Non-perturbative running of quark masses in three-flavour QCD

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in collaboration with

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ALPHA Collaboration

XXXIV International Symposium on Lattice Field Theory
Southampton, 26 July 2016

Introduction

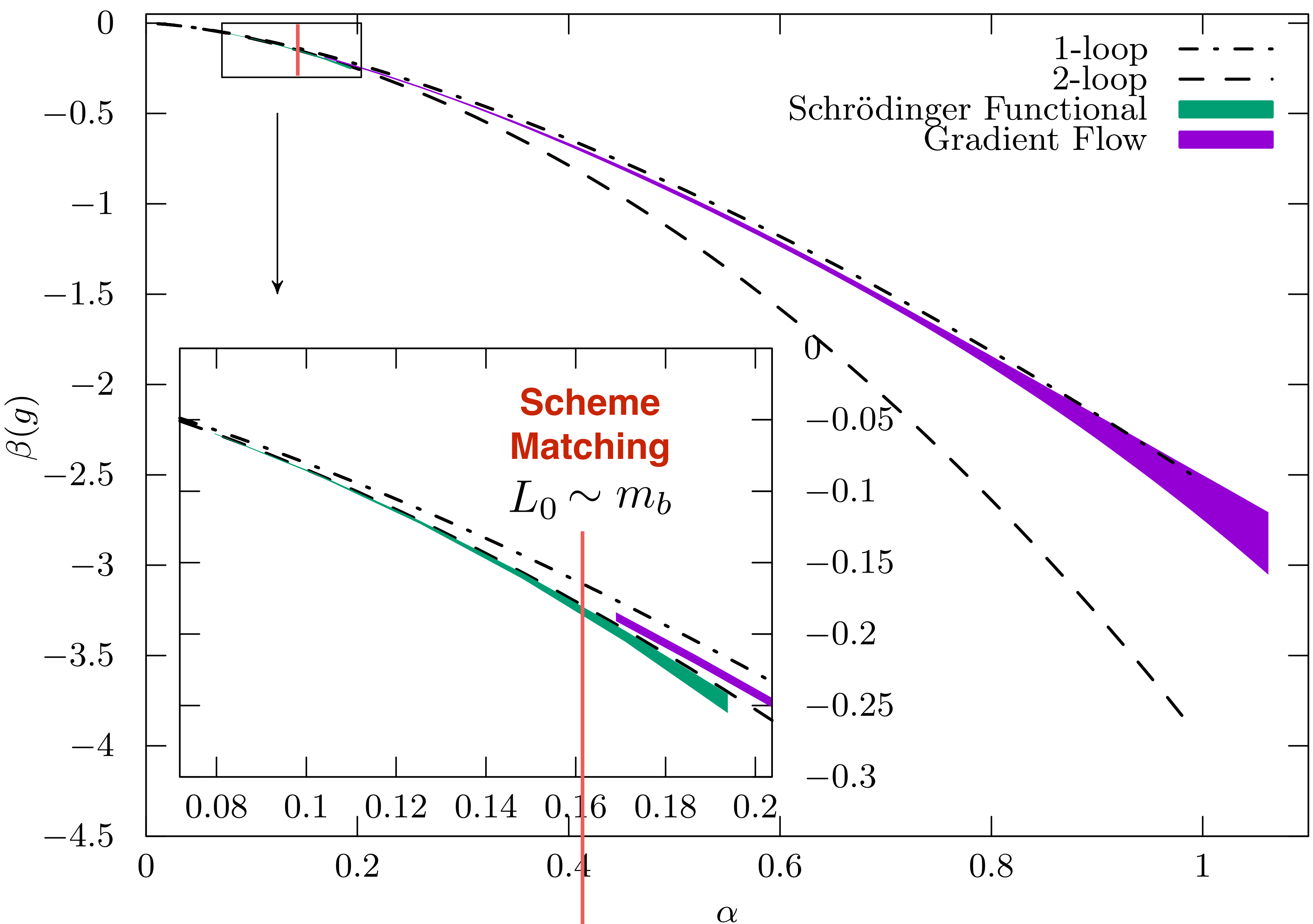
The goal of this project is to compute the **NP running of the quark masses with Nf=3 in the SF** with a crucial control on systematics and high accuracy in a large range of scales: from the EW scale down to an hadronic scale to make contact with large volume simulations.

Since this work is a joint project with the one of the **running coupling** by ALPHA we follow the **same strategy** they have been using.

The computational cost of measuring the SF coupling grows fast at low energies and in particular towards the continuum limit. Thus it is challenging to reach the low energy domain characteristic of hadronic physics, especially if one aims at maintaining the high precision.

The GF coupling seems to be better suited for this task. The relative precision of the coupling in this scheme is typically high and shows a weak dependence on both the energy scale and the cutoff.

Two Schemes, Two Regions, More fun



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A.Ramos,
1607.06423 [hep-lat]

S.Sint,
1604.06193 [hep-lat]

R.Sommer

PT (N)NLO — $\sim 60\,\mathrm{GeV}$ ⟷ $\sim 200\,\mathrm{MeV}$ — **CLS** Hadronic Simulations

SF coupling **GF coupling**

RG Equations

Renormalization Group functions for the coupling and mass are given by

$$\mu \frac{d\bar{g}}{d\mu} = \beta(\bar{g}) \qquad \boxed{\mu \frac{d\bar{m}}{d\mu} = \tau(\bar{g})\bar{m}}$$

they admit a perturbative expansion as

$$\beta(g) = -g^3(b_0 + b_1 g^2 + b_2 g^4 + \dots) \qquad \tau(g) = -g^2(d_0 + d_1 g^2 + d_2 g^4 + \dots)$$

where d_0, b_0, b_1 are the only scheme independent coefficients

We than introduce the **Renormalization Group Invariant** (RGI) quantities, formal solution of the RG equations as

$$\Lambda/\mu = (b_0 \bar{g}^2)^{-b_1/(2b_0^2)} e^{-1/(2b_0 \bar{g}^2(\mu))} \exp\left\{ - \int_0^{\bar{g}(\mu)} dx \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} \right] \right\}$$

$$M/\bar{m}(\mu) = (2b_0 \bar{g}^2(\mu))^{-d_0/(2b_0)} \exp\left\{ - \int_0^{\bar{g}(\mu)} dx \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x} \right] \right\}$$

The **RG evolution between two scales** $\mu, \mu/s$ is then

$$-\ln(s) = \int_{\sqrt{\bar{g}^2(\mu)}}^{\sqrt{\bar{g}^2(\mu/s)}} \frac{dg}{\beta(g)} \qquad \boxed{\frac{\bar{m}(\mu)}{\bar{m}(\mu/s)} = \exp\left\{ - \int_{\sqrt{\bar{g}^2(\mu)}}^{\sqrt{\bar{g}^2(\mu/s)}} \frac{\tau(g')}{\beta(g')} dg' \right\}}$$

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for $s = 2$ we have the "usual" definition of the **SSFs**

$$-\ln(2) = \int_{\sqrt{\mathrm{u}}}^{\sqrt{\sigma(\mathrm{u})}} \frac{\mathrm{dg}}{\beta(\mathrm{g})}$$

$$\boxed{\sigma_P(u) = \frac{\bar{m}(\mu)}{\bar{m}(\mu/2)} = \exp\left\{ -\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} \frac{\tau(g')}{\beta(g')} dg' \right\}}$$

SF Renormalization Scheme

M.Luscher et al.
Nucl.Phys. B384 (1992)

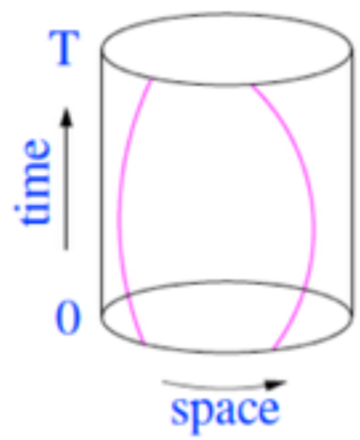
The SF renormalization condition is imposed at **vanishing quark mass**

M.Luscher et al.
Nucl.Phys. B582 (2000)

M.Della Morte et al.
Nucl.Phys. B729 (2005)

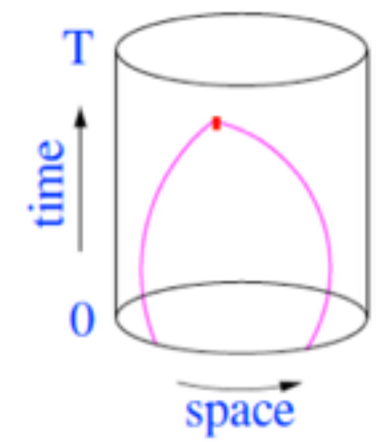
$$Z_P(g_0, L/a)\frac{f_P(L/2)}{\sqrt{3f_1}}\Bigg|_{m=0}^{\theta} = c_3(\theta, a/L) \qquad \theta = 0.5$$

