Comparison of different definitions of the topological charge: PART II

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Southampton, UK

* With C. Alexandrou, K. Cichy, A. Dromard, E. G-Ramos, K. Jansen, K. Ottnad, C. Urbach, U. Wenger and F. Zimmermann.

based on [arXiv:1411.1205] and [arXiv:1509.04259] and a forthcoming paper.

Preface

☝ Several definitions of the topological charge:

- (f) fermionic (Index, Spectral flow, Spectral Projectors).
- (g) gluonic with UV fluctuations removed via smoothing (gradient flow, cooling, smearing,...).

? How are these definitions numerically related?

☝ The gradient flow provides a well defined smoothing scheme with good renormalizability properties.

M. Lüscher [arXiv:1006.4518]

! The gradient flow is numerically equivalent to cooling!

C. Bonati and M. D'Elia [arXiv:1401.2441] and C. Alexandrou, AA and K. Jansen, [arXiv:1509.0425]

? Can this be applied to other smoothing schemes?

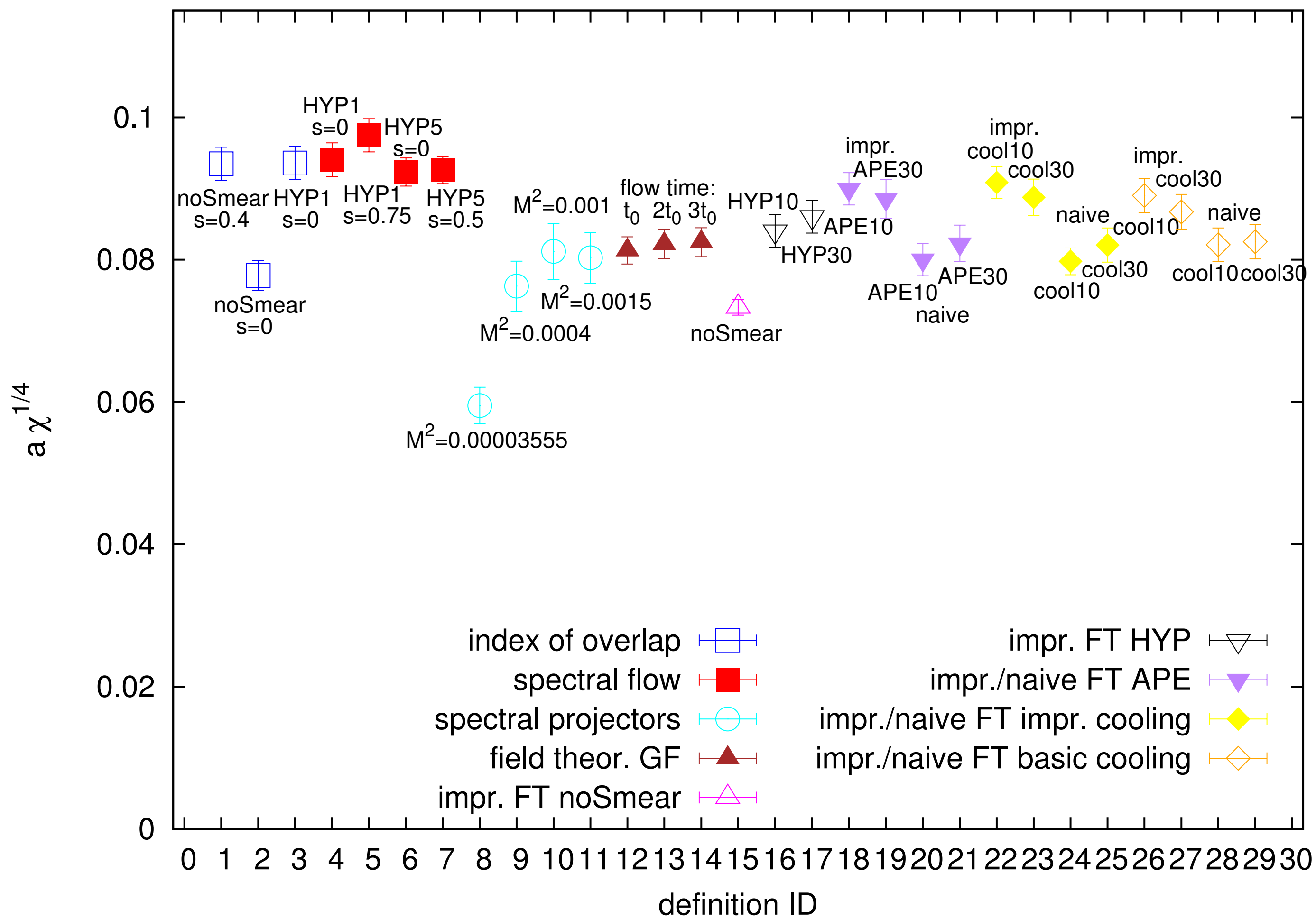
☞ Comparison of different definitions presented by Krzysztof Cichy in LATTICE 2014 ...

K. Cichy *et. al*, [arXiv:1411.1205]

- ! Most definitions are highly correlated.
- ! The topological susceptibilities are in the same region.

Overview from Lattice 2014

Continuation of Krzysztof Cichy's talk given in Lattice 2014:



Using $N_f = 2$ twisted mass configuration with:

$\beta = 3.90$, $a \simeq 0.085$fm, $r_0/a = 5.35(4)$, $m_\pi \simeq 340$ MeV, $m_\pi L = 2.5$, $L/a = 16$

