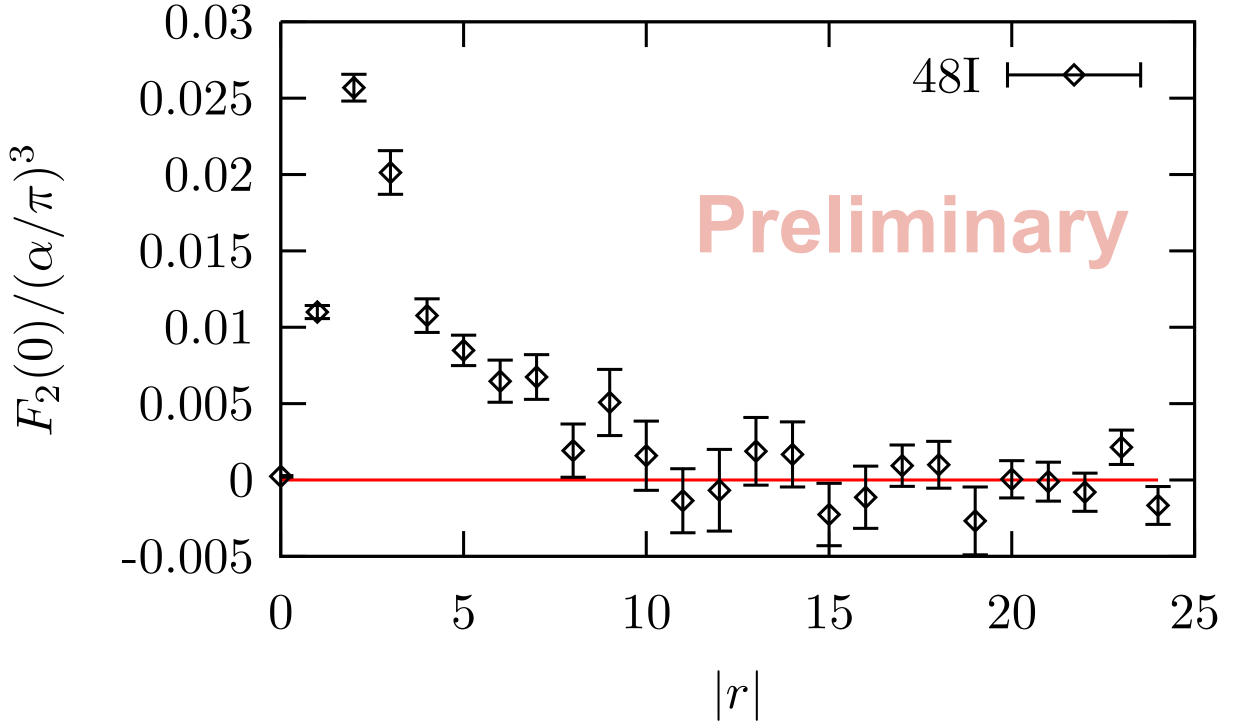

$N_f = 2+1$ flavours; DWFs

Physical pion mass; $a = 0.11$ fm

[Luchang JIN, TUE 13:20]

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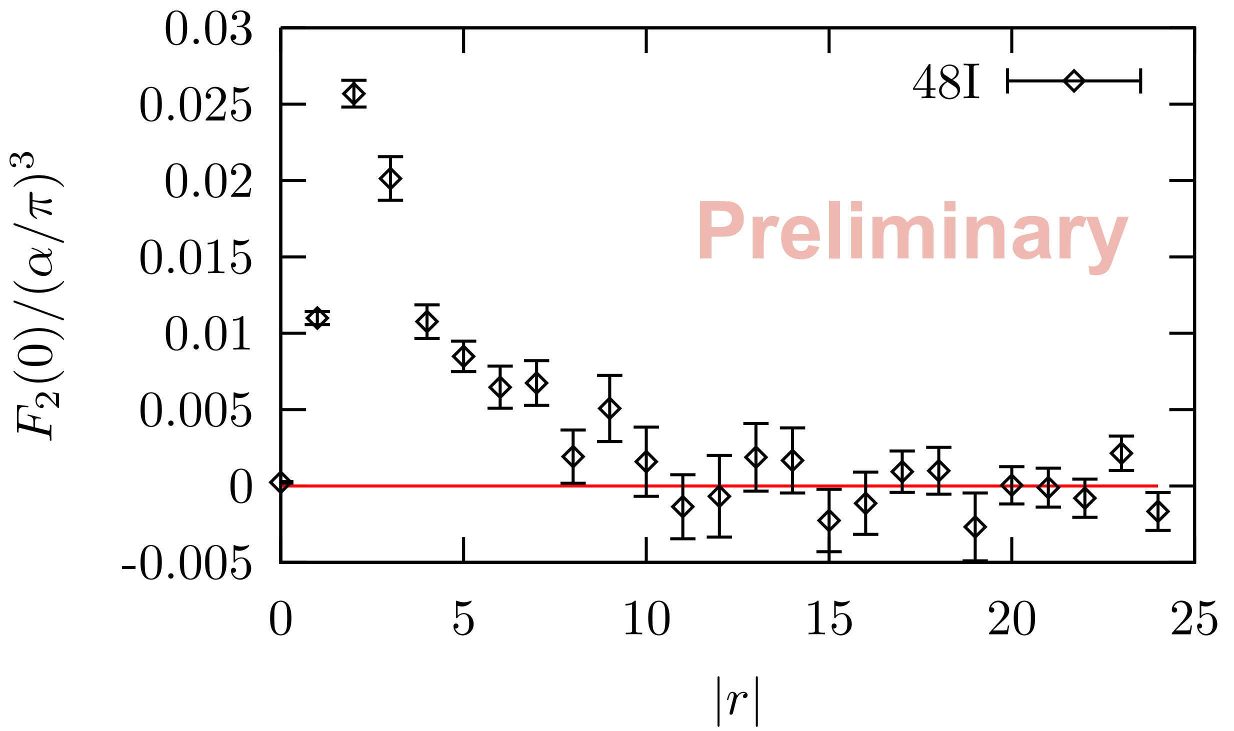


$N_f = 2+1$ flavours; DWFs

Physical pion mass; $a = 0.11$ fm

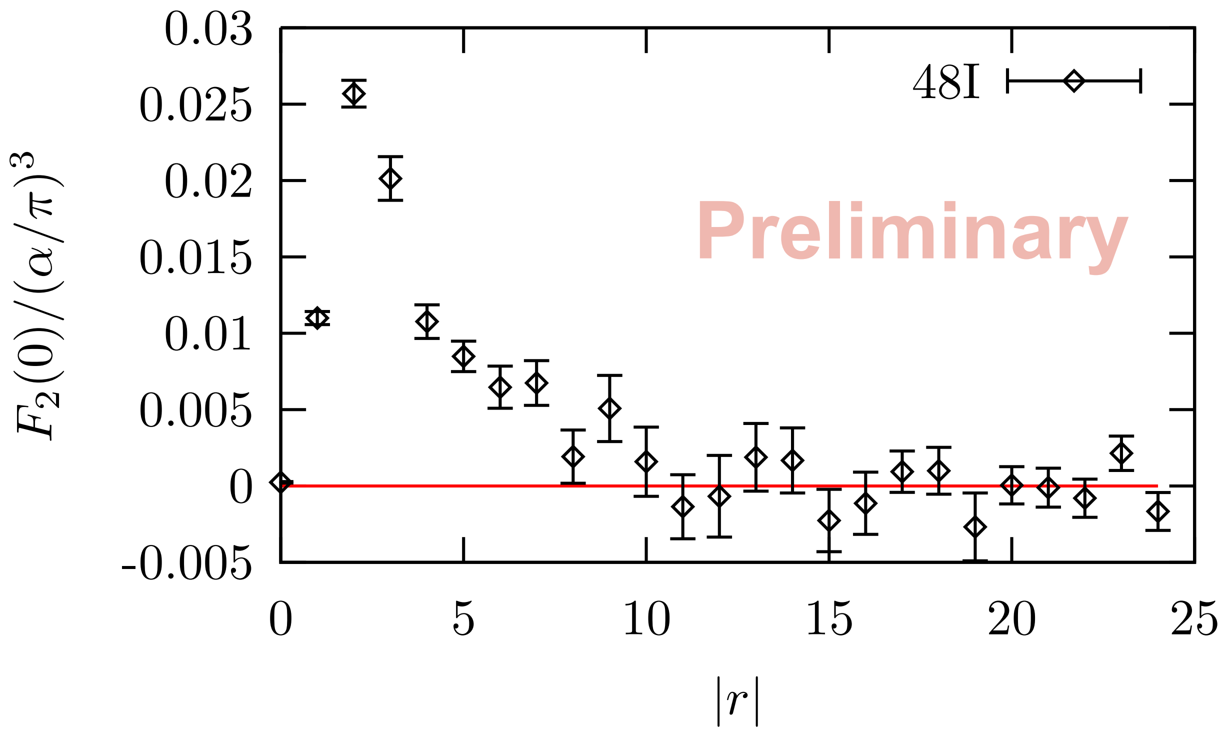
[Luchang JIN, TUE 13:20]

$$(a_\mu^{\rm hlbl})_{\rm con} = \begin{cases} (0.1054 \pm 0.0054)\,(\alpha/\pi)^3 = (132.1 \pm 6.8)\cdot 10^{-11} & (m_\pi = 171\ \text{MeV},\ \ a = 0.14\ \text{fm}) \\ (0.0933 \pm 0.0073)\,(\alpha/\pi)^3 = (116.1 \pm 9.1)\cdot 10^{-11} & (m_\pi = 139\ \text{MeV},\ \ a = 0.11\ \text{fm}) \end{cases}$$

(Connected contribution; statistical error only)

QCD + Stochastic QED

* Final result: sum over relative coordinates $|r| \equiv |(x-y)_\mu|$



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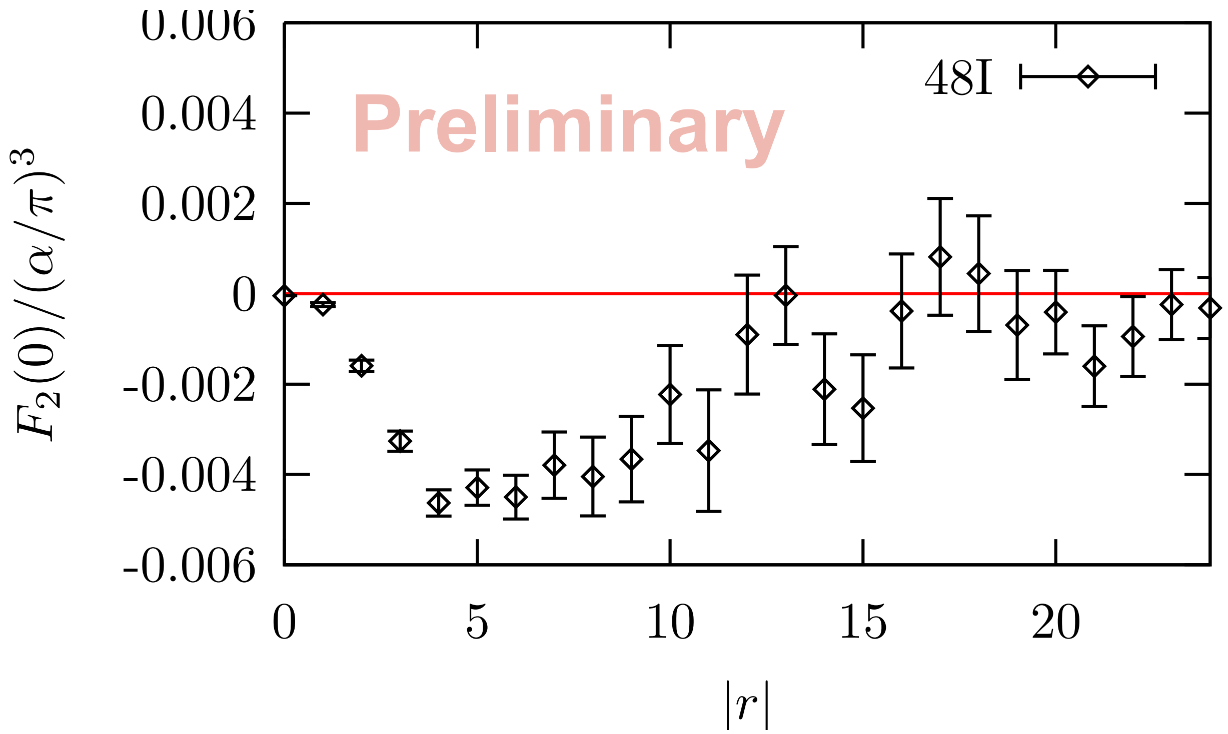
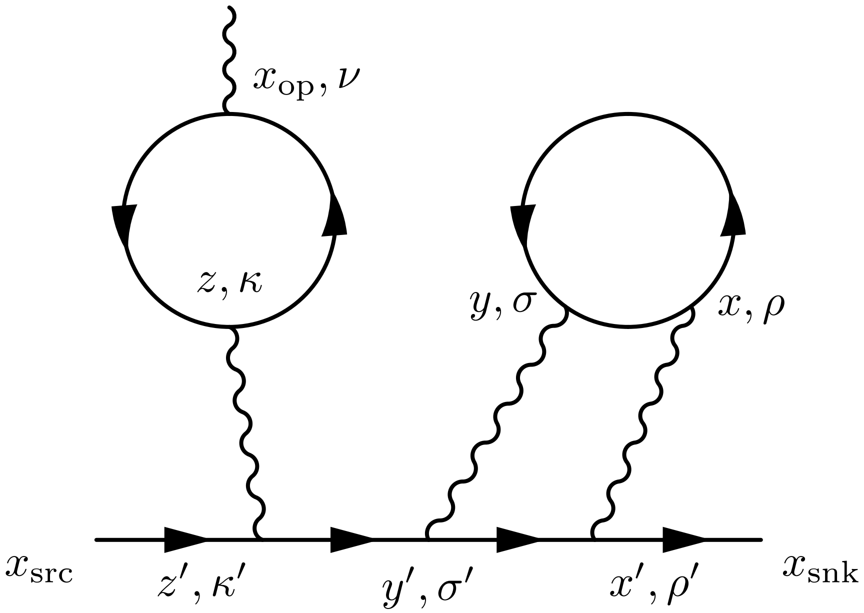
* Numerical cost: 175 Mcore-hrs for 48I ensemble