The correlation functions entering the definition above are given by



$f_P(x_0)$

f_1



The lattice version of the SSF is than defined as the ratio of renormalization constants at L and $2L$ identifying $\mu = L^{-1}$ and for $s = 2$

$$\Sigma_P(u, g_0, L/a) = \frac{Z_P(g_0, 2L/a)}{Z_P(g_0, L/a)}\Bigg|_{u=\bar{g}^2(L)} \qquad \sigma_P(u) = \lim_{a\to 0}\Sigma_P(u, g_0, L/a)$$

$$u_{SF} = [1.1100, 1.1844, 1.2565, 1.3627, 1.4808, 1.6173, 1.7943, 2.0120]$$

$$u_{GF} = [2.1257, 2.3900, 2.7359, 3.2029, 3.8643, 4.4901, 5.3010, \boxed{5.8673, 6.5489}]$$

Continuum Extrapolation SF

$$\Sigma_P(u, a/L) = \sigma_P(u) + \rho'(u)(a/L)^2$$

The 1-loop improved SSF are defined as

$$\Sigma_P^{(1)}(u, a/L) = \frac{\Sigma_P(u, a/L)}{1 + \delta(a/L)u}$$

where $\delta(a/L)$ are the 1-loop SSF cutoff effects.

Sint and Weisz
[Nucl.Phys. B545 (1999) 529-542]

We consider lattices

$$L/a = [6, 8, 12]$$
$$2L/a = [12, 16, 24]$$

Since the **switching scale** between the two schemes is defined at

$$\bar{g}_{SF}^2(L_0) = 2.012$$

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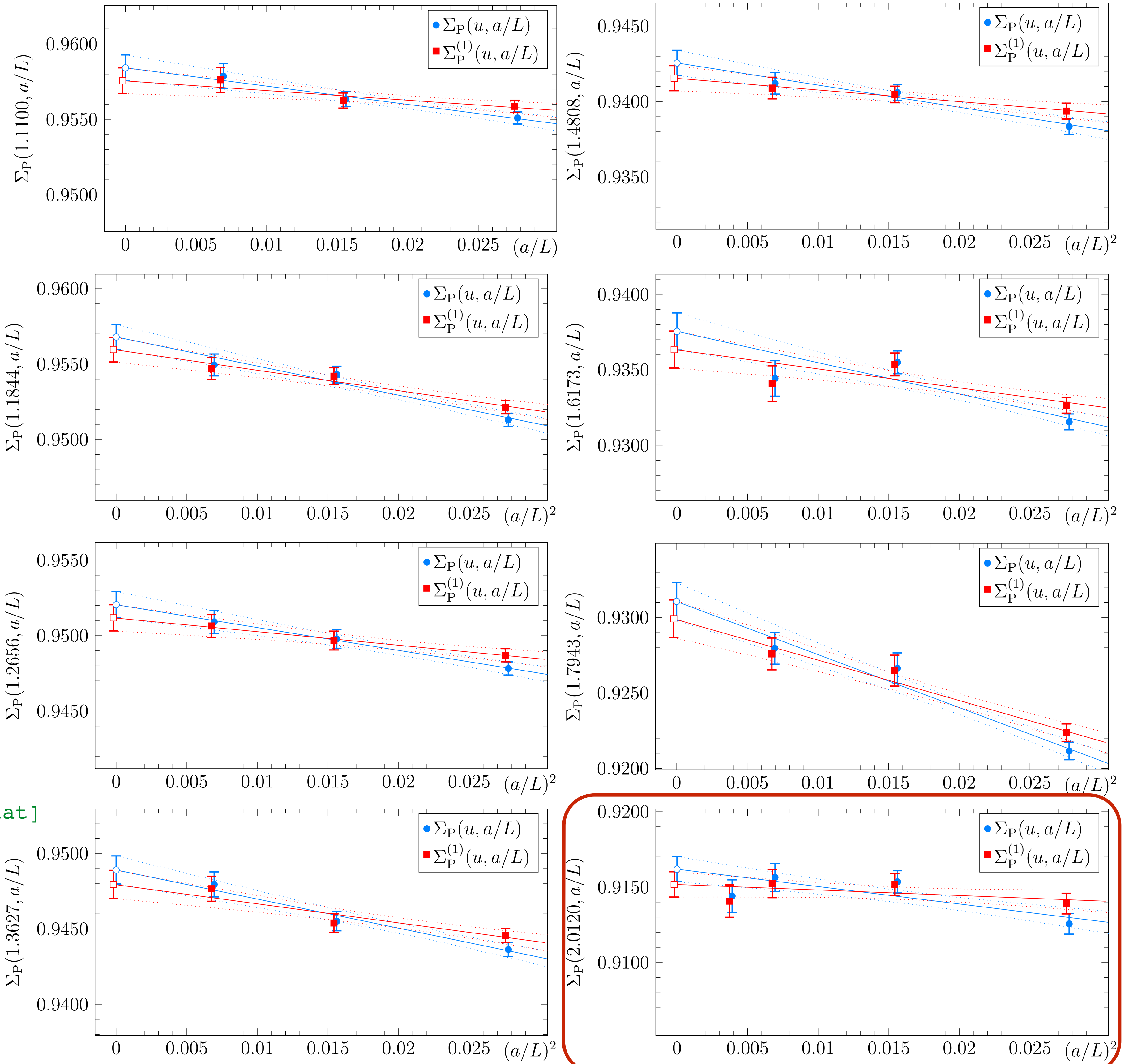
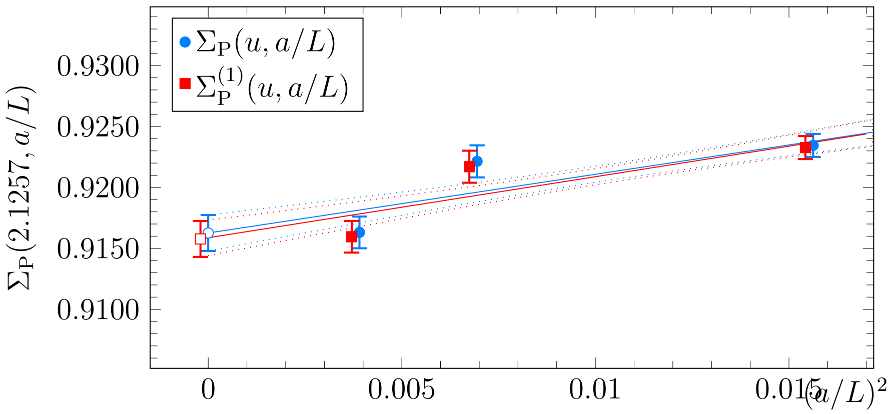
1607.06423 [hep-lat]

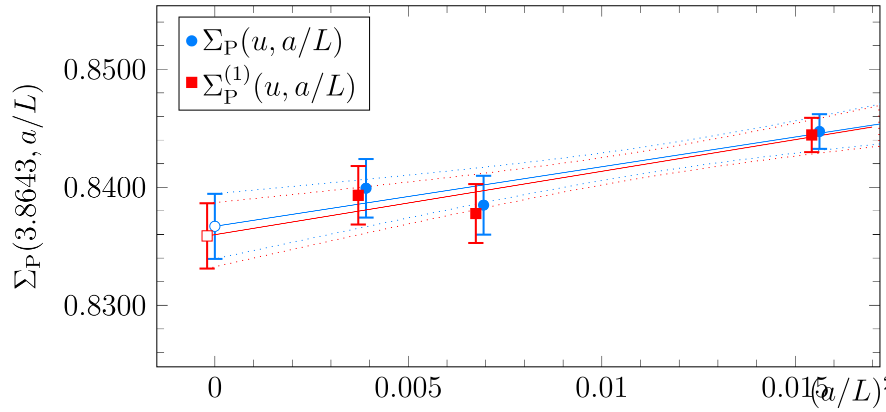
in order to have a better control on the continuum extrapolation we consider also the step $16 \to 32$