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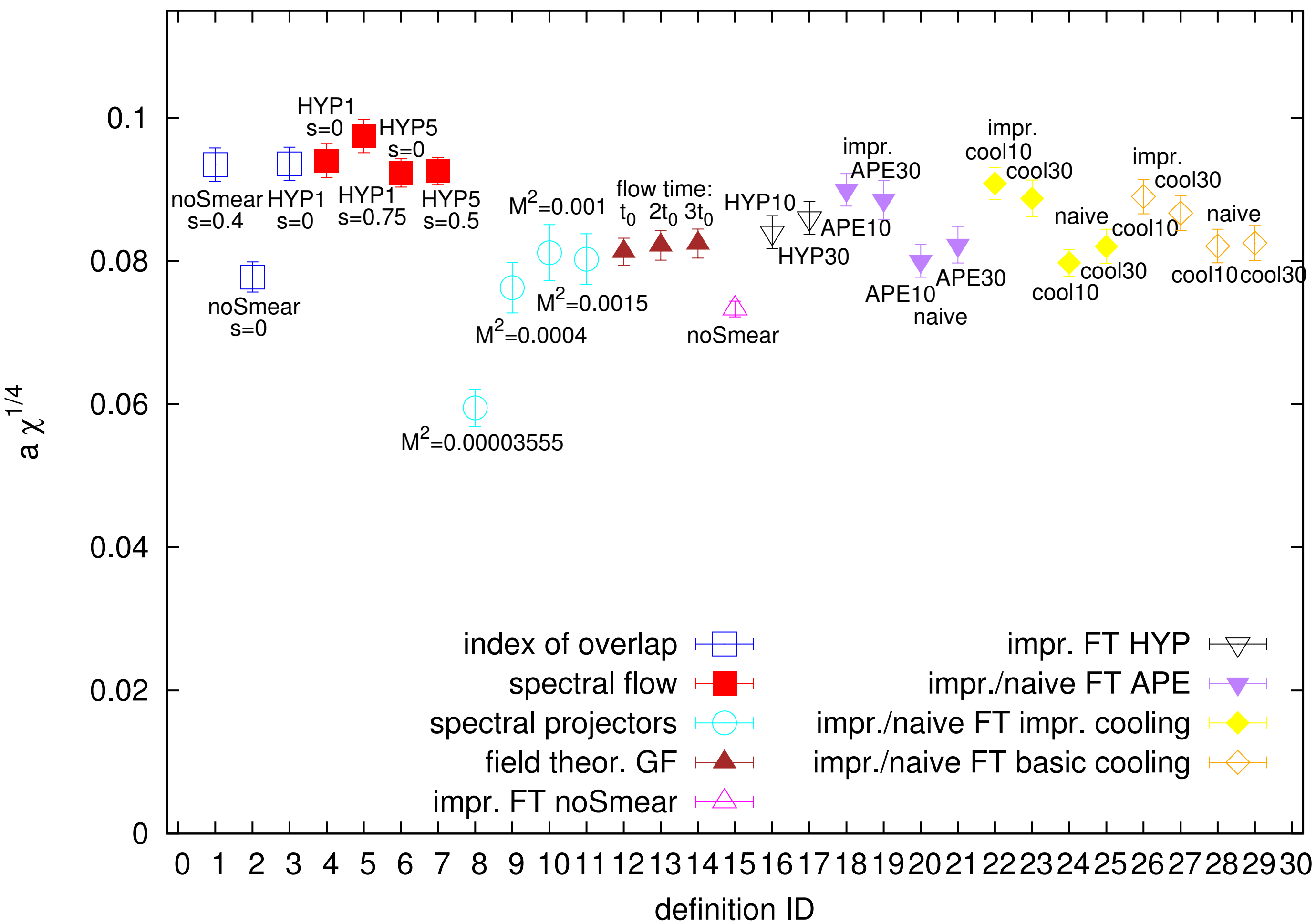

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Details of the Topological Charge Comparison

- (f) Index definition with different steps of HYP smearing.

  M. F. Atiyah and I. M. Singer, *Annals Math.* 93 (1971) 139149

- (f) Spectral-flow with different steps of HYP smearing.

  S. Itoh, Y. Iwasaki and T. Yoshie, *Phys. Rev.* D 36 (1987) 527

- (f) Spectral projectors with different cutoffs M^2.

  L. Giusti and M. Lüscher, *JHEP* 0903 (2009) 013 and M. Lüscher and F. Palombi, *JHEP* 1009 (2010) 110

- (g) The Wilson flow (also gradient flow with different actions).

  M. Lüscher, *JHEP* 1008 (2010) 071

- (g) Cooling with the Wilson plaquette action (also tlSym and Iwasaki).

  M. Teper, *Phys. Lett. B* 162 (1985) 357.

- (g) APE smearing with $\alpha_{\rm APE} = 0.4,\ 0.5,\ 0.6$.

  M. Albanese et al. [APE Collaboration], *Phys. Lett. B* 192 (1987) 163.

- (g) Stout smearing with $\rho_{\rm st} = 0.01,\ 0.05,\ 0.1$.

  C. Morningstar and M. J. Peardon, *Phys. Rev.* D69 (2004) 054501

- (g) HYP smearing with $\alpha_{\rm HYP1} = 0.75, \quad \alpha_{\rm HYP2} = 0.6 \quad \alpha_{\rm HYP3} = 0.3$.

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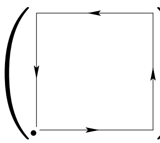
Field Theoretic Definition of the Topological Charge

- Topological charge can be defined as:

$$Q = \int d^4x \, q(x)\,, \quad \text{with} \quad q(x) = \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left\{F_{\mu\nu}F_{\rho\sigma}\right\}\,.$$

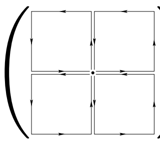
- Discretizations of $q(x)$ on the lattice:
  - Plaquette

$$q_L^{\mathrm{plaq}}(x) = \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left(C_{\mu\nu}^{\mathrm{plaq}}C_{\rho\sigma}^{\mathrm{plaq}}\right)\,, \quad \text{with} \quad C_{\mu\nu}^{\mathrm{plaq}}(x) = \mathrm{Im}\left(\quad \right)\,.$$



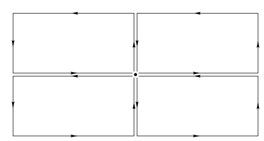
  - Clover

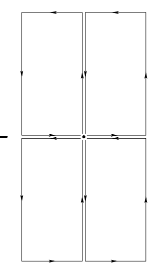
$$q_L^{\mathrm{clov}}(x) = \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left(C_{\mu\nu}^{\mathrm{clov}}C_{\rho\sigma}^{\mathrm{clov}}\right)\,, \quad \text{with} \quad C_{\mu\nu}^{\mathrm{clov}}(x) = \frac{1}{4}\mathrm{Im}\left(\quad \right)\,.$$



  - Improved

$$q_L^{\mathrm{imp}}(x) = b_0 q_L^{\mathrm{clov}}(x) + b_1 q_L^{\mathrm{rect}}(x)\,, \quad \text{with} \quad C_{\mu\nu}^{\mathrm{rect}}(x) = \frac{1}{8}\mathrm{Im}\left(\quad + \quad \right)\,.$$





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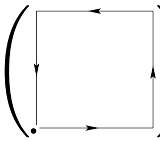

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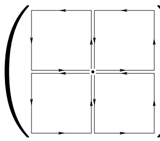

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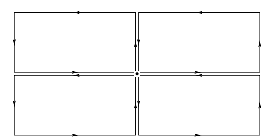
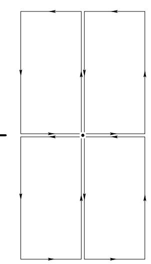




- Smoothing...