Disconnected Contributions to HLbL

* Use same setup to determine leading disconnected contribution





[Luchang JIN, TUE 13:20]

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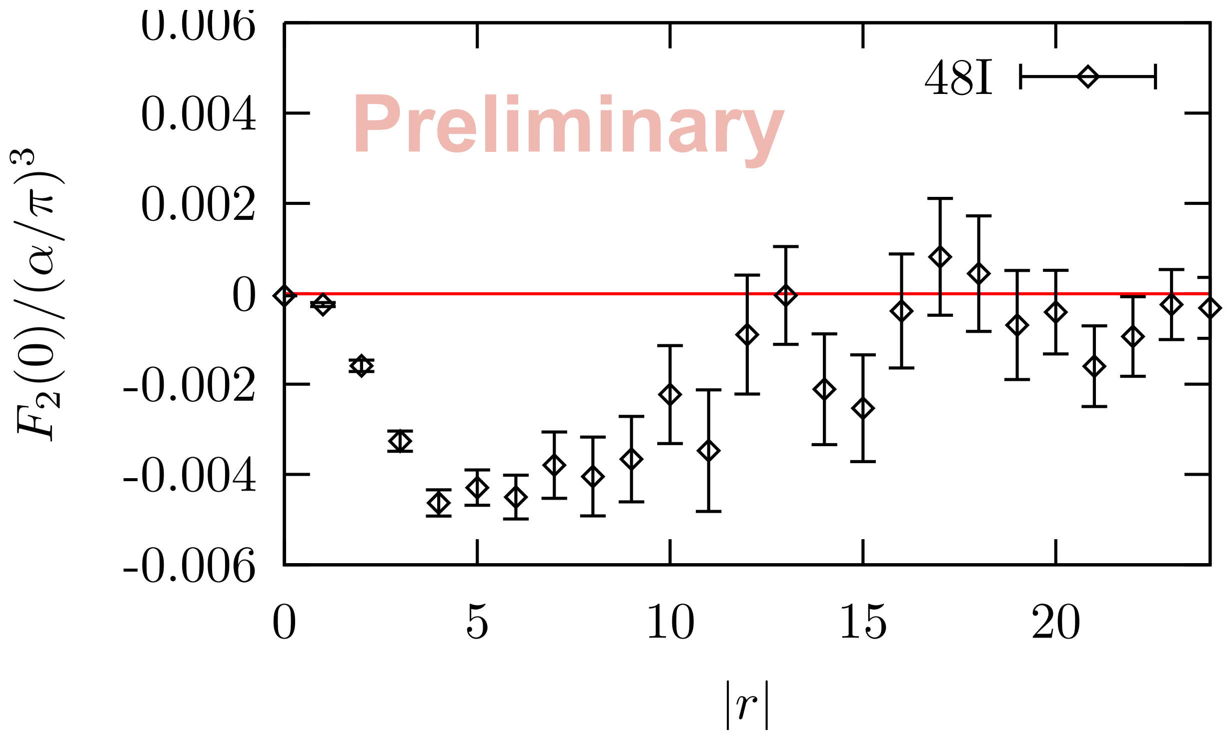

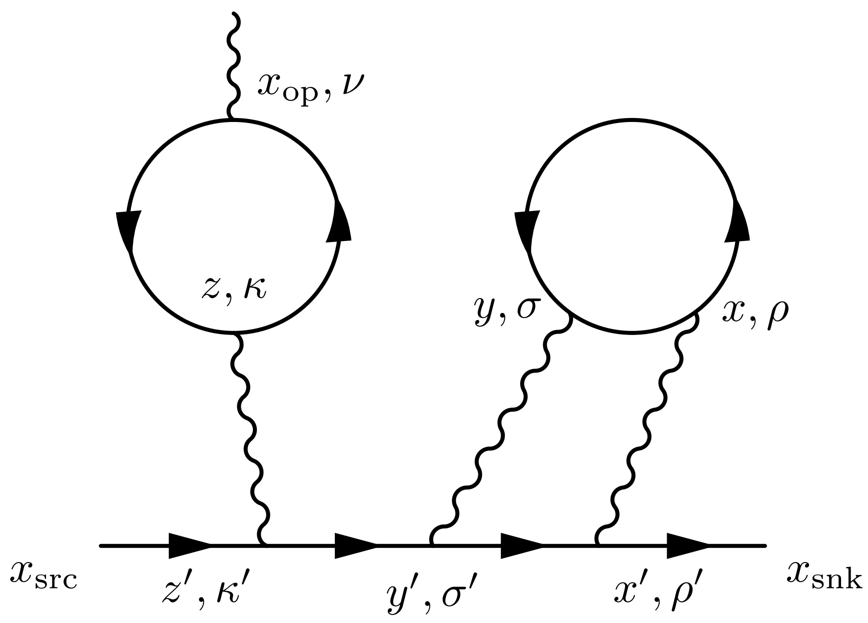

[Luchang JIN, TUE 13:20]

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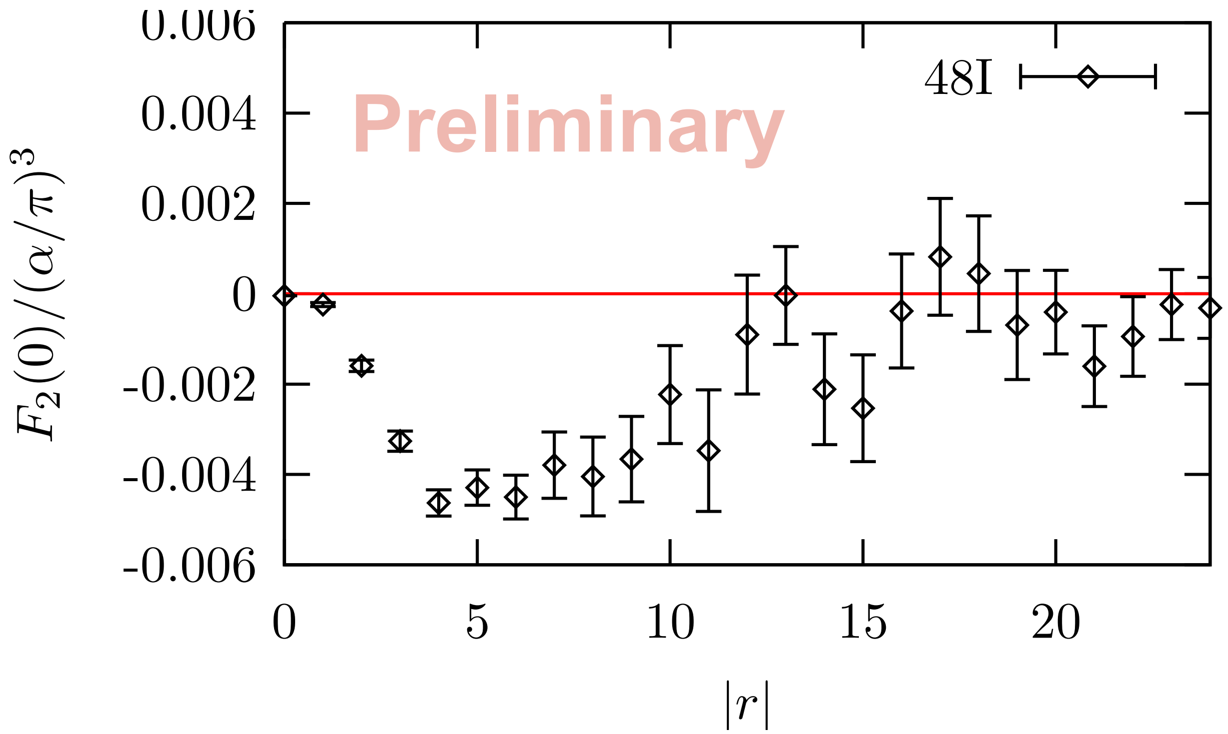

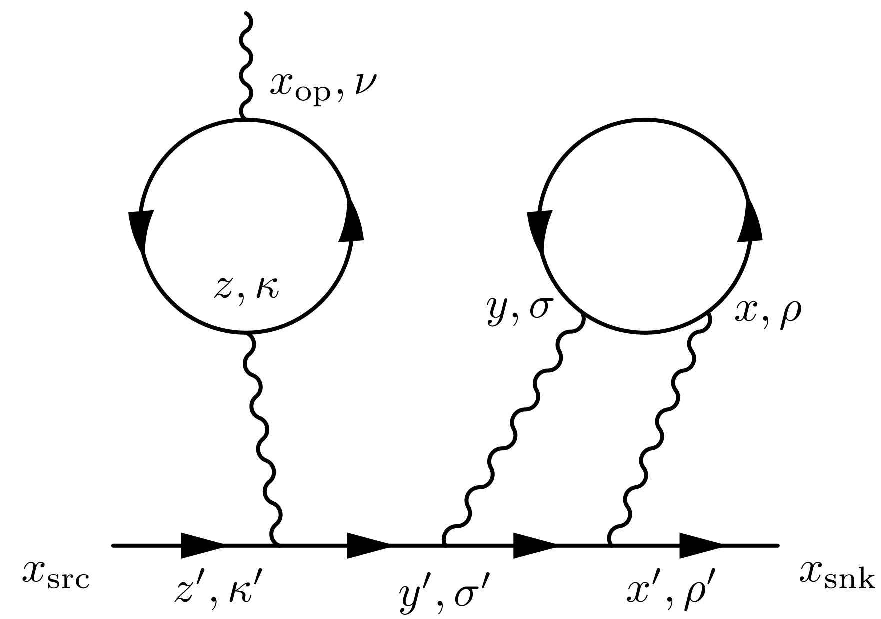

[Luchang JIN, TUE 13:20]

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(Statistical error only)

* To do: compute additional disconnected diagrams; study finite-volume effects, lattice artefacts

Exact QED kernel

* Determine QED part perturbatively in the continuum in infinite volume

⇒ no power-law volume effects

$$a_\mu^{\rm hlbl} = F_2(0) = \frac{me^6}{3}\int d^4y \int d^4x\, \overline{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\; i\Pi_{\rho;\mu\nu\lambda\sigma}(x,y)$$

* QCD four-point function: $i\Pi_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int d^4z\; z_\rho \left\langle J_\mu(x)J_\nu(y)J_\sigma(z)J_\lambda(0)\right\rangle$

[Asmussen, Green, Meyer, Nyffeler, in prep.]