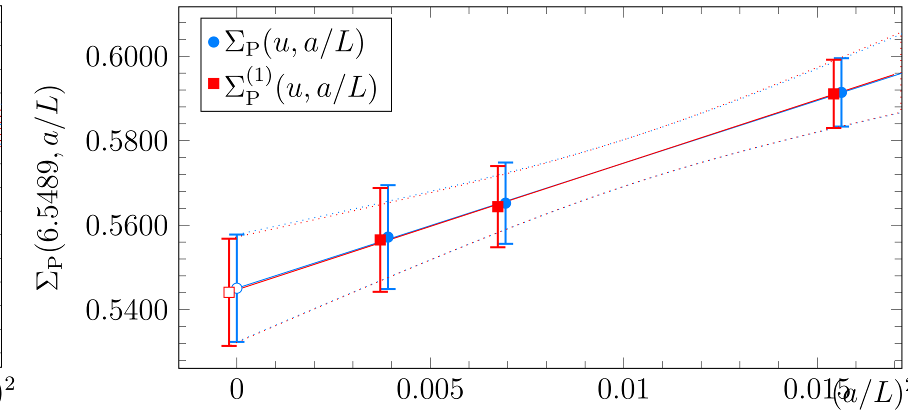


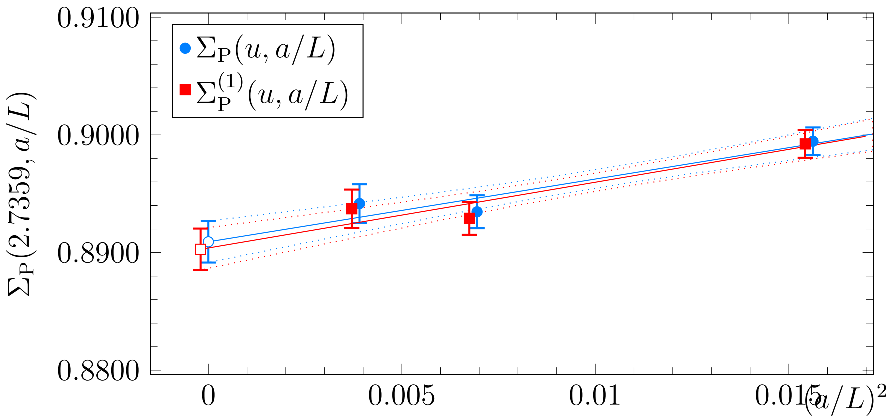
Continuum Extrapolation GF

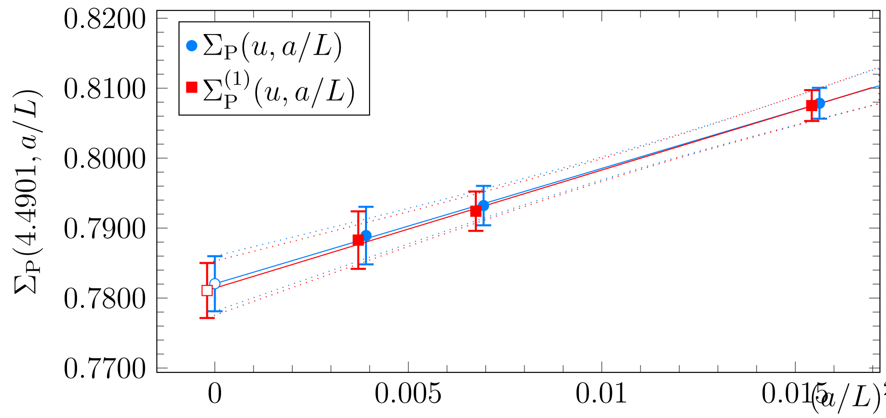
$$\Sigma_P(u,a/L) = \sigma_P(u) + \rho'(u)(a/L)^2$$

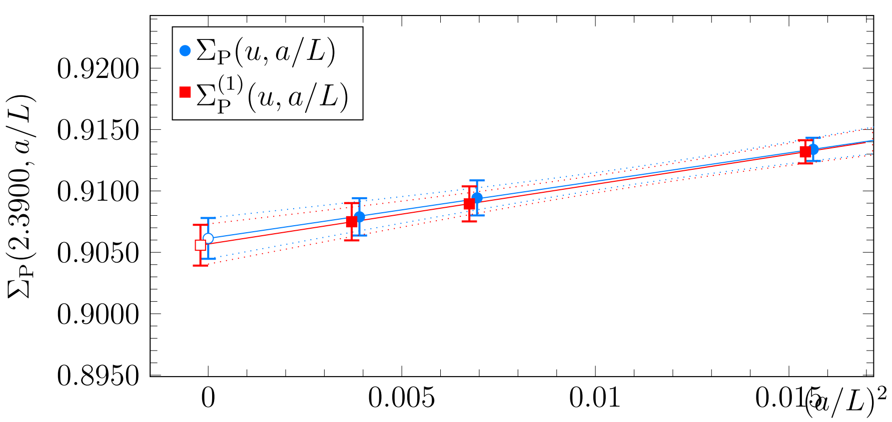


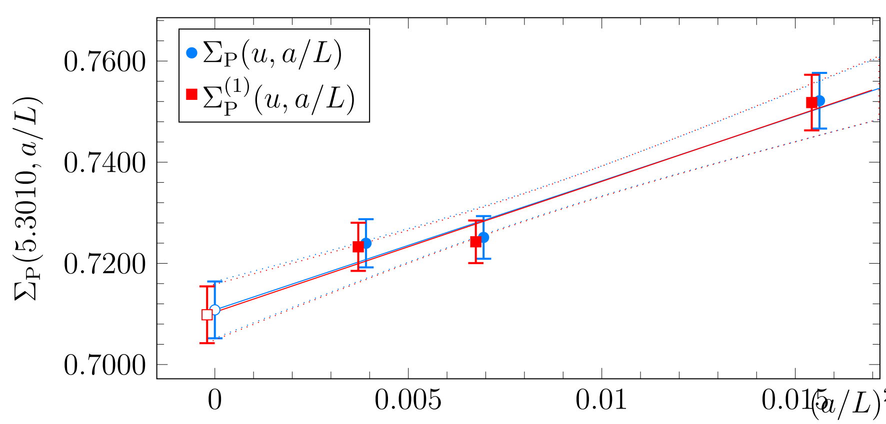


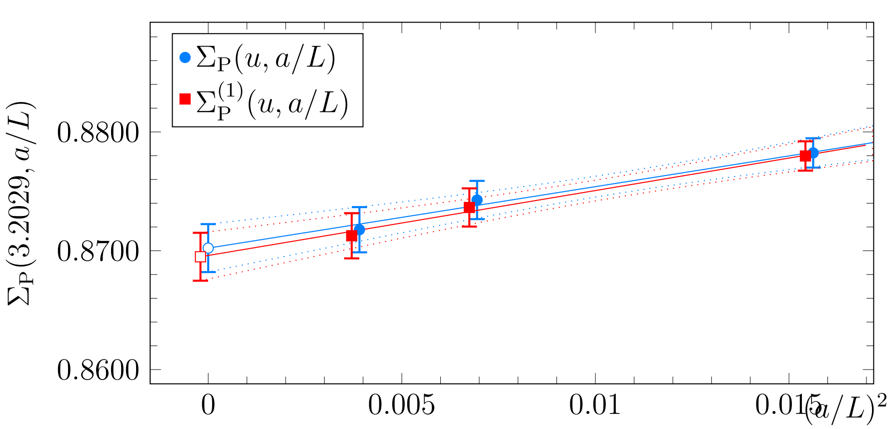


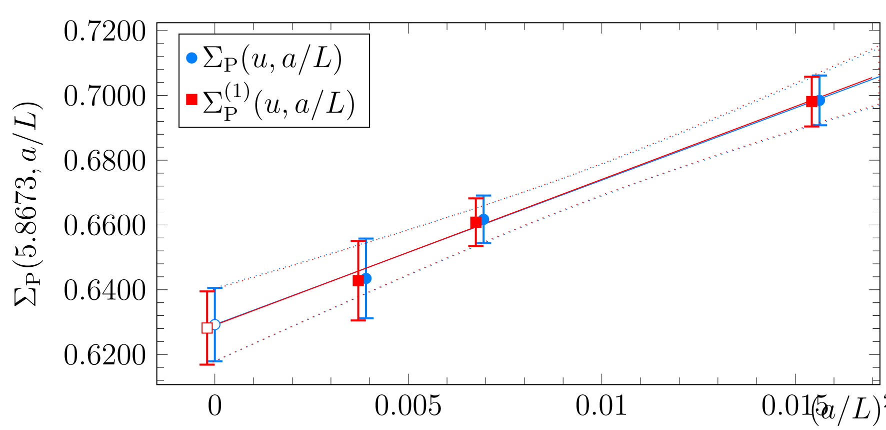












Due to the large cutoff effect induced by the GF coupling we use larger lattices respect to the ones used in SF

$$L/a = [8, 12, 16]$$

$$2L/a = [16, 24, 32]$$

From the GF side, the switching scale is defined by

$$\bar{g}^2_{GF}(2L_0) = 2.6723(64)$$

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A.Ramos,
1607.06423 [hep-lat]

Step Scaling Functions

In order to be able to compute the running we have to fit the SSF. It can be fitted as a polynomial where the first coefficient is fixed by perturbation theory

$$\sigma_P(u) = 1 + p_1 u + p_2 u^2 + p_3 u^3 + p_4 u^4 + \ldots \qquad p_1^{PT} = -d_0 \ln(2)$$

another equivalent strategy adopted in this project for the first time is to fit directly for an NP τ with the universal coefficient fixed to PT.