Example: The Wilson flow Vs. Cooling

Gradient Flow

- Solution of the evolution equations:

$$\begin{aligned} \dot{V}_\mu(x,\tau) &= -g_0^2 \left[\partial_{x,\mu} S_G(V(\tau))\right] V_\mu(x,\tau) \\ V_\mu(x,0) &= U_\mu(x) \,, \end{aligned}$$

- With link derivative defined as:

$$\begin{aligned} \partial_{x,\mu} S_G(U) &= i \sum_a T^a \frac{\mathrm{d}}{\mathrm{d}s} S_G\left(e^{isY^a} U\right)\Bigg|_{s=0} \\ &\equiv i \sum_a T^a \partial^{(a)}_{x,\mu} S_G(U) \,, \end{aligned}$$

- Total gradient flow time: τ
- Reference flow time t_0 such that $t^2 \langle E(t) \rangle|_{t=t_0} = 0.3$ with $t = a^2\tau$ and $E(t) = -\frac{1}{2V} \sum_x \mathrm{Tr}\left\{F_{\mu\nu}(x,t) F_{\mu\nu}(x,t)\right\}$

Cooling

- Cooling $U_\mu(x) \in SU(N)$: $U_\mu^{\mathrm{old}}(x) \to U_\mu^{\mathrm{new}}(x)$ with

$$P(U) \propto e^{(\lim_{\beta\to\infty} \beta \frac{1}{N} \mathrm{ReTr} X_\mu{}^\dagger U_\mu)}.$$

- Choose a $U_\mu^{\mathrm{new}}(x)$ that maximizes:

$$\mathrm{ReTr}\{ U_\mu^{\mathrm{new}}(x) X_\mu^\dagger(x)\}.$$

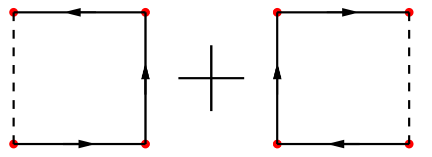
- One full cooling iteration $n_c = 1$

Perturbative expansion of links

☞ A link variable which has been smoothed can been written as:

$$U_\mu(x, j_{\rm sm}) \simeq \mathbb{1} + i \sum_a u_\mu^a(x, j_{\rm sm}) T^a \, .$$

☞ Simple staples are written as:



☞ For the Wilson flow with $\Omega_\mu(x) = U_\mu(x) X_\mu^\dagger(x)$

$$g_0^2 \partial_{x,\mu} S_G(U)(x) = \frac{1}{2}\left(\Omega_\mu(x) - \Omega_\mu^\dagger(x)\right) - \frac{1}{6}{\rm Tr}\left(\Omega_\mu(x) - \Omega_\mu^\dagger(x)\right) .$$

where

$$g_0^2 \partial_{x,\mu} S_G(U) \simeq i \sum_a \left[6 u_\mu^a(x,\tau) - w_\mu^a(x,\tau)\right] T^a \, .$$

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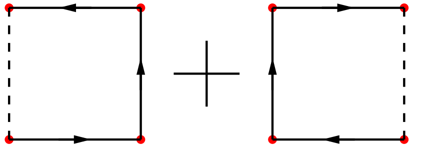

per space-time slice, thus.

$$X_\mu(x, j_{\rm sm}) \simeq 6 \cdot \mathbb{1} + i\sum_a w^a_\mu(x, j_{\rm sm}) T^a \, .$$

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Perturbative matching: Wilson flow Vs. Cooling

☞ Evolution of the Wilson flow by an infinitesimally small flow time ϵ:

$$u_\mu^a(x,\tau+\epsilon) \simeq u_\mu^a(x,\tau) - \epsilon\left[6u_\mu^a(x,\tau) - w_\mu^a(x,\tau)\right] \, .$$

where $U_\mu(x,\tau+\epsilon) \simeq \mathbb{1} + i\sum_a u_\mu^a(x,\tau+\epsilon)T^a$

☞ After a cooling step:

$$u_\mu^a(x,n_c+1) \simeq \frac{w_\mu^a(x,n_c)}{6} \, .$$

☞ Wilson flow would evolve the same as cooling if $\epsilon = 1/6$.

\+ Cooling has an additional speed up of two.

! Hence, cooling has the same effect as the Wilson flow if:

$$\tau \simeq \frac{n_c}{3} \, .$$

Result extracted by C. Bonati and M. D'Elia, *Phys. Rev.* **D89** (2014), 105005 [arXiv:1401.2441]

☞ Generalization of this result for smoothing actions with rectangular terms (b_1):

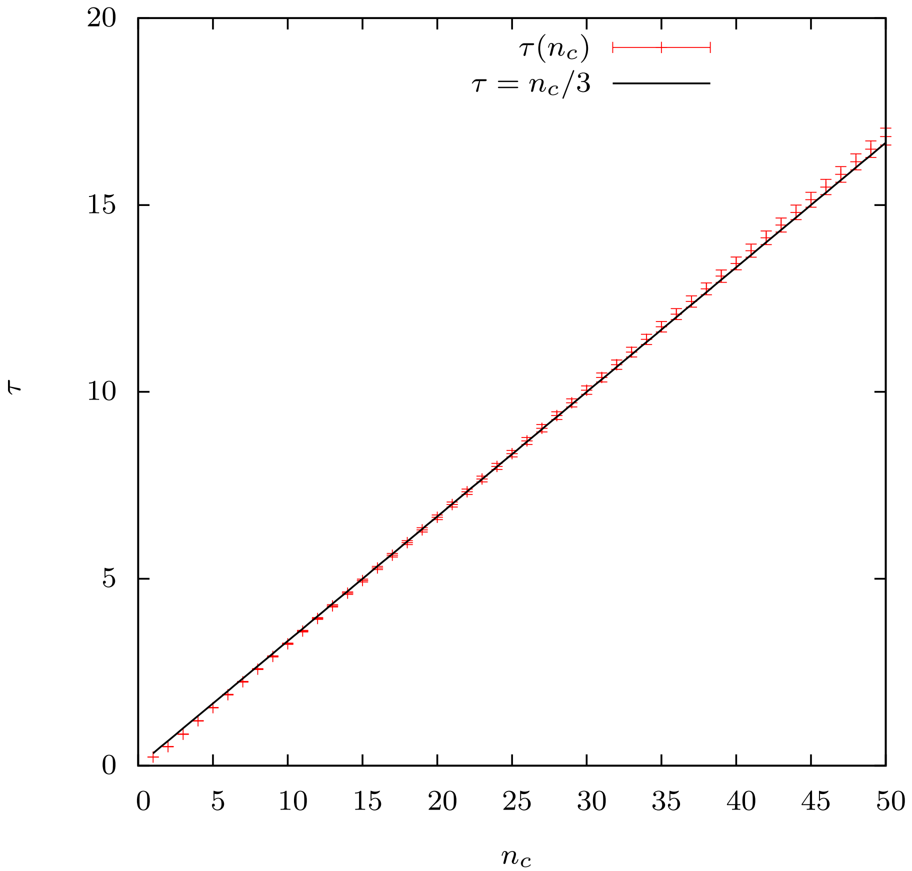
$$\tau \simeq \frac{n_c}{3 - 15b_1} \, .$$