[Nils ASMUSSEN, TUE 15:40]

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[Asmussen, Green, Meyer, Nyffeler, in prep.]

[Nils ASMUSSEN, TUE 15:40]

- Infra-red finite; can be computed analytically
- Admits a tensor decomposition in terms of six form factors which depend on $x^2, \quad y^2, \quad x \cdot y$

⇒ 3D integration instead of $\int d^4x \int d^4y$

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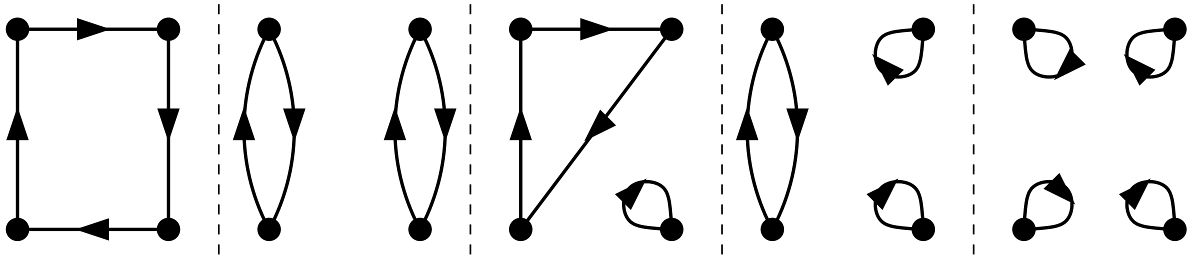
⇒ 3D integration instead of $\int d^4x \int d^4y$

* Form factors computed and stored on disk

HLbL four-point correlator

* Four-point correlator of one local and three conserved vector currents



* Fully connected contribution with summed fixed kernels:

$$\Pi^{\text{pos}'}_{\mu_1\mu_2\mu_3\mu_4}(x_4; f_1, f_2) = \sum_{x_1,x_2} f(x_1)\, f(x_2)\, \Pi^{\text{pos}}_{\mu_1\mu_2\mu_3\mu_4}(x_1, x_2, 0, x_4)$$

(computable in terms of sequential and double-sequential propagators)

* Euclidean four-point function in momentum space:

$$\Pi^{E}_{\mu_1\mu_2\mu_3\mu_4}(p_4; p_1, p_2) = \sum_{x_4} \mathrm{e}^{-ip_4\cdot x_4}\, \Pi^{\text{pos}'}_{\mu_1\mu_2\mu_3\mu_4}(x_4; p_1, p_2)$$

[Green et al., Phys Rev Lett 115 (2015) 222003]

Forward light-by-light amplitude

* Forward kinematics: $Q_1 \equiv p_2 = -p_1, \quad Q_2 \equiv p_4, \quad \nu = -Q_1 \cdot Q_2$

[Green et al., Phys Rev Lett 115 (2015) 222003]

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photon virtualities

[Green et al., Phys Rev Lett 115 (2015) 222003]

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* Forward scattering of transversely polarised virtual photons:

$$\mathcal{M}_{TT}(-Q_1^2, -Q_2^2, \nu) = \frac{e^4}{4} R_{\mu_1\mu_2} R_{\mu_3\mu_4} \, \Pi^E_{\mu_1\mu_2\mu_3\mu_4}(-Q_2; -Q_1, Q_1)$$

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- Related to $\sigma_{0,2}(\gamma^* \gamma^* \longrightarrow \text{hadrons})$ via subtracted dispersion relation:

$$\mathcal{M}_{TT}(-Q_1^2, -Q_2^2, \nu) - \mathcal{M}_{TT}(-Q_1^2, -Q_2^2, 0) = \frac{2\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \, \frac{\sqrt{\nu'^2 - Q_1^2 Q_2^2}}{\nu'(\nu'^2 - \nu^2 - i\epsilon)} \left[\sigma_0(\nu') + \sigma_2(\nu')\right]$$

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⇒ Compare lattice data to phenomenological expectations, e.g. leading contribution to a_μ^{hlbl} from π^0 exchange diagrams

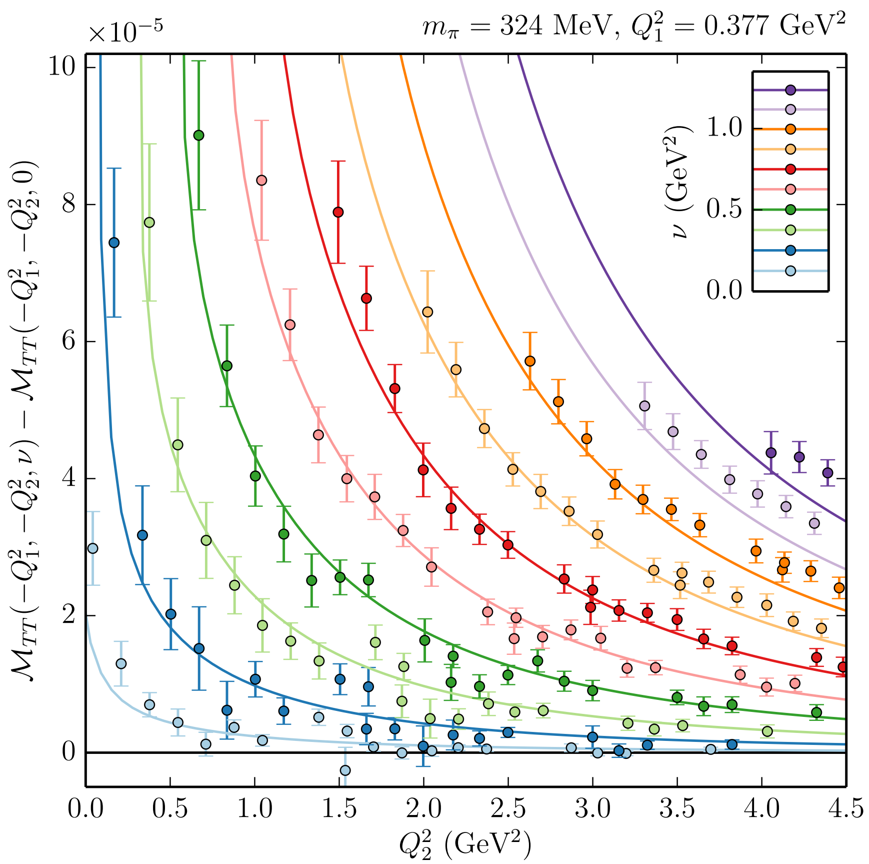
[Green et al., Phys Rev Lett 115 (2015) 222003]

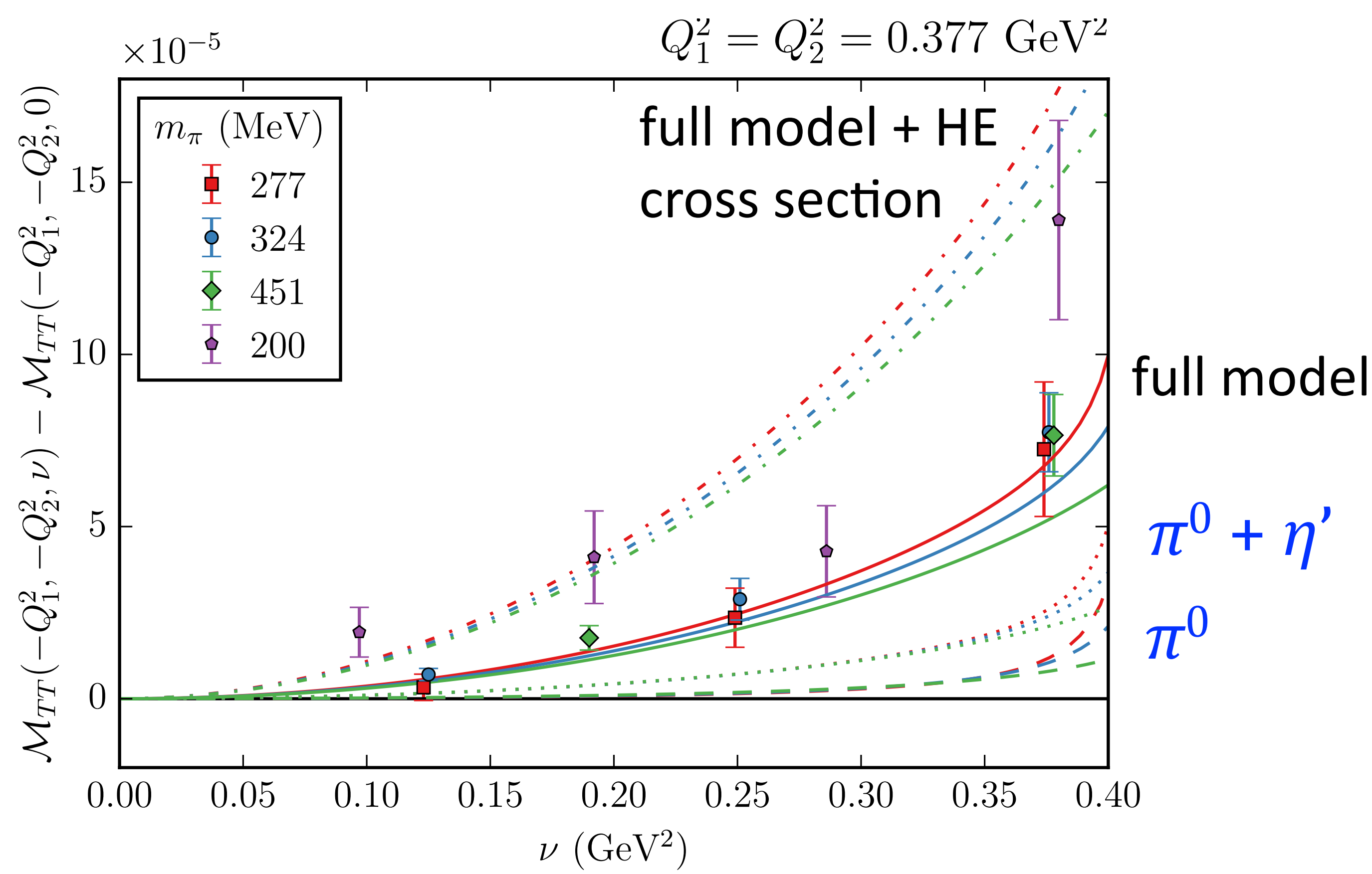
Forward light-by-light amplitude

* Test in two-flavour QCD (CLS ensembles):

* Cross sections: $\sigma_0 + \sigma_2 = \sum_{\pi^0,\eta',a_0,f_0,\ldots} \sigma(\gamma^*\gamma^* \to M) + \sigma(\gamma^*\gamma^* \to \pi^+\pi^-)$

* Compare lattice data to dispersion relation and model for cross sections:

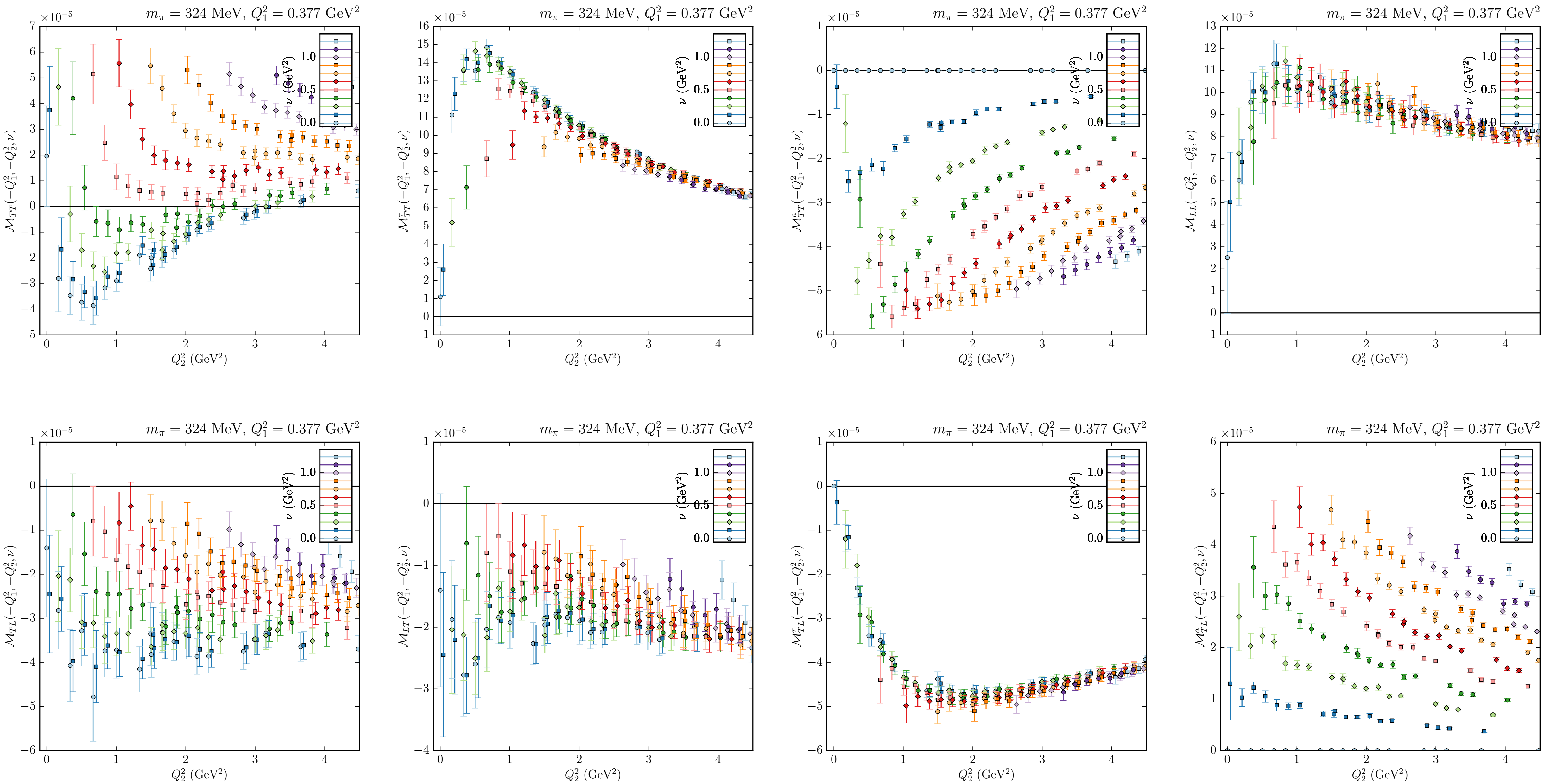




[Green et al., Phys Rev Lett 115 (2015) 222003, and in prep.]

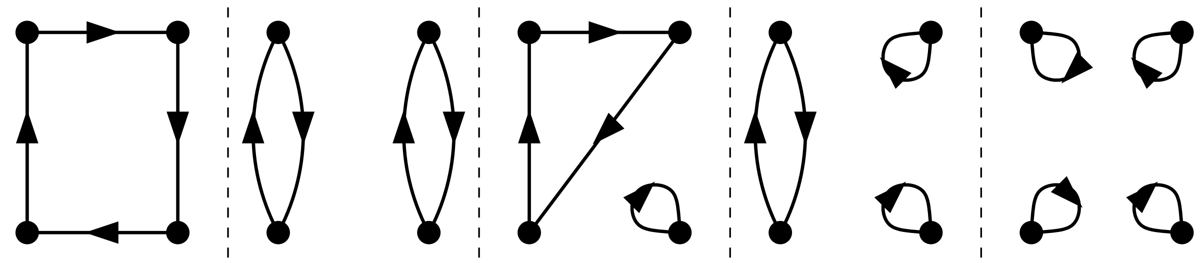
Forward-scattering amplitudes

* Full set of eight forward-scattering amplitudes:



* Comparison with model under way — much stronger constraints

Quark-disconnected contributions

* Quark contractions: 

[Green et al., in preparation]

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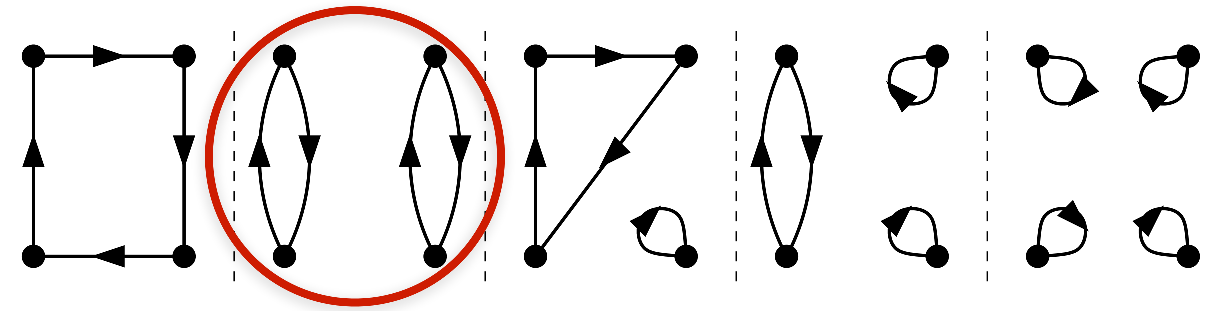

* Enhancement of (2,2) disconnected diagram by charge factors

[Green et al., in preparation]

Quark-disconnected contributions

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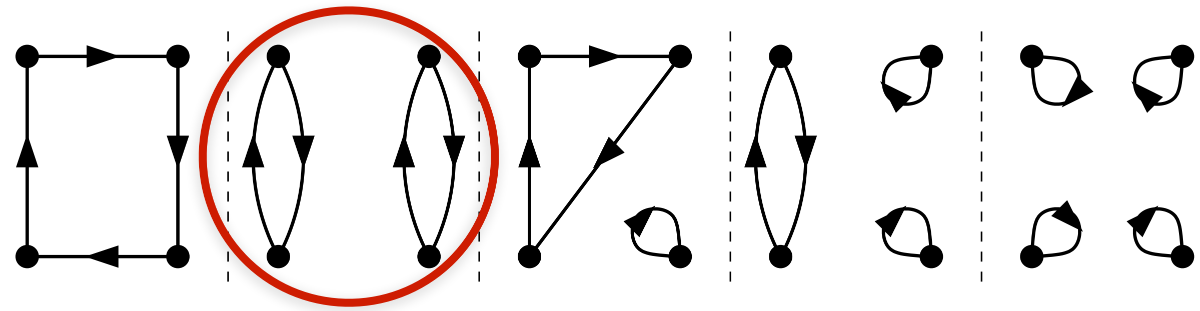
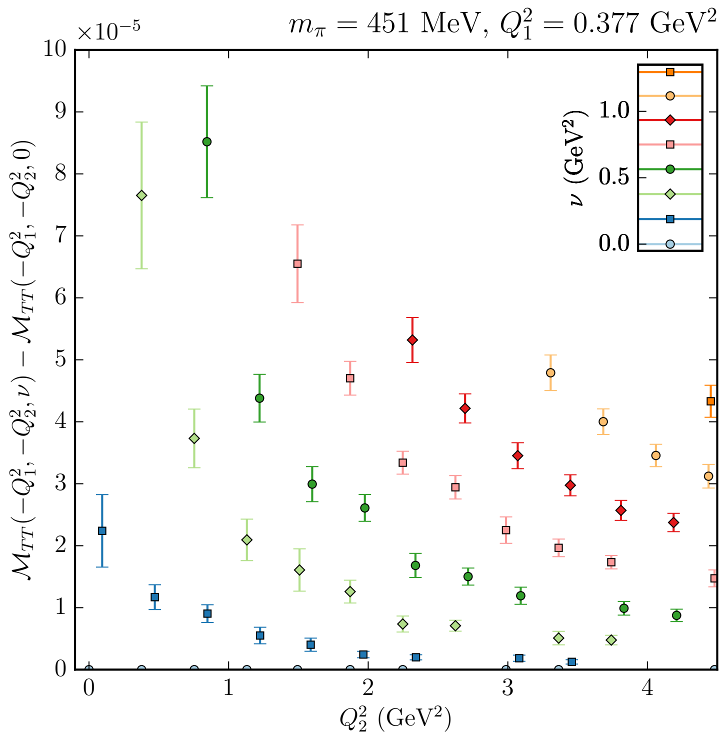
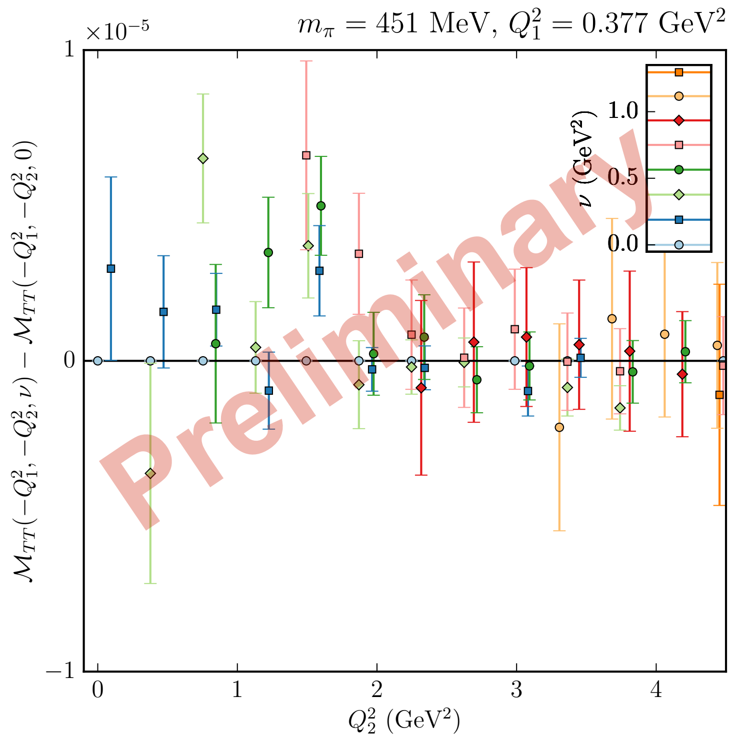


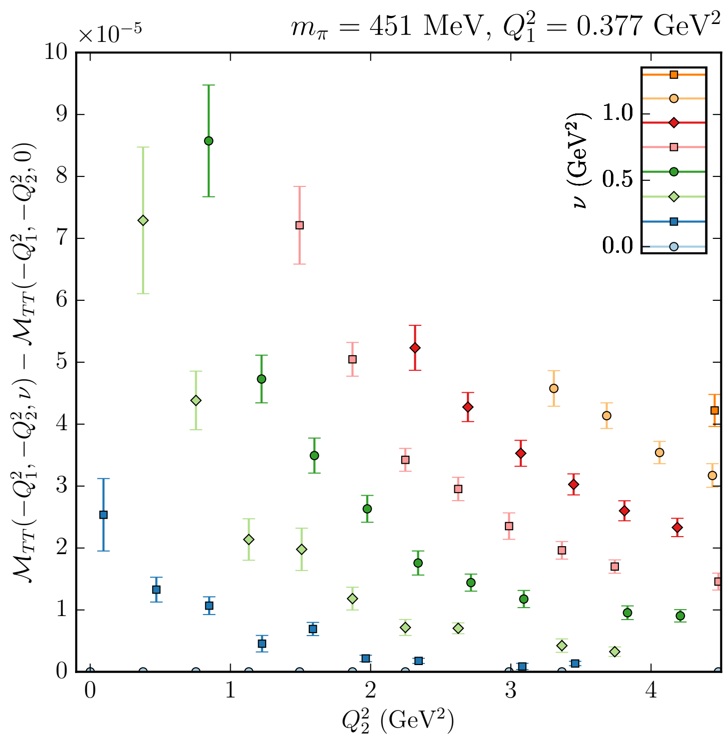
* Results for subtracted forward amplitude:



fully connected



(2,2)-disconnected

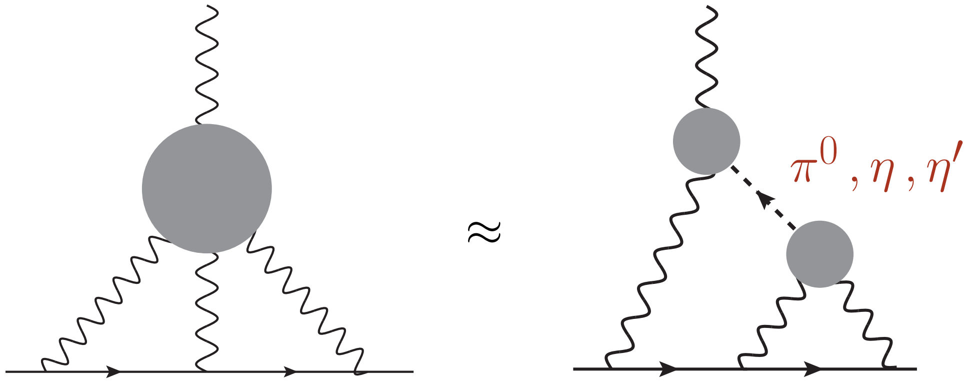


sum

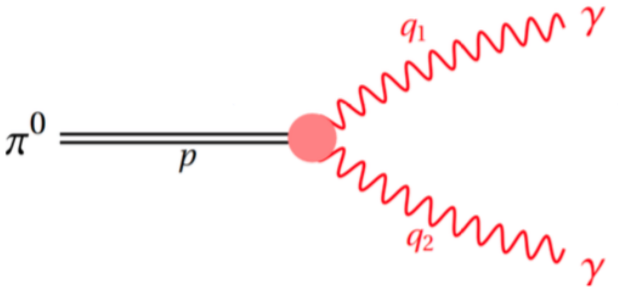
[Green et al., in preparation]