$$\sigma_P(u) = \exp\left\{ -\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} dg \frac{\tau^{NP}(g)}{\beta^{NP}(g)} \right\} \qquad -\ln(2) = \int_{\sqrt{\mathrm{u}}}^{\sqrt{\sigma(\mathrm{u})}} \frac{\mathrm{dg}}{\beta^{\mathrm{NP}}(\mathrm{g})}$$

where the NP effective β is used as an input in this analysis:

$$\beta^{NP}(g) = \begin{cases} -g^3 P(g) & \mathrm{SF} \\ \frac{-g^3}{\tilde{P}(g)} & \mathrm{GF} \end{cases} \qquad P(g) = \sum b_k g^{2k}$$

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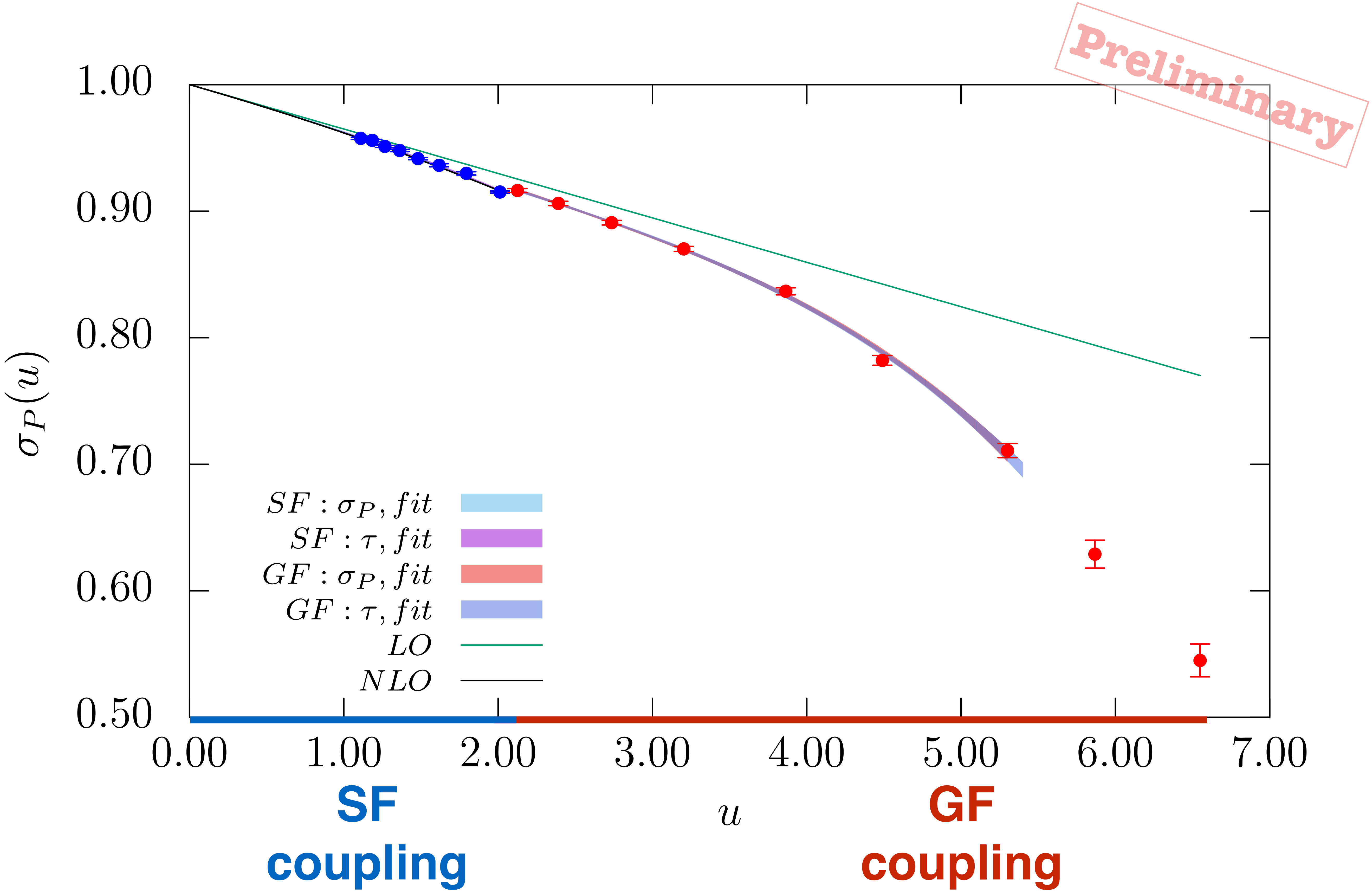
1607.06423 [hep-lat]

and the anomalous dimension is fitted as a polynomial with LO and NLO coefficients fixed to PT in the SF side.

$$\tau^{NP}(g) = \begin{cases} -g^2(d_0 + d_1 g^2 + t_2 g^4 + t_3 g^6) & \mathrm{SF} \\ -g^2(t_0 + t_1 g^2 + t_2 g^4 + t_3 g^6) & \mathrm{GF} \end{cases}$$