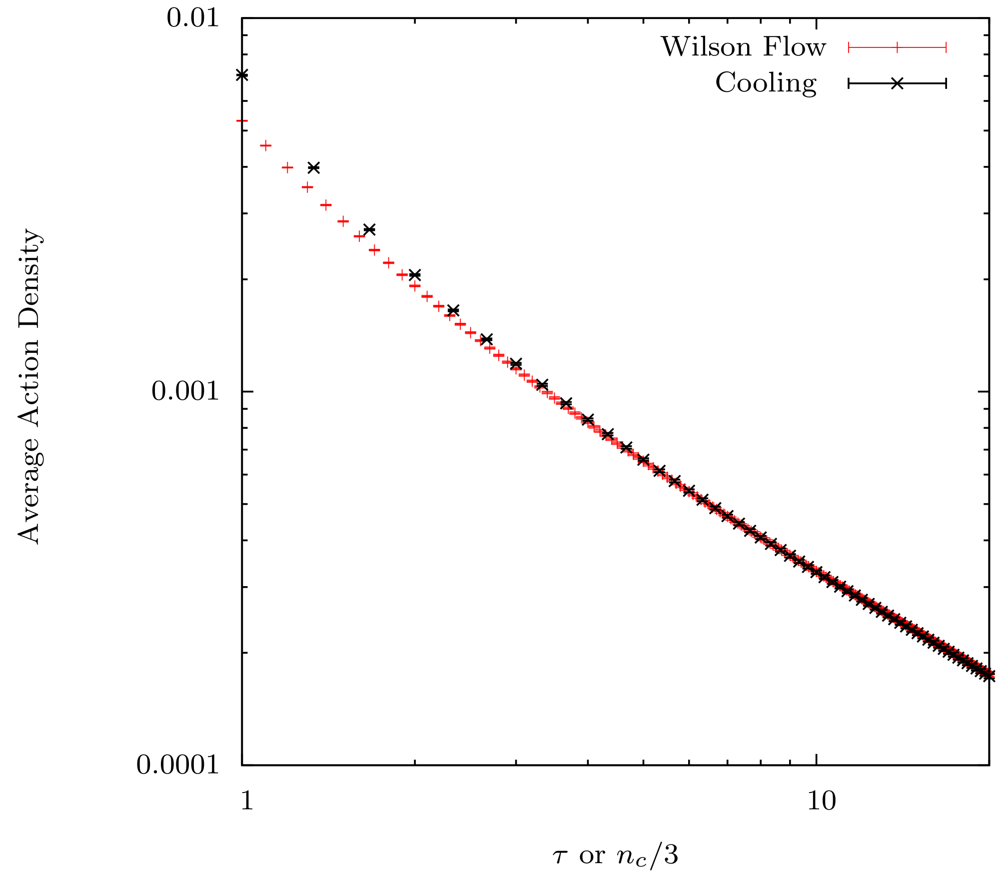
Result extracted by C. Alexandrou, AA and K. Jansen, *Phys. Rev.* **D92** (2015), 125014 [arXiv:1509.0425]

Numerical matching: Wilson flow Vs. Cooling

Matching condition: $\tau \simeq \frac{n_c}{3}$.

Define function $\tau(n_c)$ such as τ and n_c change the action by the same amount.





Perturbative matching: Wilson flow Vs. APE

☞ According to the APE operation:

$$U_\mu^{(n_{\rm APE}+1)}(x) = Proj_{SU(3)}\left[(1-\alpha_{\rm APE})\,U_\mu^{(n_{\rm APE})}(x) + \frac{\alpha_{\rm APE}}{6}X_\mu^{(n_{\rm APE})}(x)\right] .$$

☞ Evolution of the Wilson flow by an infinitesimally small flow time ϵ is expressed as:

$$u_\mu^a(x,\tau+\epsilon) \simeq u_\mu^a(x,\tau) - \epsilon\left[6u_\mu^a(x,\tau) - w_\mu^a(x,\tau)\right] .$$

☞ Evolution of the APE smearing with parameter $\alpha_{\rm APE}$ is expressed as:

$$u_\mu^a(x,n_{\rm APE}+1) \simeq u_\mu^a(x,n_{\rm APE}) - \frac{\alpha_{\rm APE}}{6}\left[6u_\mu^a(x,n_{\rm APE}) - w_\mu^a(x,n_{\rm APE})\right] .$$

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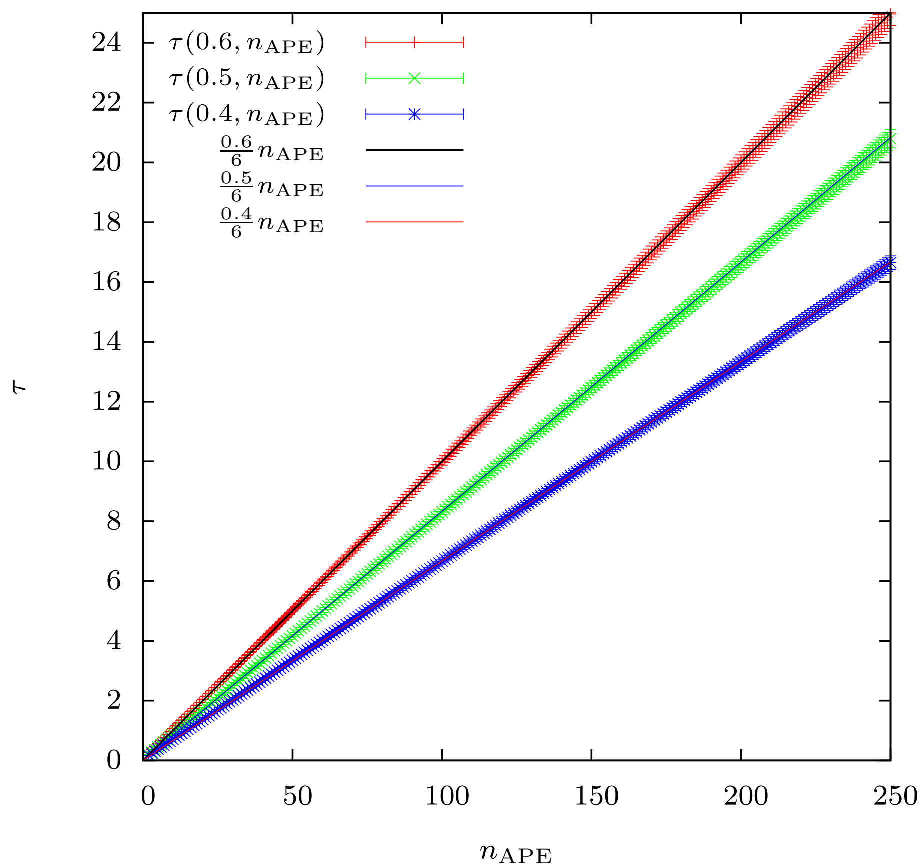
☝ Hence, APE has the same effect as the Wilson flow if:

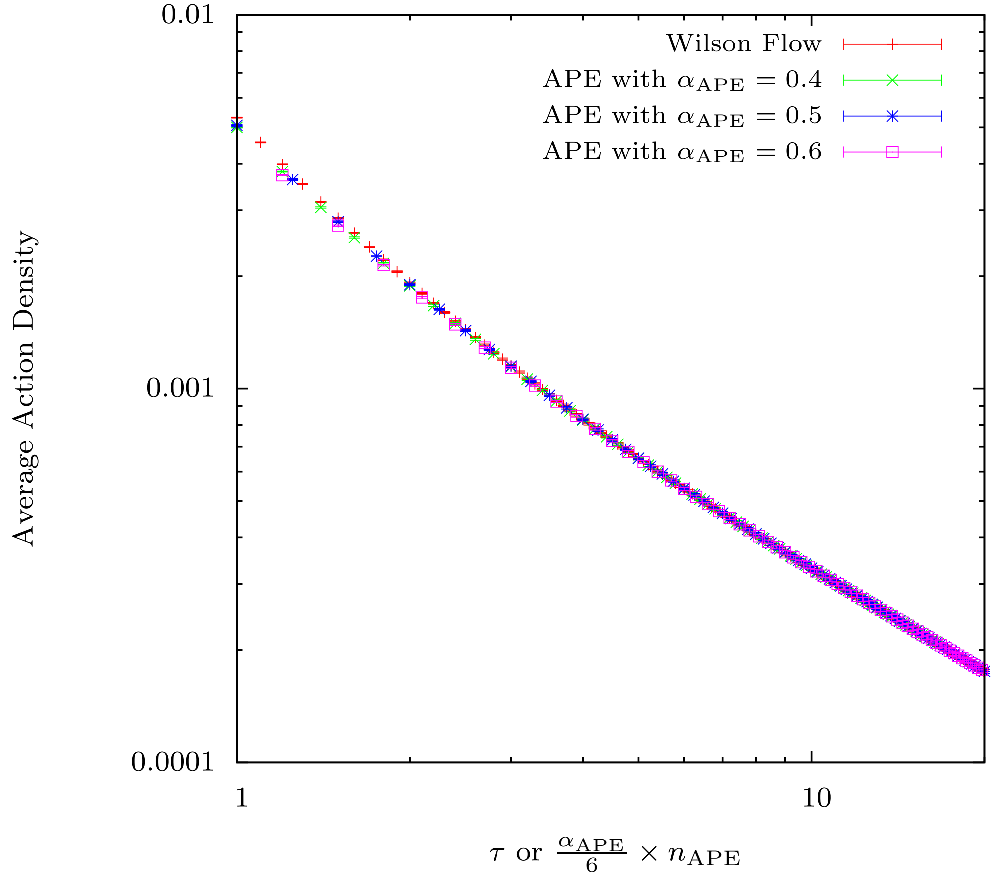
$$\tau \simeq \frac{\alpha_{\rm APE}}{6}n_{\rm APE} .$$

Numerical matching: Wilson flow Vs. APE

Matching condition: $\tau \simeq \frac{\alpha_{\rm APE}}{6} n_{\rm APE}$.

Define function $\tau(\alpha_{\rm APE}, n_{\rm APE})$ such as τ and $n_{\rm APE}$ changes action by the same amount





Perturbative matching: Wilson flow Vs. stout

☞ According to the stout smearing operation:

$$U_\mu^{(n_{\rm st}+1)}(x) = \exp\left(iQ_\mu^{n_{\rm st}}(x)\right) U_\mu^{(n_{\rm st})}(x)\,.$$

with

$$Q_\mu(x) = \frac{i}{2}\left(\Xi_\mu^\dagger(x) - \Xi_\mu(x)\right) - \frac{i}{6}{\rm Tr}\left(\Xi_\mu^\dagger(x) - \Xi_\mu(x)\right)\,, \quad \text{with} \quad \Xi_\mu(x) = \rho_{\rm st} X_\mu(x) U_\mu^\dagger(x)$$

☞ Evolution of the Wilson flow by an infinitesimally small flow time ϵ is expressed as:

$$u_\mu^a(x,\tau+\epsilon) \simeq u_\mu^a(x,\tau) - \epsilon\left[6u_\mu^a(x,\tau) - w_\mu^a(x,\tau)\right]\,.$$

☞ Evolution of the stout smearing with parameter $\rho_{\rm st}$ is expressed as:

$$u_\mu^a(x,n_{\rm st}+1) \simeq u_\mu^a(x,n_{\rm st}) - \rho_{\rm st}\left[6u_\mu^a(x,n_{\rm st}) - w_\mu^a(x,n_{\rm st})\right]\,.$$