HLbL from transition $\pi^0 \longrightarrow \gamma^*\gamma^*$

- Pseudoscalar meson exchange expected to dominate LbL scattering:
  [Nyffeler, EPJ Web Conf 118 (2016) 01024, arXiv:1602.03737]



- Compute transition form factor between π^0 and two off-shell photons:



$$\epsilon_{\mu\nu\alpha\beta}\, q_1^\alpha q_2^\beta\, \mathcal{F}_{\pi^0\gamma^*\gamma^*}(m_\pi^2; q_1^2, q_2^2) \equiv M_{\mu\nu}$$

[Antoine GÉRARDIN, FRI 18:10]

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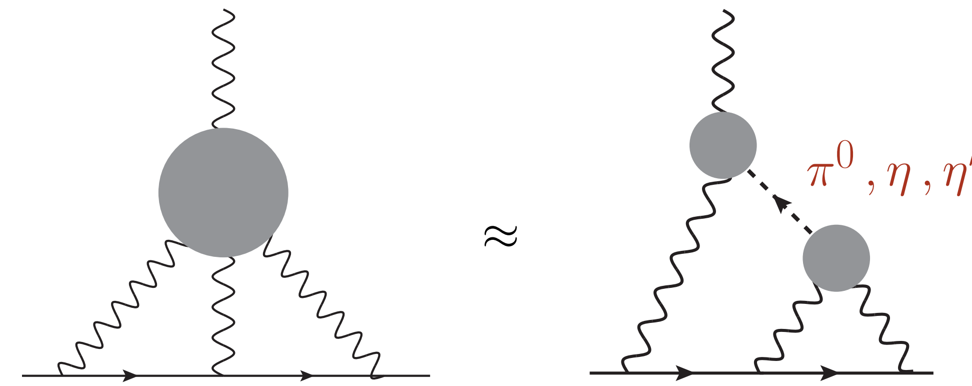

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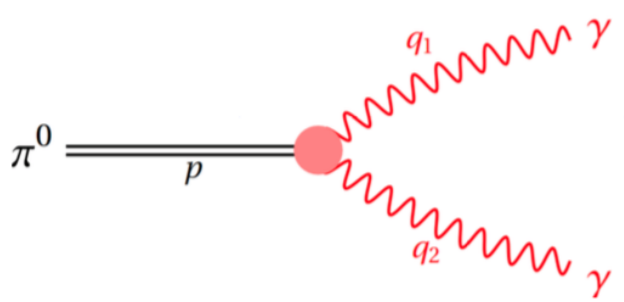

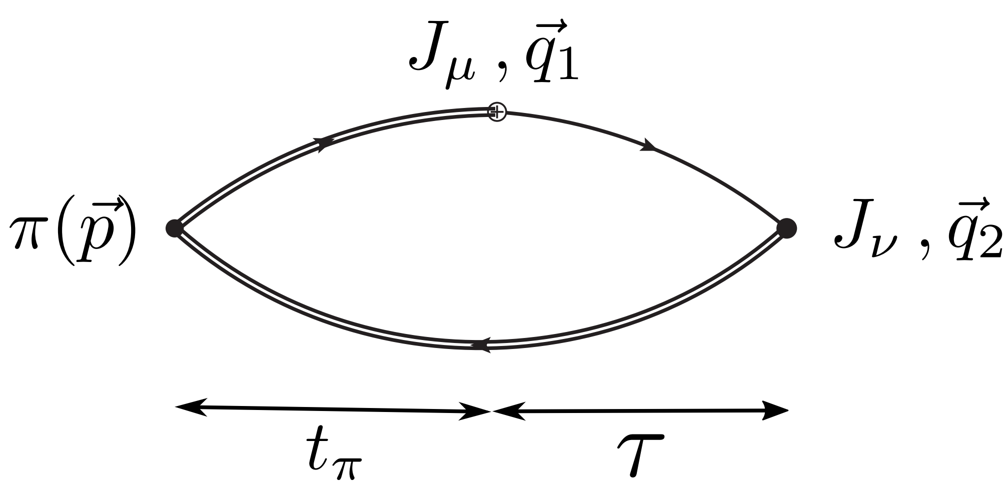

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$$M_{\mu\nu} \sim C^{(3)}_{\mu\nu}(\tau, t_\pi; \vec{p}, \vec{q}_1, \vec{q}_2) =$$

$$\sum_{\vec{x},\vec{z}} \left\langle T\left\{J_\nu(\vec{0}, \tau + t_\pi) J_\mu(\vec{z}, t_\pi) P(\vec{x}, 0)\right\}\right\rangle \, \mathrm{e}^{i\vec{p}\cdot\vec{x}} \mathrm{e}^{-i\vec{q}_1\cdot\vec{z}}$$

[Antoine GÉRARDIN, FRI 18:10]

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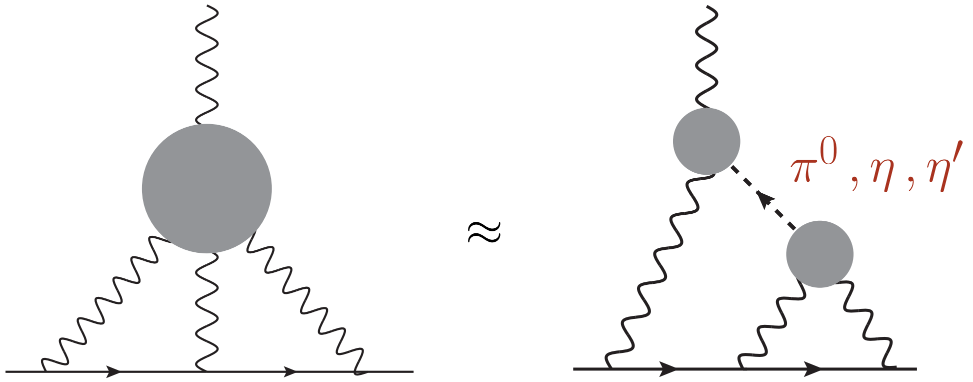

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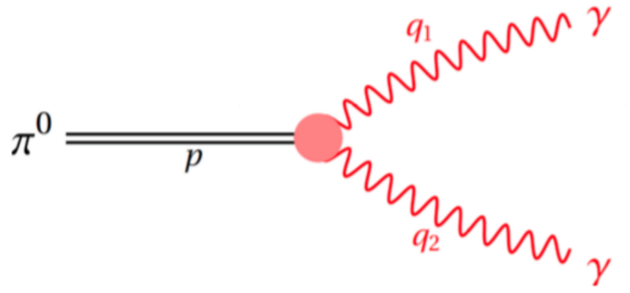

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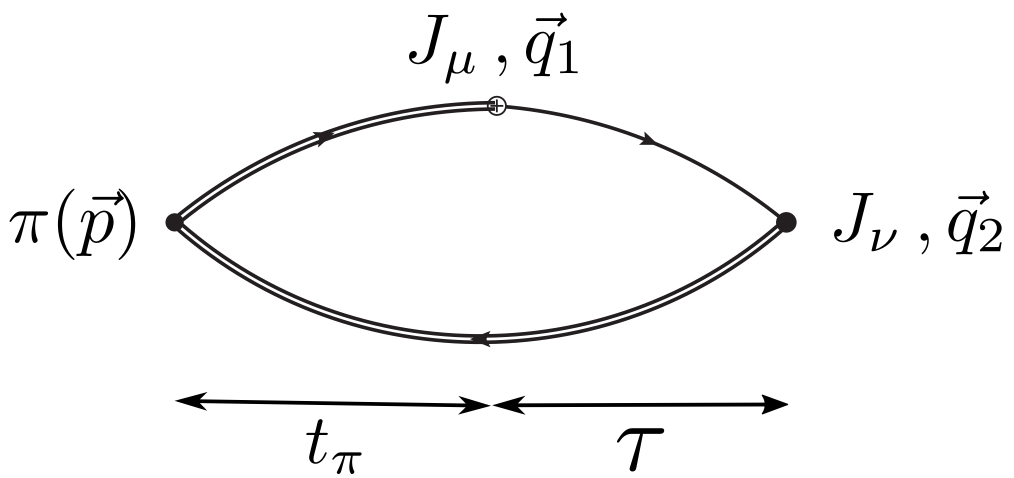

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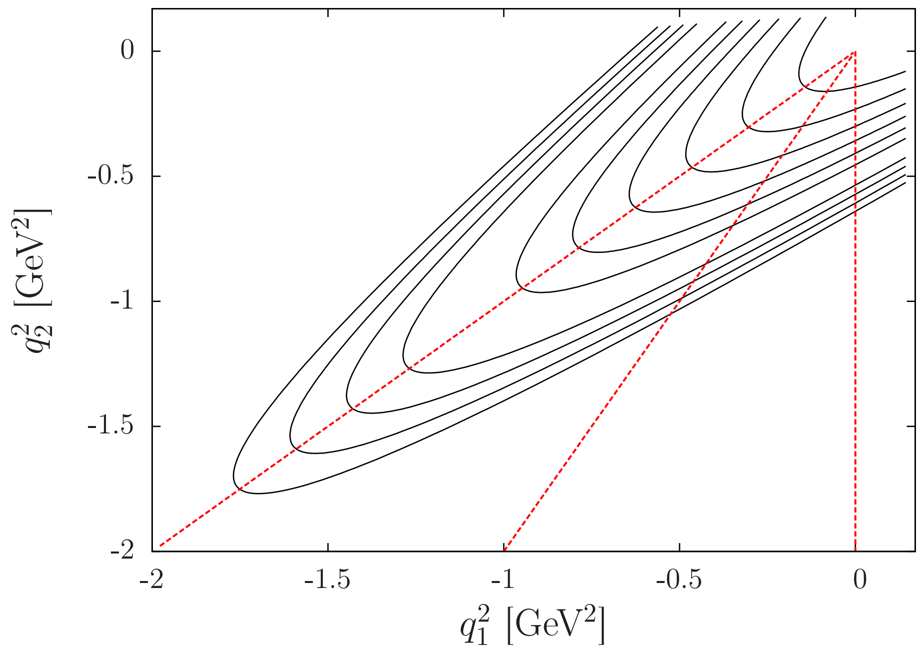
* Kinematics: $\vec{p} = 0, \quad q_1^2 = \omega_1^2 - |\vec{q}_1|^2, \quad q_2^2 = (m_\pi - \omega_1)^2 - |\vec{q}_1|^2$

[Antoine GÉRARDIN, FRI 18:10]

Transition form factor $\pi^0 \longrightarrow \gamma^*\gamma^*$

* Kinematical range:

[Antoine GÉRARDIN, FRI 18:10]



Fit ansatz:

Lowest meson dominance (LMD) model:

$$\mathcal{F}^{\mathrm{LMD}}_{\pi^0\gamma^*\gamma^*} = \frac{\alpha M_{\mathrm{V}}^4 + \beta(q_1^2 + q_2^2)}{(M_{\mathrm{V}}^2 - q_1^2)(M_{\mathrm{V}}^2 - q_2^2)}$$

(refinement: LMD-V model)