Step Scaling Functions



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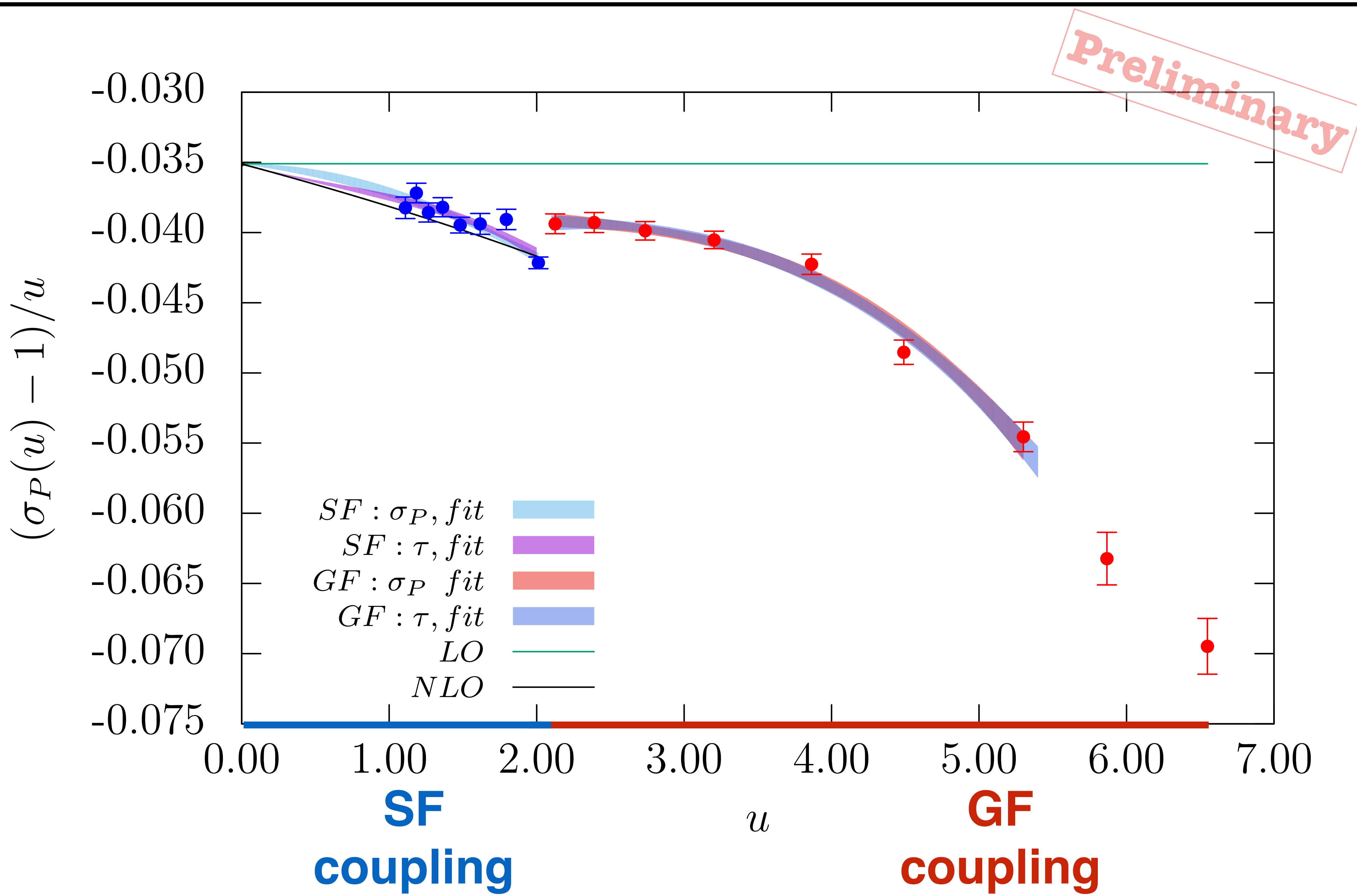
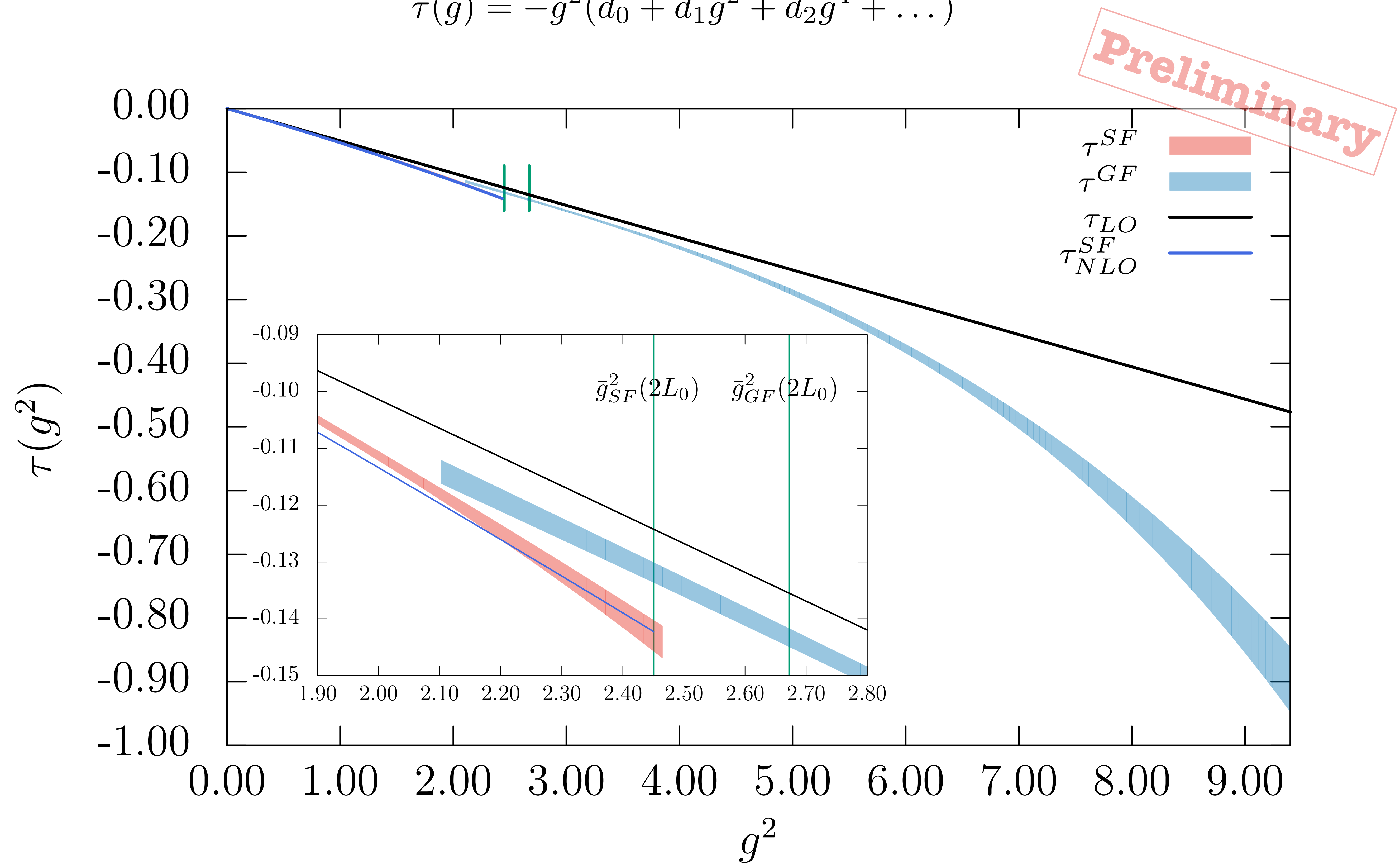


NP Mass Anomalous Dimension SF & GF

$$\tau(g) = -g^2(d_0 + d_1 g^2 + d_2 g^4 + \dots)$$



NP Running: strategy

The running from an hadronic scale identified by L_{had} can be written as

$$\frac{M}{\bar{m}(L_{had})} = \left.\frac{M}{\bar{m}(L_{pt})}\right|_{SF} \left.\frac{\bar{m}(L_{pt})}{\bar{m}(L_0)}\right|_{SF} \left.\frac{\bar{m}(L_0)}{\bar{m}(2L_0)}\right|_{SF} \left.\frac{\bar{m}(2L_0)}{\bar{m}(L_{had})}\right|_{GF}$$

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where each term explicitly is computed as it follows

$$\frac{M}{\bar{m}(L_{pt})} = \left(2b_0\bar{g}^2(L_{pt})\right)^{-d_0/(2b_0)} \exp\left\{-\int_0^{\bar{g}(L_{pt})} dx \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x}\right]\right\} \qquad L_{pt} = \frac{L_0}{2^N}$$

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$$\frac{\bar{m}(L_{pt})}{\bar{m}(L_0)} = \frac{\bar{m}(L_{pt})}{\bar{m}(L_0/2^{N-1})} \frac{\bar{m}(L_0/2^{N-1})}{\bar{m}(L_0/2^{N-2})} \cdots \frac{\bar{m}(L_0/2)}{\bar{m}(L_0)} = \prod_{i=1}^{N} \sigma_P(u_i) \qquad \begin{gathered} \sigma(u_{i+1}) = u_i \\ u_i = \bar{g}^2(2^{-i}L_0) \end{gathered}$$

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$$\frac{\bar{m}(L_0)}{\bar{m}(2L_0)}\bigg|_{SF} = \sigma_P(u_0) \quad u_0 = \bar{g}^2(L_0) = 2.012 \longleftarrow$$

Connection with GF scheme

NP Running: strategy

The running from an hadronic scale identified by L_{had} can be written as

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$$\left.\frac{\bar{m}(L_0)}{\bar{m}(2L_0)}\right|_{SF} = \sigma_P(u_0) \quad u_0 = \bar{g}^2(L_0) = 2.012 \longleftarrow$$

Connection with GF scheme

$$\frac{\bar{m}(2L_0)}{\bar{m}(L_{had})} = \exp\left\{-\int_{g(2L_0)}^{g(L_{had})} dg\,\frac{\tau_{GF}^{NP}(g)}{\beta_{GF}^{NP}(g)}\right\}$$

(this procedure is alternative to the usual SSF recursion)

NP Running: results

Preliminary

Two strategy for computing the NP running have been applied, the usual fit of the SSF and the extraction of the effective anomalous dimension for the mass. Both procedure agree.

SF

SSF recursion $\quad \dfrac{\bar{m}(L_{pt})}{\bar{m}(L_0)} = 0.7949(13) \qquad \dfrac{M}{\bar{m}(L_0)} = 1.9163(42) \quad \text{rel err} = 0.2\%$

NP $\quad \tau, \beta \quad \dfrac{\bar{m}(L_{pt})}{\bar{m}(L_0)} = 0.7946(21) \qquad \dfrac{M}{\bar{m}(L_0)} = 1.9156(50) \quad \text{rel err} = 0.3\%$

The connection with PT at NLO has been performed at

$$u_{pt} = \bar{g}^2_{SF}(L_{pt} = 2^{-4}L_0) = 1.19187(510) \sim 64\,\mathrm{GeV}$$

In order to go below $2L_0$ it is than required to switch the scheme to the one denoted by the GF coupling.

The largest renormalized GF coupling we are considering that will allow us to make contact with CLS large volume simulations have been estimated to be $u_{had} = 5.3010$

SF+GF

$$\bar{g}^2(L_{had}) = \sigma(u_{had}) = 9.3812 \qquad L_{had} \sim 213\,\mathrm{MeV}$$

NP $\quad \tau, \beta \quad \dfrac{\bar{m}(2L_0)}{\bar{m}(L_{had})} = 0.5184(42) \qquad \dfrac{M}{\bar{m}(L_{had})} = 0.9088(78) \quad \text{rel err} = 0.9\%$

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1607.06423 [hep-lat]
&
R.Sommer

$$1/L_0 \sim 4\,\mathrm{GeV} \qquad \frac{L_{had}}{L_0} = 18.74(26) \qquad \frac{L_{had}}{L_{pt}} = 300(4)$$

Conclusions & Outlook

- We have computed the **NP running quark mass for Nf=3** between $\sim 200\,\mathrm{MeV}$ and $\sim 60\,\mathrm{GeV}$ with an uncertainty $\lesssim 1\%$

- For the first time we dealt with **two schemes**, providing a strategy for a **NP matching** between them at the intermediate scale of $\sim\ 2\,\mathrm{GeV}$

- We are also providing for the first time an **”effective” NP anomalous dimension** for both SF and GF-based schemes allowing to chose L_{had} in a broad range of values. What has been showed here is just one illustration pushing toward the lowest possible value of the hadronic matching scale.