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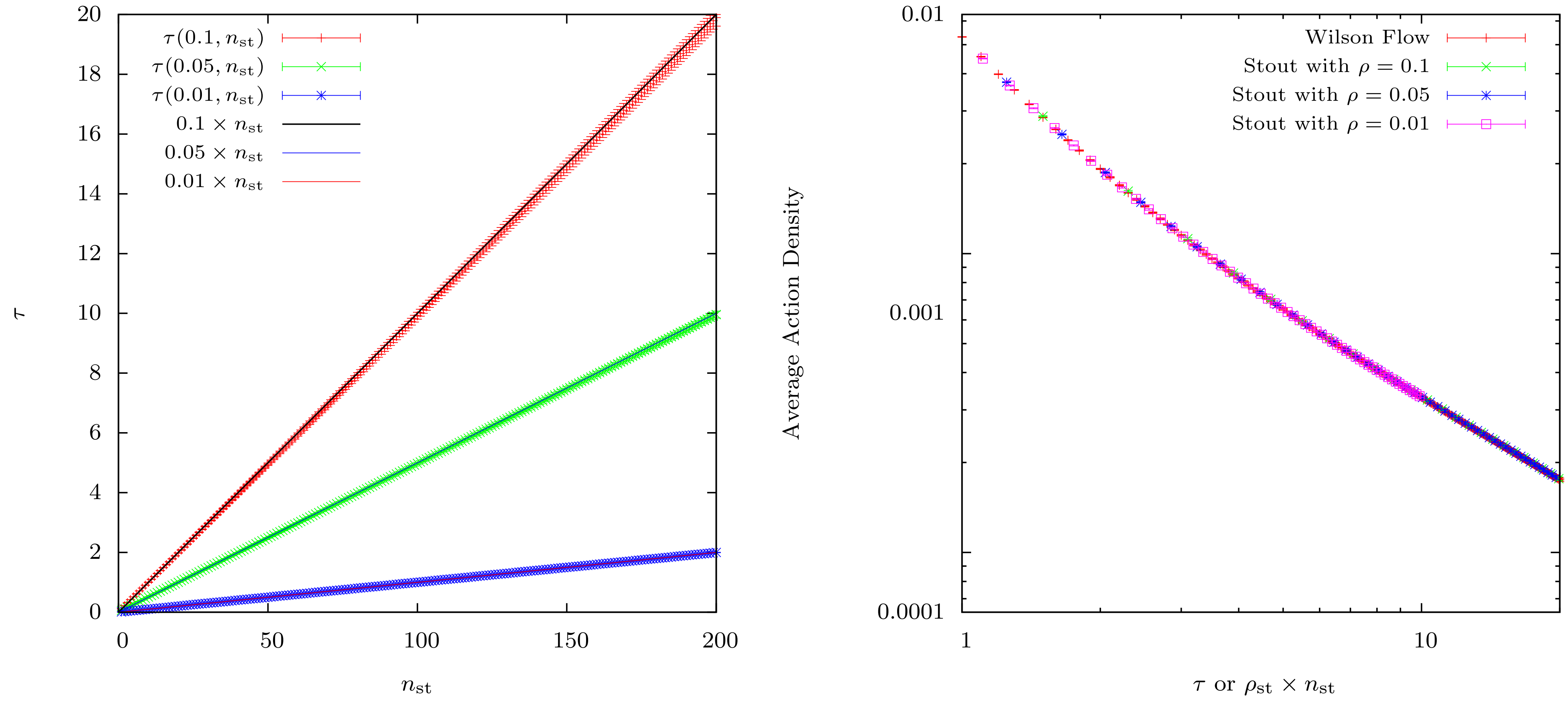
- Hence, stout smearing has the same effect as the Wilson flow if

$$\tau \simeq \rho_{\rm st} n_{\rm st}\,.$$

Numerical matching: Wilson flow Vs. stout

Matching condition: $\tau \simeq \rho_{\rm st} n_{\rm st}$.

Define function $\tau(\rho_{\rm st}, n_{\rm st})$ such as τ and $n_{\rm st}$ changes action by the same amount



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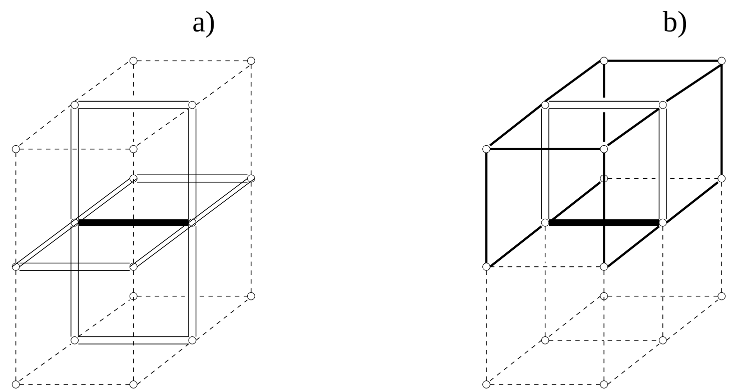
☞ We considered HYP smearing with parameters:

$$\alpha_{\mathrm{HYP1}} = 0.75, \quad \alpha_{\mathrm{HYP2}} = 0.6 \quad \alpha_{\mathrm{HYP3}} = 0.3$$

☟ HYP staples not the same as $X_\mu(x)$ (A. Hasenfratz and F. Knechtli, *Phys. Rev.* D64 (2001) 034504).



☞ Define function $\tau_{\mathrm{HYP}}(n_{\mathrm{HYP}})$ and fit using the ansatz:

$$\tau_{\mathrm{HYP}}(n_{\mathrm{HYP}}) = A\, n_{\mathrm{HYP}} + B\, n_{\mathrm{HYP}}^2 + C\, n_{\mathrm{HYP}}^3 \,.$$

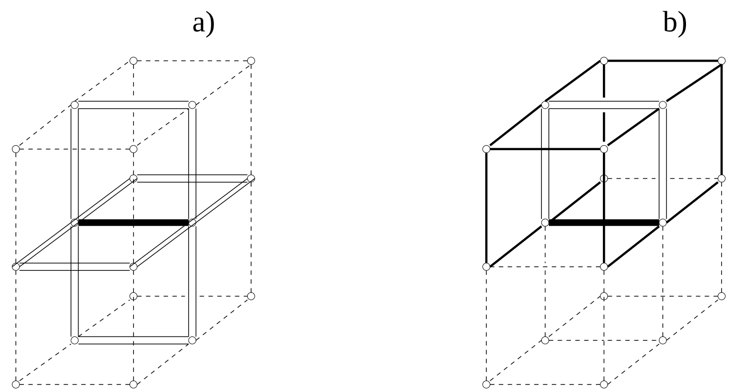
with $A = 0.25447(32)$, $B = -0.001312(90)$, $C = 1.217(91) \times 10^{-5}$

Numerical matching: Wilson flow Vs. HYP

☞ We considered HYP smearing with parameters:

$$\alpha_{\rm HYP1} = 0.75, \quad \alpha_{\rm HYP2} = 0.6 \quad \alpha_{\rm HYP3} = 0.3$$

☟ HYP staples not the same as $X_\mu(x)$ (A. Hasenfratz and F. Knechtli, *Phys. Rev.* D64 (2001) 034504).