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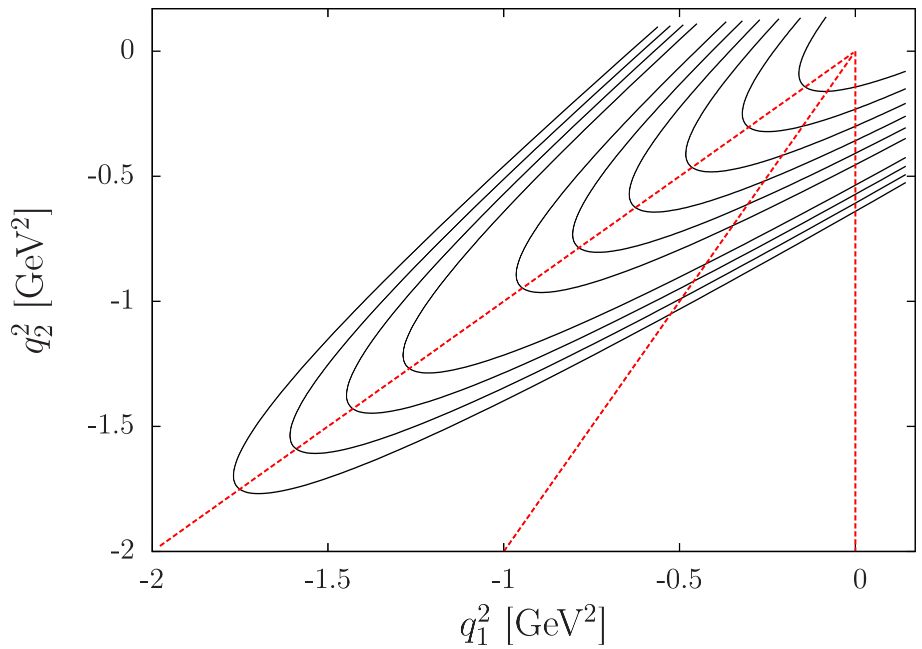
[Antoine GÉRARDIN, FRI 18:10]



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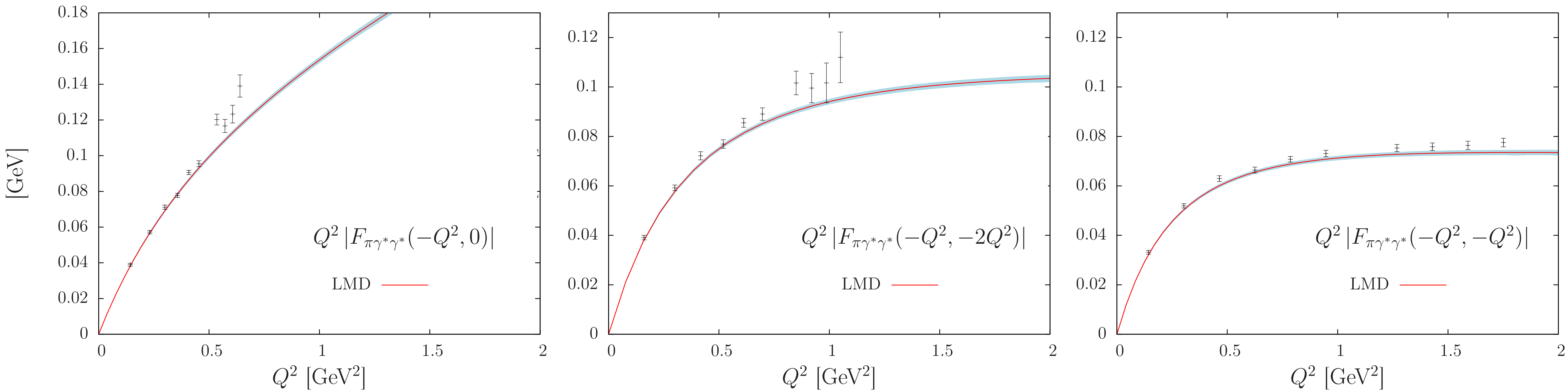

HLbL contribution from $\pi^0 \longrightarrow \gamma^*\gamma^*$

* Results for $N_f = 2$ flavours of $O(a)$ improved Wilson fermions:

$$\alpha = \begin{cases} 0.275(18)\,\mathrm{GeV}^{-1} & \text{(LMD)} \\ 0.273(24)\,\mathrm{GeV}^{-1} & \text{(LMD+V)} \end{cases}$$

Preliminary

(combined chiral and continuum extrapolation)

⇒ agrees well with theoretical expectation $\alpha = 1/(4\pi^2 F_\pi) = 0.274\,\mathrm{GeV}^{-1}$

[Antoine GÉRARDIN, FRI 18:10]

HLbL contribution from $\pi^0 \longrightarrow \gamma^*\gamma^*$

* Results for $N_f = 2$ flavours of $O(a)$ improved Wilson fermions:

$$\alpha = \begin{cases} 0.275(18)\,\mathrm{GeV}^{-1} & \text{(LMD)} \\ 0.273(24)\,\mathrm{GeV}^{-1} & \text{(LMD+V)} \end{cases}$$

Preliminary

(combined chiral and continuum extrapolation)

⇒ agrees well with theoretical expectation $\alpha = 1/(4\pi^2 F_\pi) = 0.274\,\mathrm{GeV}^{-1}$

* Results for π^0 contribution to hadronic light-by-light scattering:

$$(a_\mu^{\mathrm{hlbl}})_{\pi^0} = \begin{cases} (68.2 \pm 7.4) \cdot 10^{-11} & \text{(LMD)} \\ (65.0 \pm 8.3) \cdot 10^{-11} & \text{(LMD+V)} \end{cases}$$

Preliminary

⇒ agrees well with phenomenological studies

[Antoine GÉRARDIN, FRI 18:10]

Summary

Can lattice QCD deliver estimates with sufficient accuracy?

$$\delta a_\mu^{\rm hvp}/a_\mu^{\rm hvp} < \ 0.5\%, \qquad \delta a_\mu^{\rm hlbl}/a_\mu^{\rm hlbl} \lesssim \ 10\%$$

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- Statistical accuracy limited by disconnected diagrams
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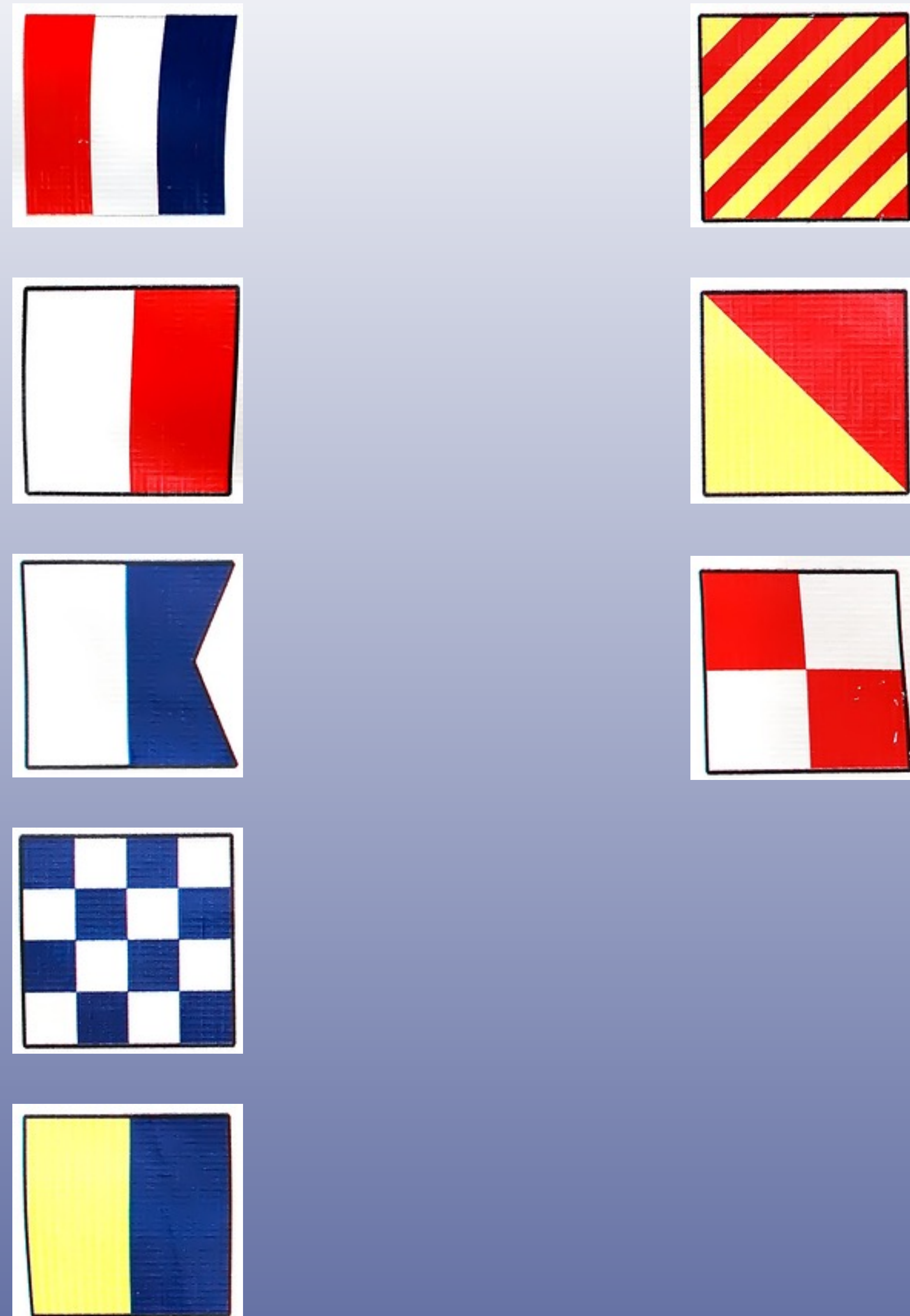
Hadronic Light-by-Light Scattering:

- Statistical accuracy: $\approx 10\%$ (connected)
- Disconnected contributions can be resolved
- Phenomenological models can be verified