- The next point in the project is the **matching with large volume betas**

- Along with the mass project we have collected data for applying the same strategy to the **Tensor current** the only other bilinear with an independent anomalous dimension.

Backup

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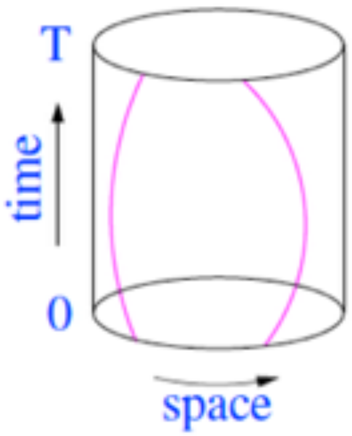
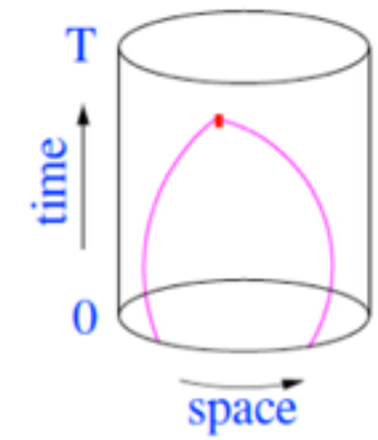


$$f_P(x_0) = -\frac{1}{3}\int d^3\mathbf{y}d^3\mathbf{z}\langle\bar{\psi}(x)\gamma_5\frac{1}{2}\tau^a\psi(x)\bar{\zeta}(\mathbf{y})\gamma_5\frac{1}{2}\tau^a\zeta(\mathbf{z})\rangle$$

$$f_1 = -\frac{1}{3L^6}\int d^3\mathbf{u}d^3\mathbf{v}d^3\mathbf{y}d^3\mathbf{z}\langle\bar{\zeta}(\mathbf{u})\gamma_5\frac{1}{2}\tau^a\zeta(\mathbf{v})\bar{\zeta}(\mathbf{y})\gamma_5\frac{1}{2}\tau^a\zeta(\mathbf{z})\rangle$$



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Two Schemes, Two Regions, More fun

The peculiarity of this work is to consider two **different renormalization scheme for the couplings**

$$u_{SF} = [1.1100, 1.1844, 1.2565, 1.3627, 1.4808, 1.6173, 1.7943, 2.0120]$$ **High**

$$u_{GF} = [2.1257, 2.3900, 2.7359, 3.2029, 3.8643, 4.4901, 5.3010, 5.8673, 6.5489]$$ **Low**

but **same renormalization condition for the mass**!

A change of scheme for both coupling and mass can be written in terms of the differences of finite parts χ

$$g'_R = g_R\sqrt{\chi_g(g_R)} \qquad \beta'(g'_R) = \left\{\beta(g_R)\frac{\partial g'_R}{\partial g_R}\right\}$$

$$m'_R = m_R\chi_m(g_R) \qquad \tau'(g'_R) = \left\{\tau(g_R) + \beta(g_R)\frac{\partial}{\partial g_R}\ln\chi_m(g_R)\right\}_{g_R=g_R(g'_R)}$$

At 1-loop for instance one can easily see how the NLO anomalous dimension vary from one scheme to another due to a change of scheme in the renormalized coupling through the finite parts $\chi_g^{(1)}$

$$\chi(g_R) \overset{g_R\to 0}{\sim} 1 + \sum_{k=1}^{\infty}\chi^{(k)}g_R^{2k} \qquad d'_1 = d_1 + 2b_0\chi_m^{(1)} - d_0\chi_g^{(1)}$$

Sint and Weisz [Nucl.Phys. B545 (1999) 529-542]

Since we do not know the perturbative finite parts from GF and we do not want to rely on PT at $2L_0 \sim m_b/2$ we perform a **NP matching**

Systematic Errors

- $c_t, \tilde{c}_t$ do not contribute at 1-loop for $\sigma_P(u)$ but still under investigation

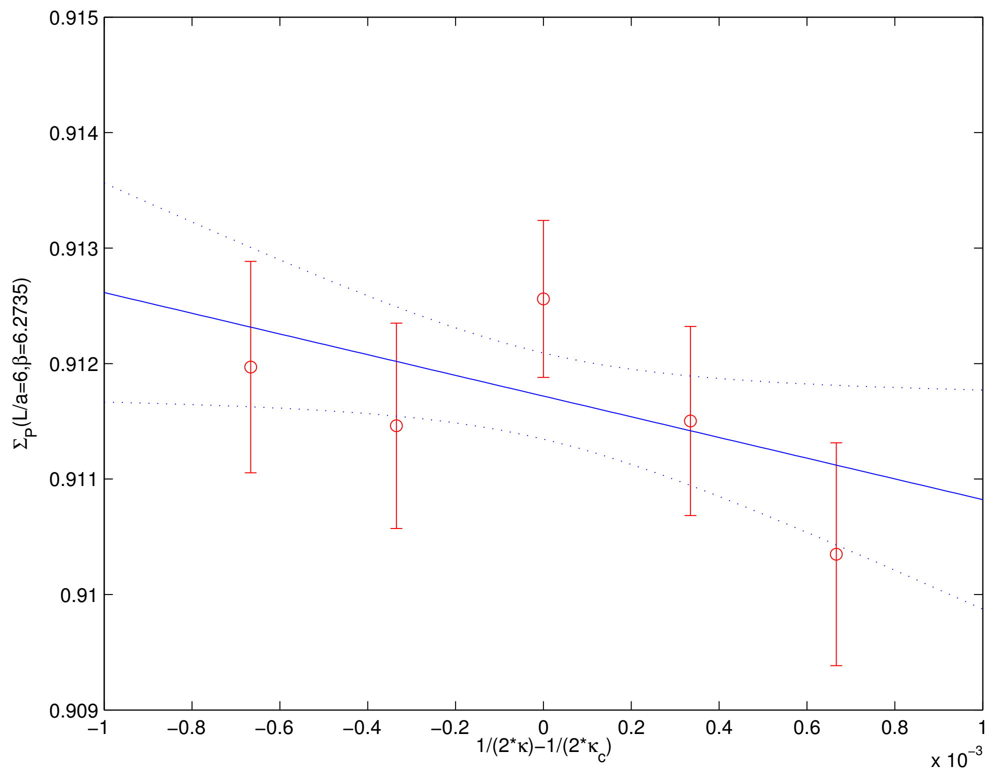
Sint and Weisz [Nucl.Phys. B545 (1999) 529-542]

- δu_i Still under investigation

- $\delta \kappa_c$

$$\Sigma'_P = \frac{1}{L} \left. \frac{\partial \Sigma_P}{\partial m} \right|_{u,L}$$

N_f	u	Σ'_P
2	0.9793	-0.0755(10)
2	2.4792	-0.1130(27)
3	2.012	-0.149(145)



Given the tuning precision to be $|Lm| < 0.001$ resulting systematic error coming from the tuning procedure is then $\sim 0.00015(14)$ that is negligible respect to the the statistical error

Systematics Tuning

Systematic uncertainties from tuning:

- κ_c $\qquad\qquad$ $Lm < 0.001$

1-loop estimate:

$$\Delta Z = g_0^2 \left| \frac{1}{f_P^{(0)}} \frac{\partial f_P^{(0)}}{\partial m_0} - \frac{1}{2 f_1^{(0)}} \frac{\partial f_1^{(0)}}{\partial m_0} \right| \Delta m \sim \mathcal{O}(10^{-4})$$