☞ Define function $\tau_{\rm HYP}(n_{\rm HYP})$ and fit using the ansatz:

$$\tau_{\rm HYP}(n_{\rm HYP}) = A\, n_{\rm HYP} + B\, n_{\rm HYP}^2 + C\, n_{\rm HYP}^3\,.$$

with $A = 0.25447(32)$, $B = -0.001312(90)$, $C = 1.217(91) \times 10^{-5}$

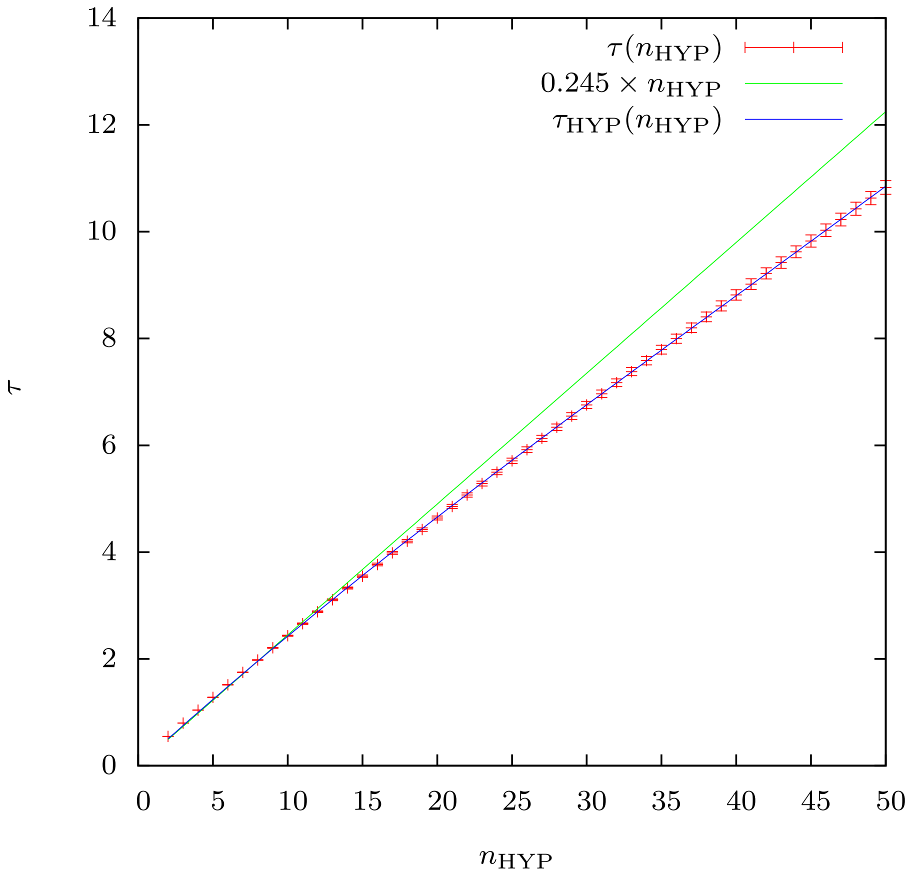
☝ Hence, HYP smearing has the same effect as the Wilson flow if

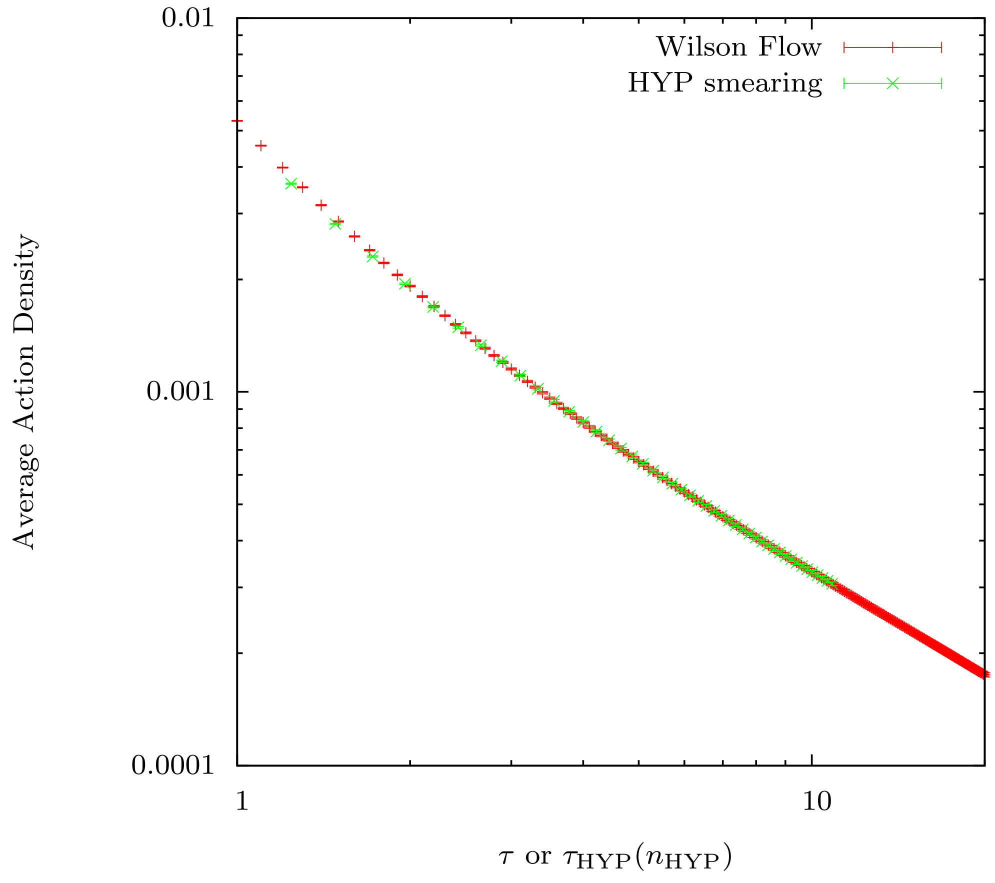
$$\tau \simeq \tau_{\rm HYP}(n_{\rm HYP})\,.$$

Numerical matching: Wilson flow Vs. HYP

Numerical matching condition: $\tau_{\mathrm{HYP}}(n_{\mathrm{HYP}}) = A\ n_{\mathrm{HYP}} + B\ n_{\mathrm{HYP}}^2 + C\ n_{\mathrm{HYP}}^3$.