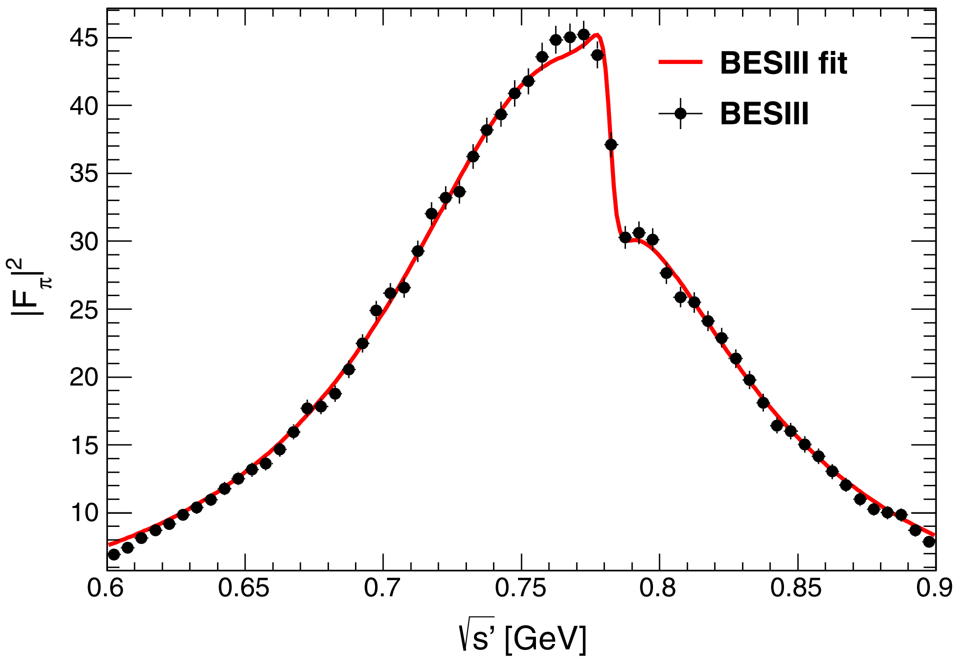
Spares

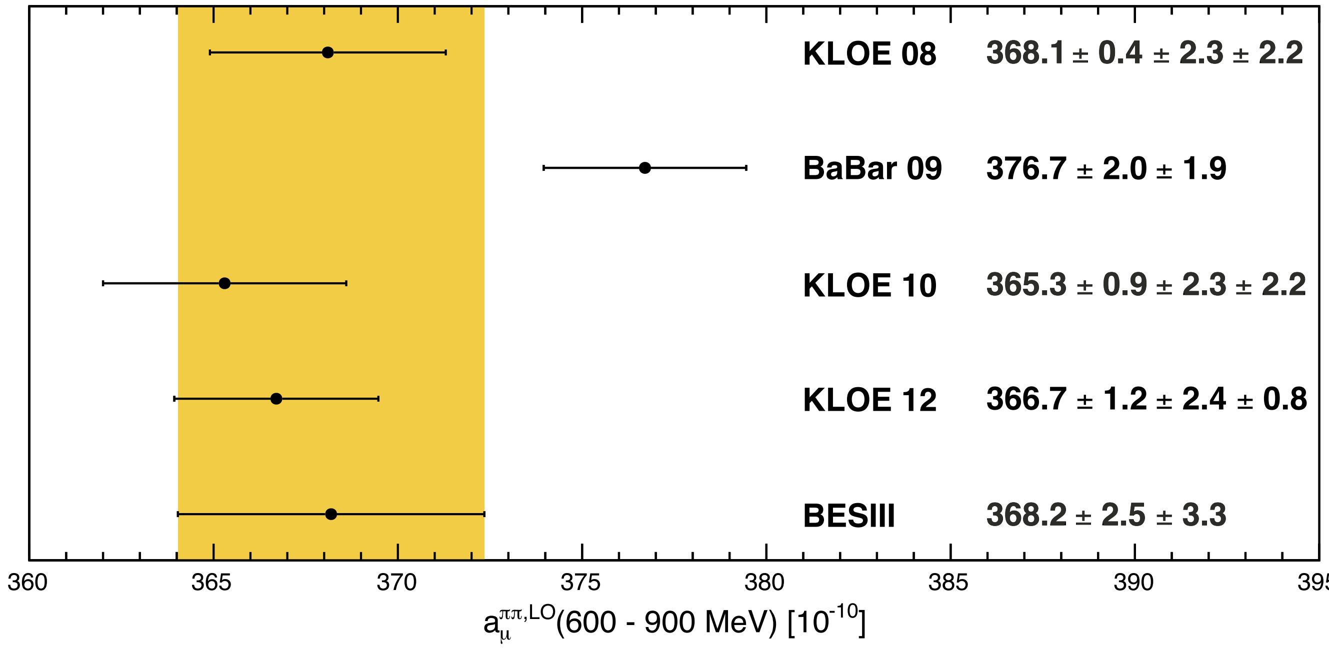
Anomalous magnetic moment of the muon

Hadronic Vacuum Polarisation and Dispersion Theory

$$a_\mu^{\rm hvp} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left\{ \int_{m_\pi^2}^{E_{\rm cut}^2} ds\, \frac{R_{\rm had}^{\rm data}(s)\hat{K}(s)}{s^2} + \int_{E_{\rm cut}^2}^{\infty} ds\, \frac{R_{\rm had}^{\rm pQCD}(s)\hat{K}(s)}{s^2} \right\}$$

* Relies on experimental data for hadronic cross section $R_{\rm had}(e^+e^- \to \text{hadrons})$





* New measurements of pion form factor by BESIII confirm 3.6σ tension

[BESIII Collaboration (M. Ablikim et al.), Phys Lett B753 (2016) 629]

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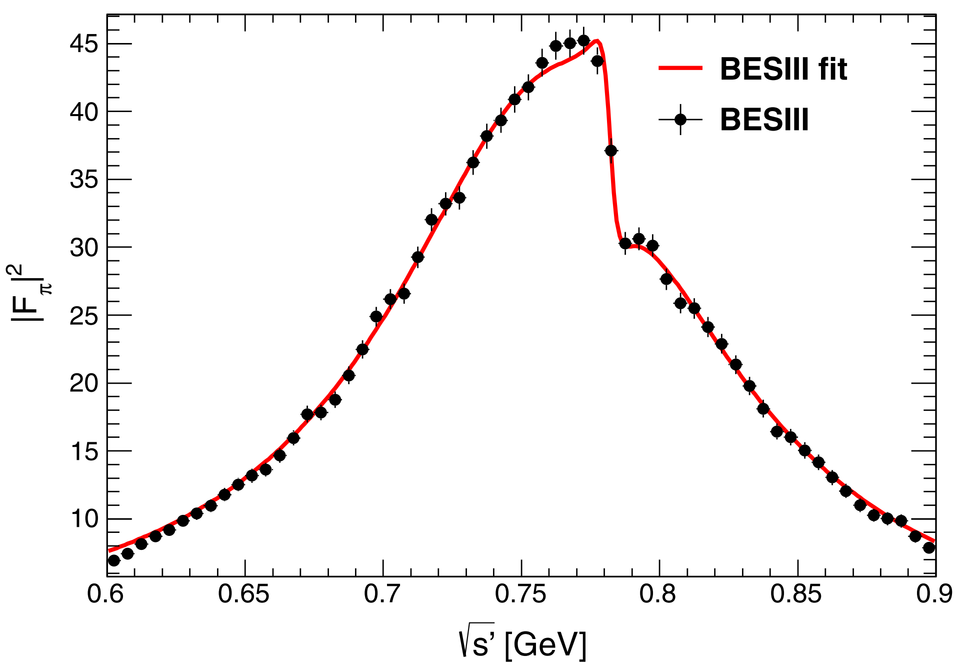
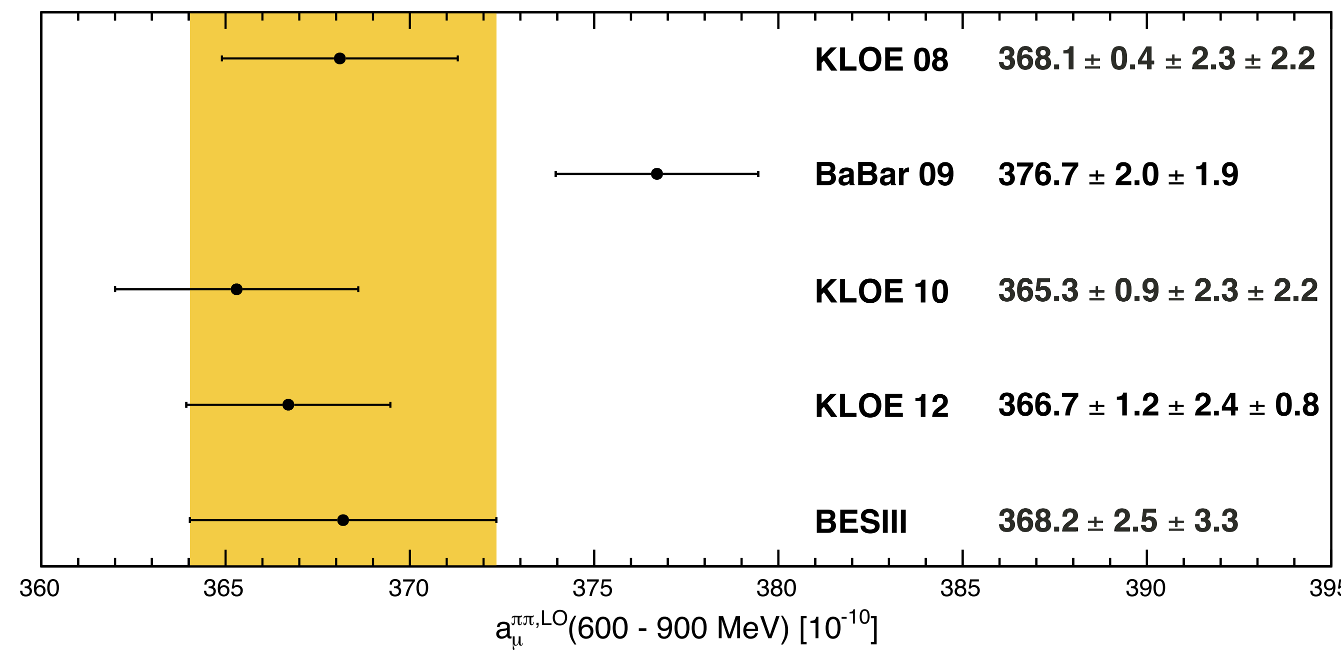




* Re-analysis of BaBar data in progress

[BESIII Collaboration (M. Ablikim et al.), Phys Lett B753 (2016) 629]

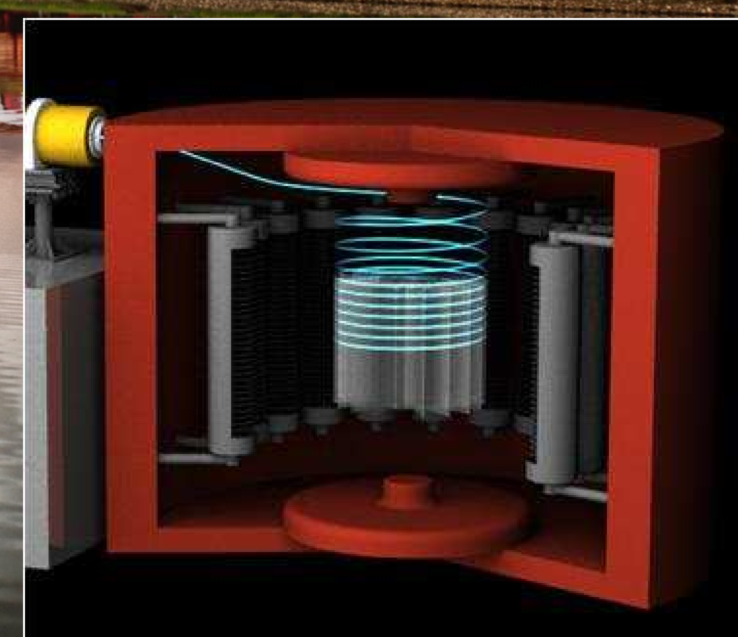
Future measurements

∗ Fermilab E989 (Storage ring of BNL E821)

- 20 times larger data sample
- better field calibration
- target precision of **0.14 ppm**

∗ J-PARC E34

- ultra-cold muon beam
- 66cm magnetic storage ring
- measure a_μ alongside d_μ
- target precision of **0.1 ppm**



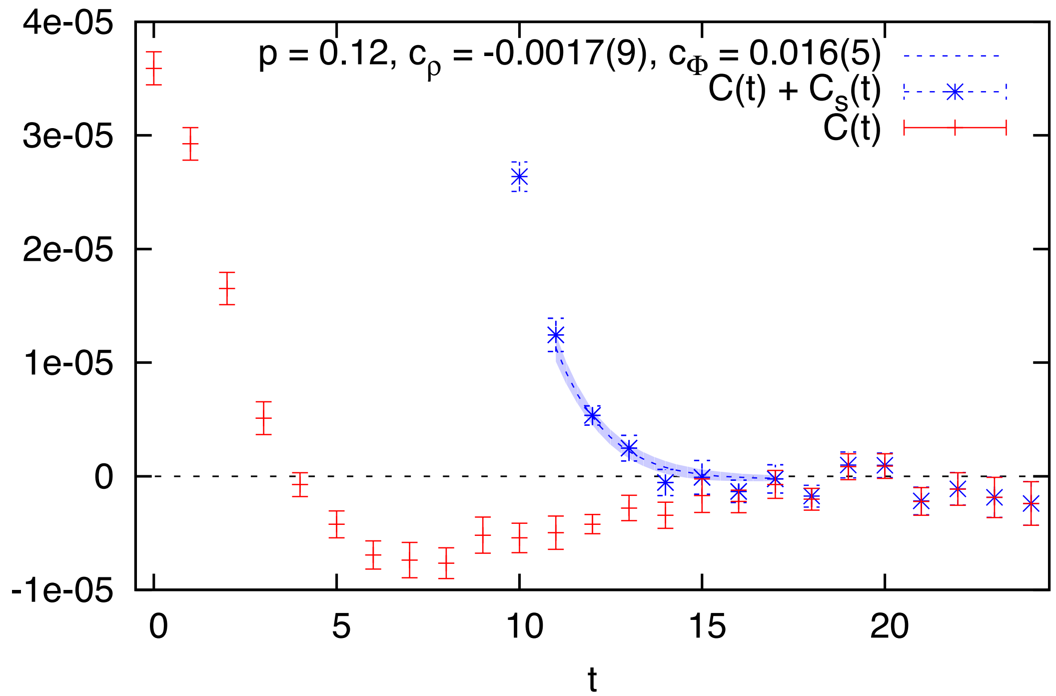
Disconnected Contributions

✱ Monitor saturation of:

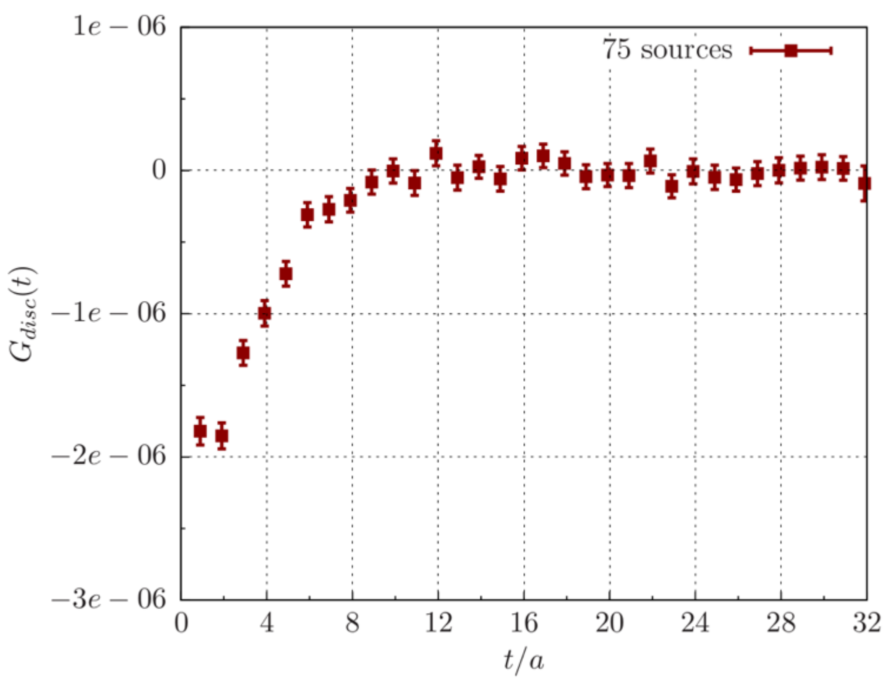
$$L_T = \sum_{x_0=0}^{T} w(x_0)\, D^{\ell s}(x_0) \xrightarrow{T\to\infty} (a_\mu^{\rm hvp})_{\rm disc}$$

$$w(x_0) = 4\alpha^2 \int_0^\infty \frac{dQ^2}{Q^2}\, f(Q^2)\left\{Q^2x_0^2 - 4\sin^2\left(\tfrac{1}{2}Qx_0\right)\right\}$$

[T. Blum et al., PRL 116 (2016) 232002]



[Gülpers et al., in preparation]



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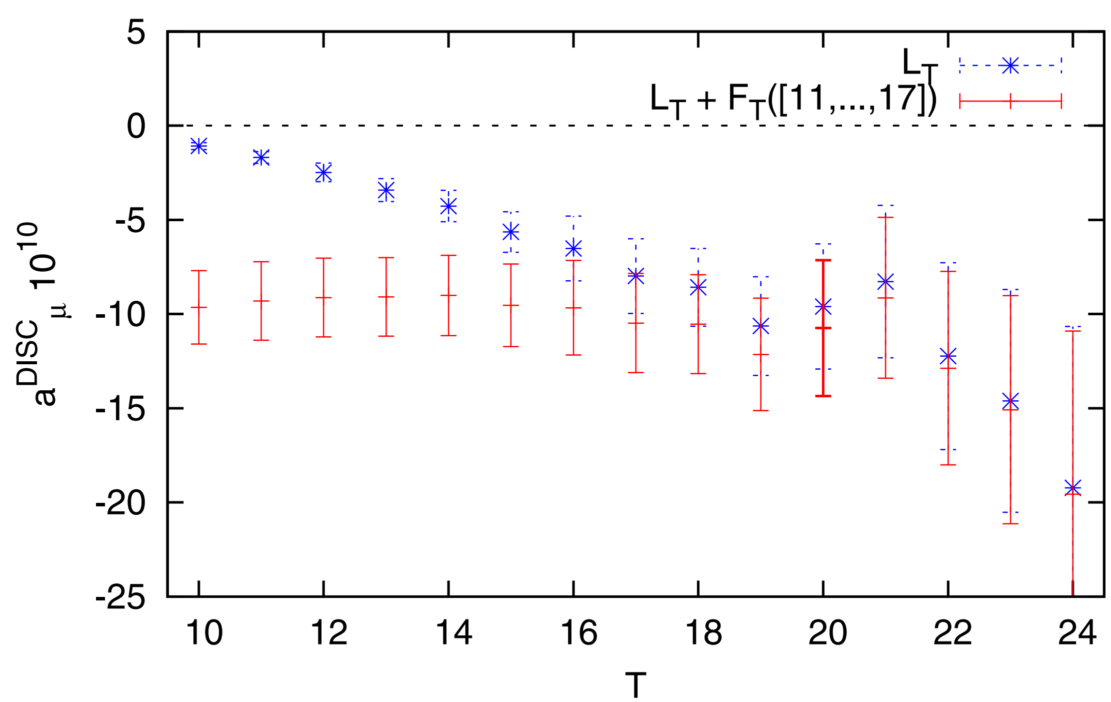

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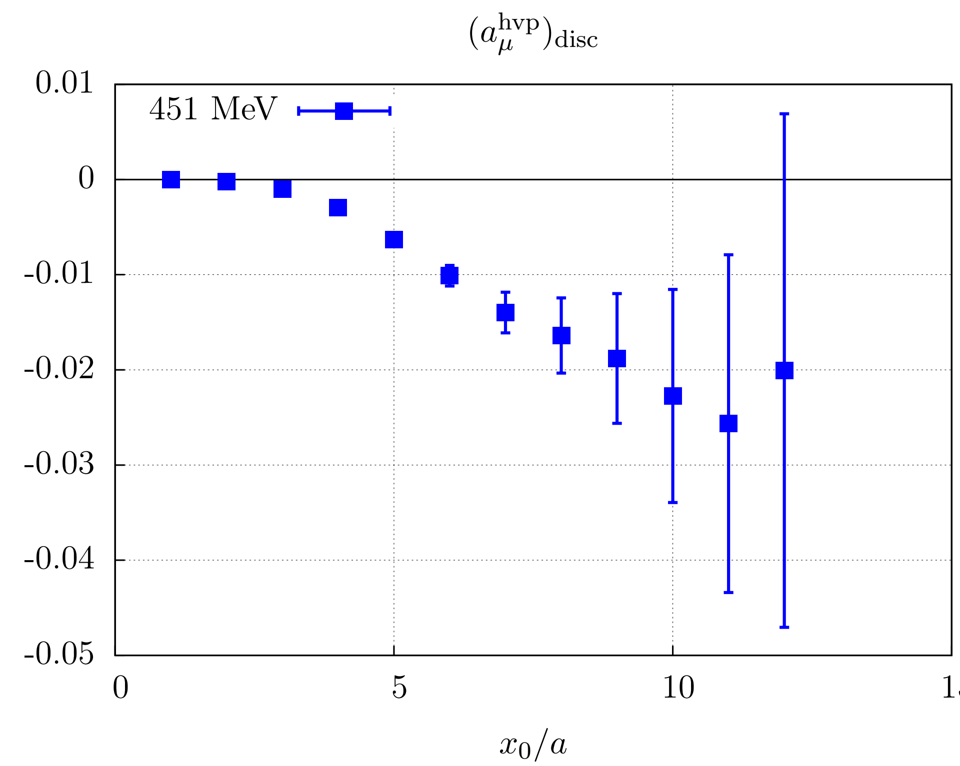


Disconnected Contributions: Results Summary

- Non-zero disconnected contribution can be resolved

	$(a_\mu^{\rm hvp})_{\rm disc}/(a_\mu^{\rm hvp})_{\rm con}^{\ell\ell}$	$(a_\mu^{\rm hvp})_{\rm disc}\cdot 10^{10}$	$\Pi^{\rm disc}/\Pi^{\rm con}$	Statistics
HPQCD	$-0.14(5)\%$	≈ -0.84		$N_{\rm cfg} = 553$, $N_{\rm dist} = 162$
RBC/UKQCD	$-1.6(7)\%$	$-(9.6 \pm 3.3 \pm 2.3)$		$N_{\rm inv} = 11.3$ k, $N_{\rm ev} = 2000$
CLS/Mainz	$-0.0032(11)\%$	$-(0.019 \pm 0.07)$		$N_{\rm inv} = 4800$ k
Bali & Endrődi			$-(3.6 \pm 4.5)\cdot 10^{-4}$	$N_{\rm inv} = 20$ k

HPQCD: Anisotropic Clover action; $m_\pi = 391$ MeV; $a_s \approx 0.12$ fm; Distillation

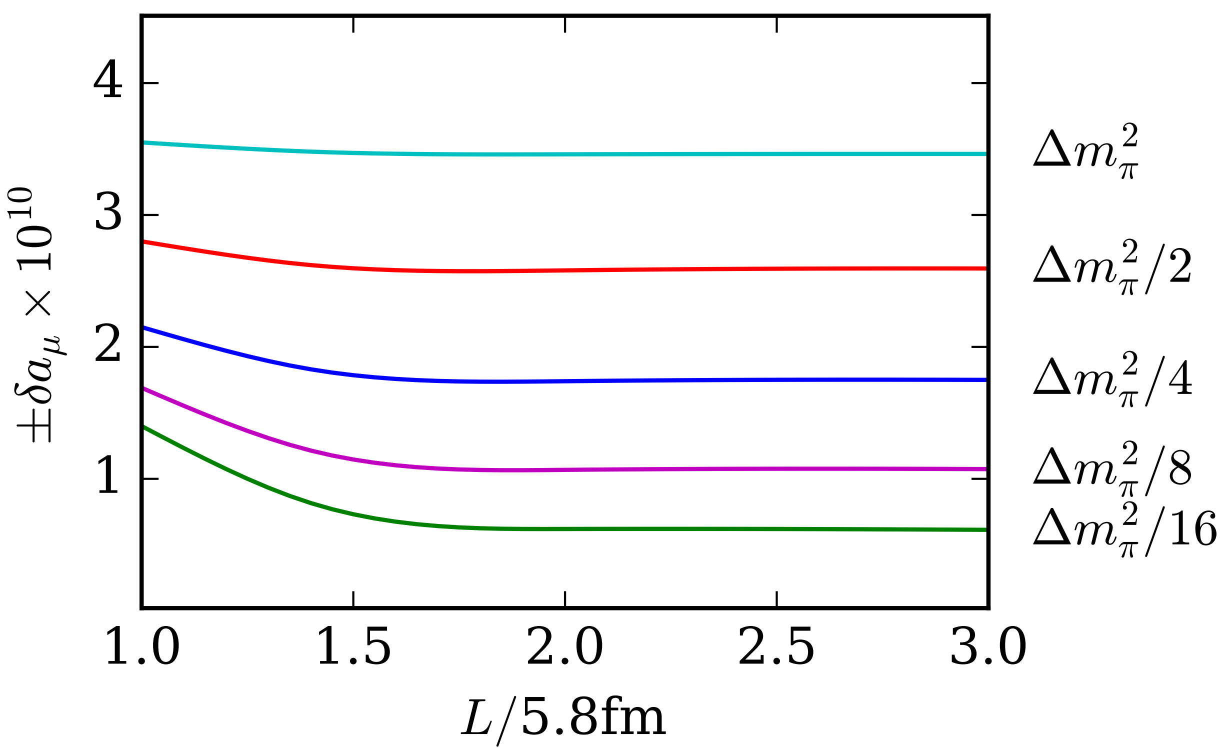
RBC/UKQCD: Domain wall fermions; physical pion mass; $a \approx 0.11$ fm, $m_\pi L \approx 3.9$; Low-mode averaging, AMA

CLS/Mainz: $N_{\rm f} = 2$ Clover fermions; $m_\pi = 311, 437$ MeV; $a = 0.063$ fm; HPE; 3D stochastic noise sources

Bali & Endrődi: Rooted staggered fermions; physical pion mass; $a = 0.1 - 0.29$ fm; 4D stochastic noise sources

Finite-volume effects: taste breaking

* Uncertainty in finite-volume shifts as a function of average taste splitting:



Scaling: $\Delta m_\pi^2 \sim a^2$

* Physical pion mass: $a_\mu^{\mathrm{hvp}}(\infty) - a_\mu^{\mathrm{hvp}}(L) = (7.0 \pm 0.7)\%$

[Chakraborty et al., arXiv:1601.03071, Christine DAVIES, poster session]