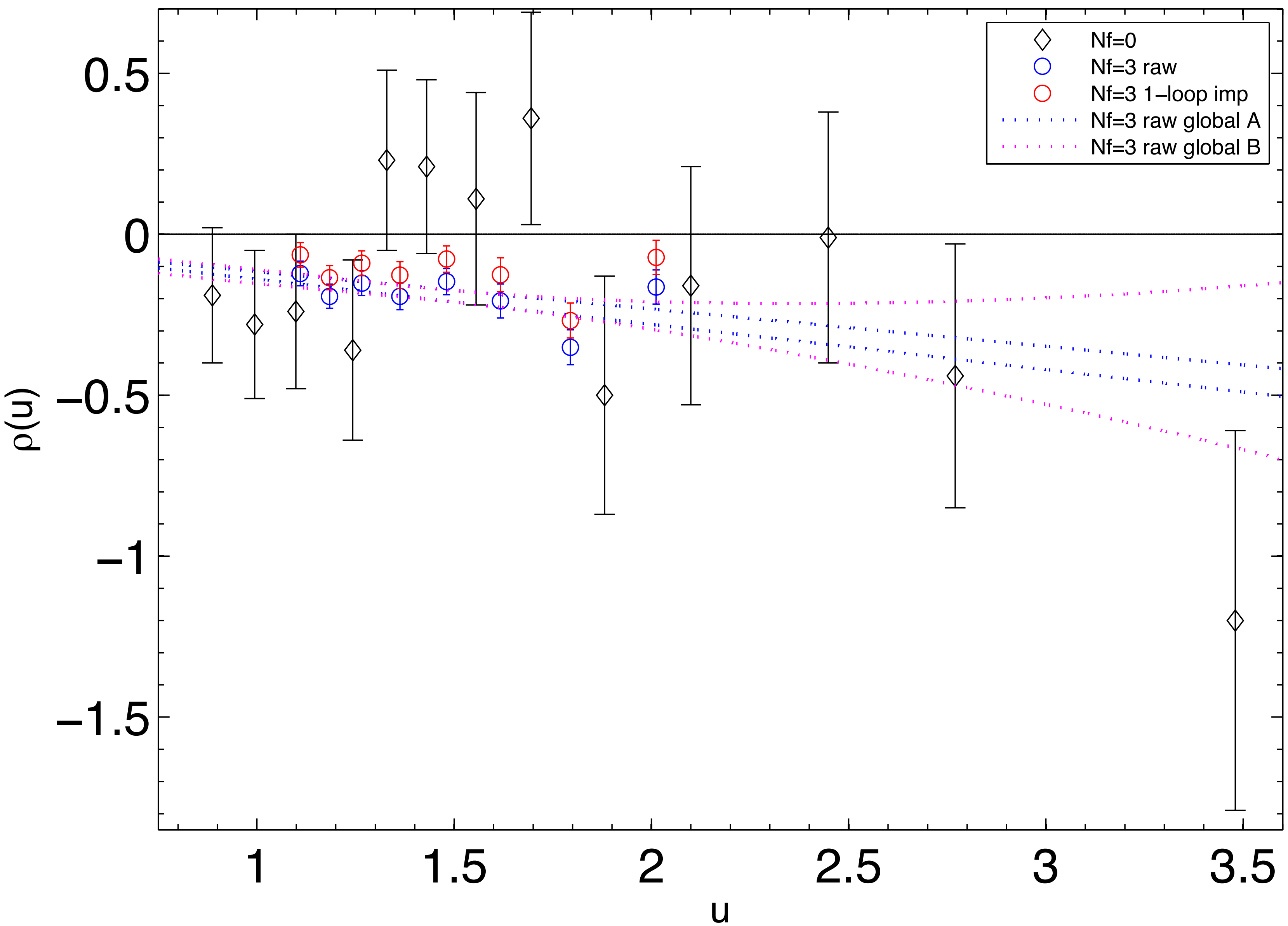
Same order as the error on Z
BUT:

$$\Sigma(u, a/L) = 1 + g_0^2 (Z^{(1)}(a/2L) - Z^{(1)}(a/L)) + \mathcal{O}(g_0^4)$$

$$\Delta(\Sigma(u, a/L)) \sim \mathcal{O}(10^{-5})$$

Continuum Extrapolation SF

$$\Sigma_P(u, a/L) = \sigma_P(u) + \rho'(u)(a/L)^2$$



Continuum Extrapolation

Nf=0 $\Sigma_P(u,a/L) = \sigma_P(u) + \rho'(u)(a/L)^2$

u	$\sigma_{\mathrm{P}}(u)$	$\rho'(u)$	χ^2/n_{df}
0.8873	0.9683(21)	−0.19(21)	0.51
0.9944	0.9672(23)	−0.28(23)	0.47
1.0989	0.9622(24)	−0.24(24)	0.09
1.2430	0.9579(29)	−0.36(28)	0.13
1.3293	0.9470(28)	0.23(28)	0.27
1.4300	0.9407(30)	0.21(27)	0.40
1.5553	0.9382(33)	0.11(33)	3.01
1.6950	0.9297(32)	0.36(33)	0.00
1.8811	0.9284(36)	−0.50(37)	0.21
2.1000	0.9168(37)	−0.16(37)	1.10
2.4484	0.8942(38)	−0.01(39)	3.04
2.7700	0.8781(42)	−0.44(41)	1.44
3.48	0.8451(55)	−1.20(59)	1.05
3.48*	0.8483(50)	−0.94(48)	0.26

Table 7: Continuum extrapolations of Σ_{P} using Fit B

$$\frac{\Delta(\sigma_P(2.100))}{\sigma_P(2.100)} = 4‰$$

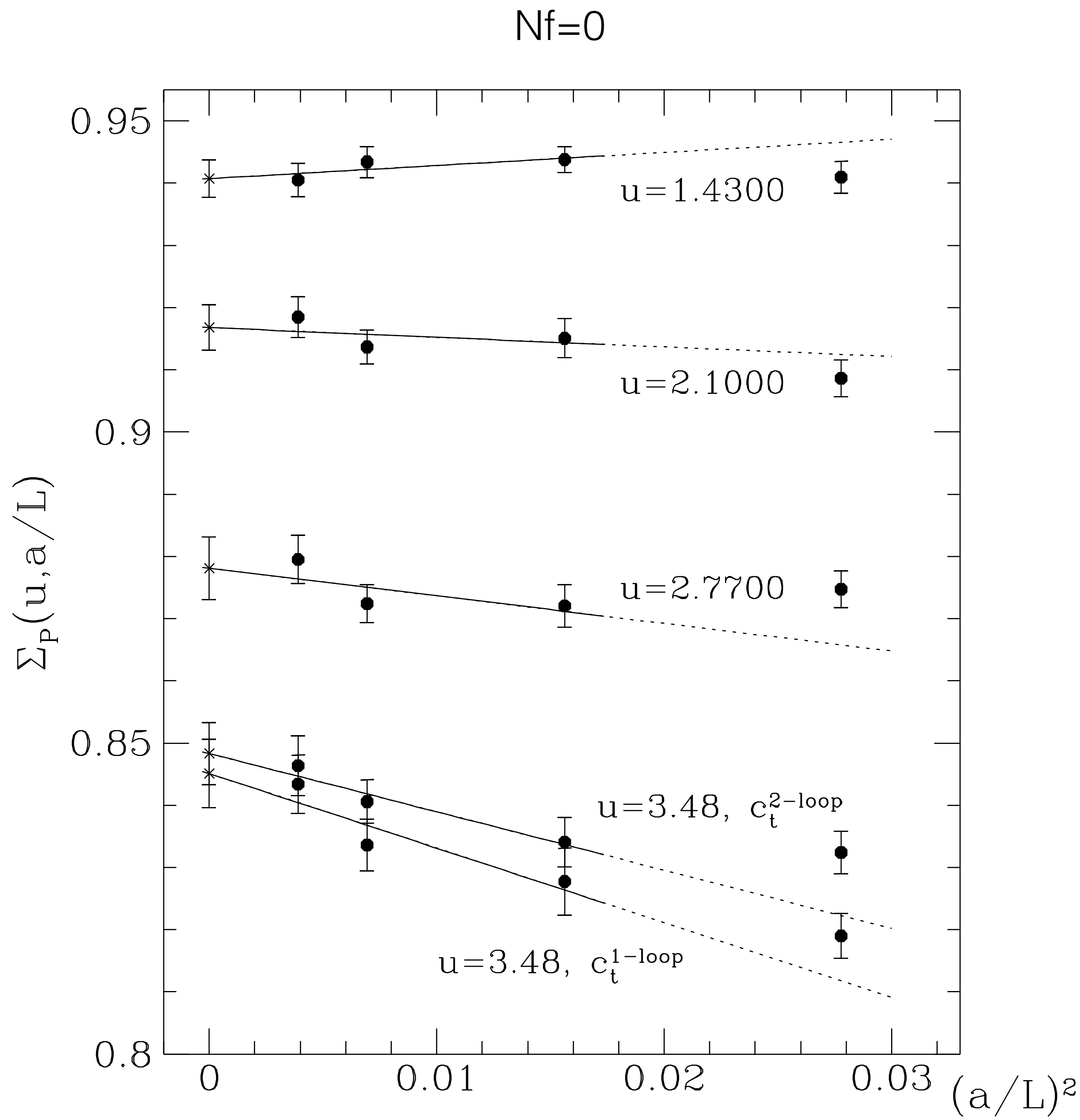


Figure 8: Examples of continuum extrapolations of Σ_{P} using Fit B. The dotted lines are the continuation of the fit functions to the data points for $L/a = 6$, which have been excluded from the fit.

Continuum Extrapolation

Nf=2 $\Sigma_P(u,a/L) = \sigma_P(u) + \rho'(u)(a/L)^2$

u	$\sigma_{\mathrm{P}}(u)$	χ^2/n_{df}
0.9793	0.9654(9)(11)	2.16
1.1814	0.9527(11)(6)	0.47
1.5031	0.9413(16)(2)	0.01
2.0142	0.9174(16)(24)	3.58
2.4792	0.8871(23)(18)	0.54
3.3340	0.8384(35)(12)	0.20

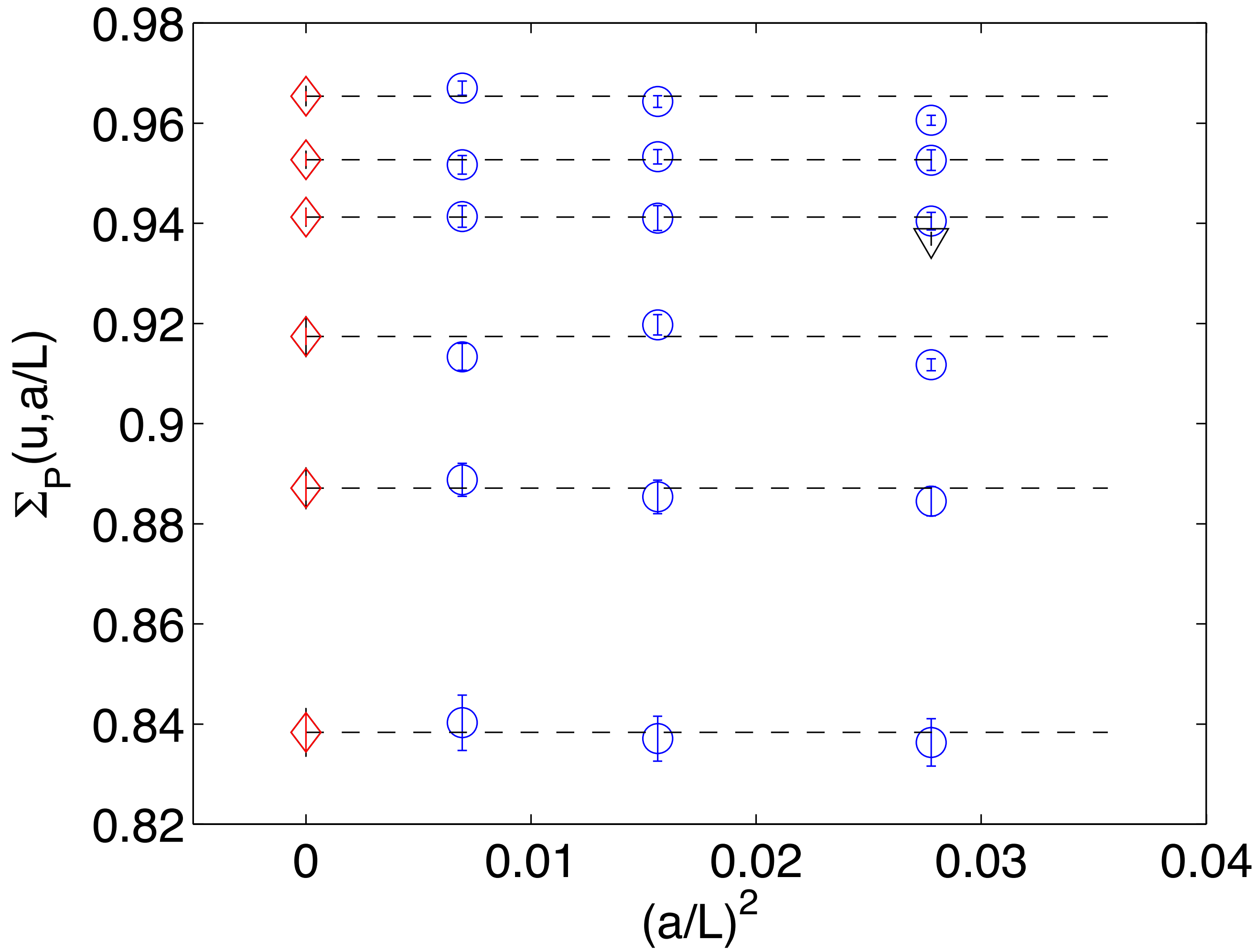
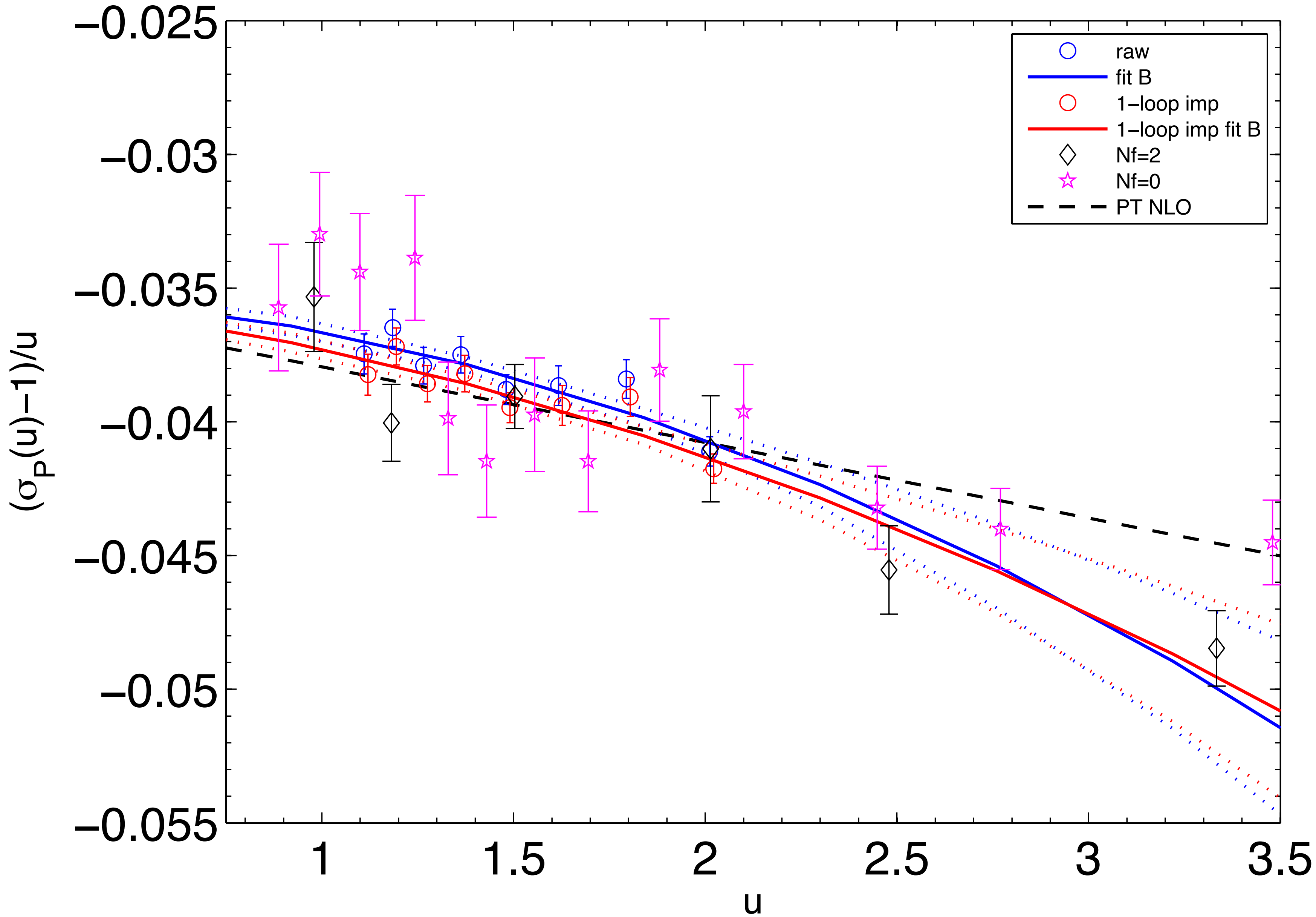


Table 1: Continuum extrapolations of Σ_{P} fitting the $L/a = 8$ and $L/a = 12$ data to a constant. The first error is statistical. The second error is the difference between the fit and the $L/a = 8$ results and will be added linearly as a systematic error.

$$\frac{\Delta(\sigma_P(2.0142))}{\sigma_P(2.0142)} = 4.4‰$$

Total error = stat+syst

SSF Nf=3 && Nf=2 && Nf=0



$$\sigma_P(u) = 1 + p_1 u + p_2 u^2 + p_3 u^3 + p_4 u^4 + \dots$$