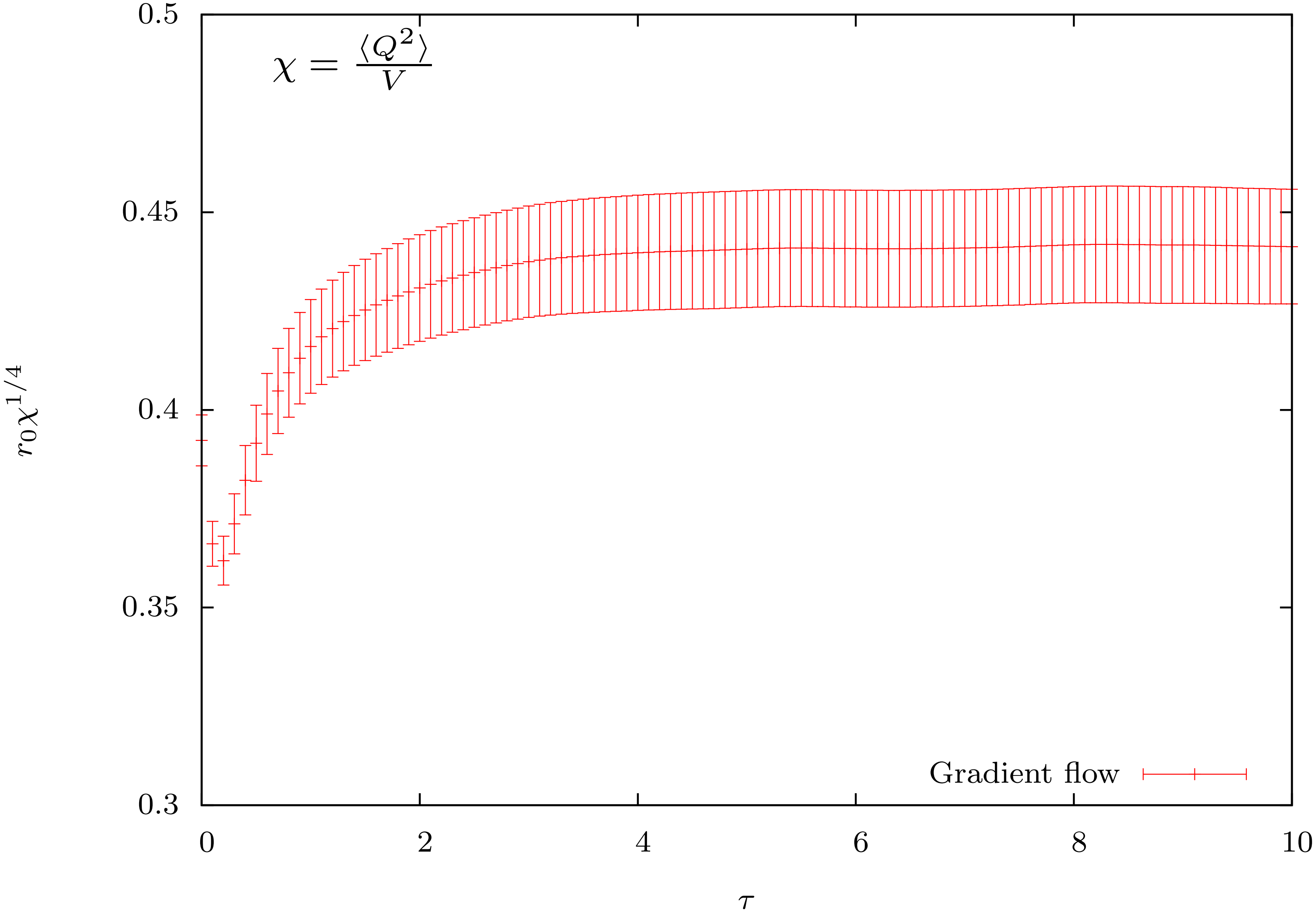
Define function $\tau(n_{\mathrm{HYP}})$ such as τ and n_{HYP} changes action by the same amount





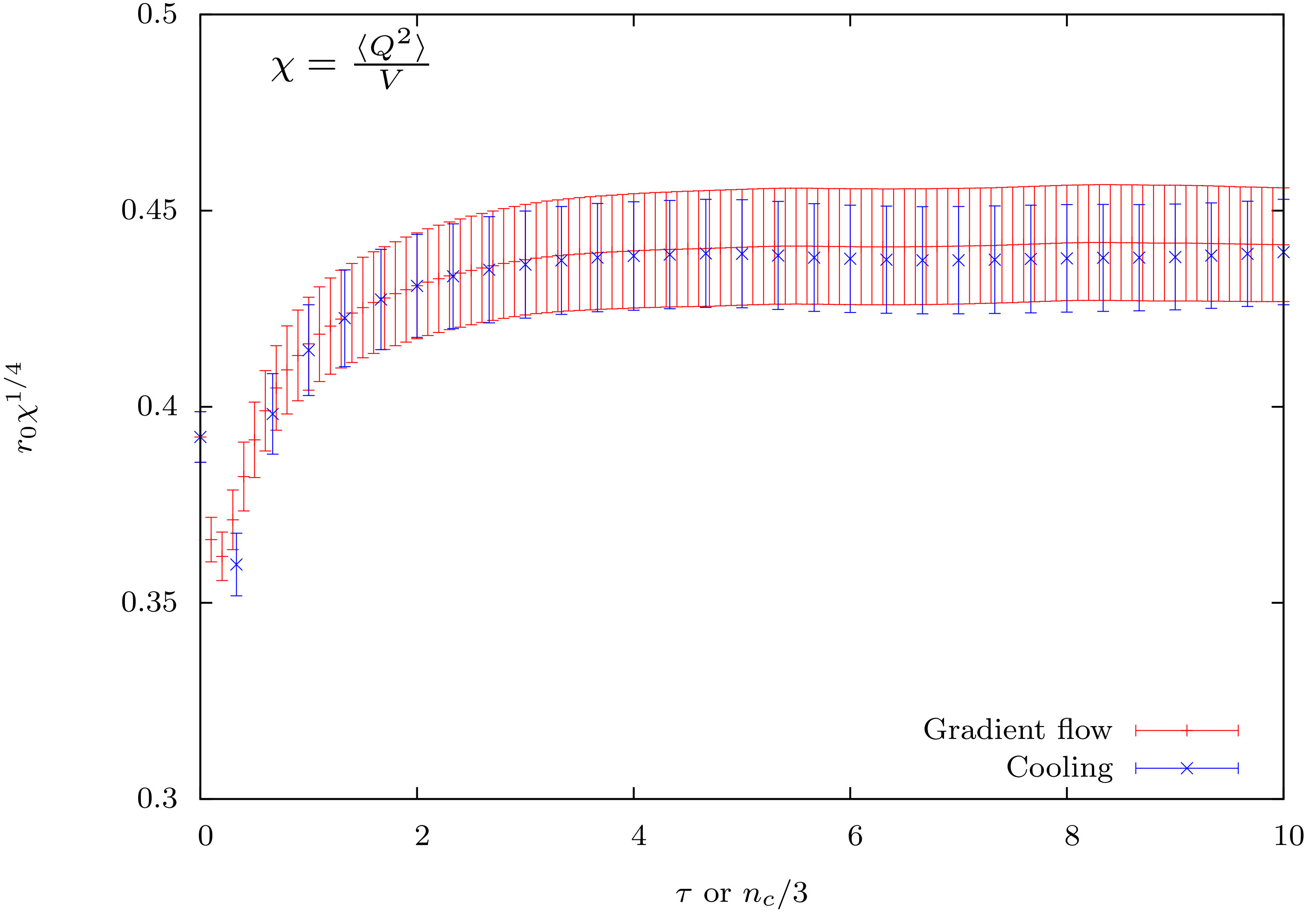
Topological Susceptibility - The Wilson flow

The Wilson flow time $t_0 \simeq 2.5a^2$



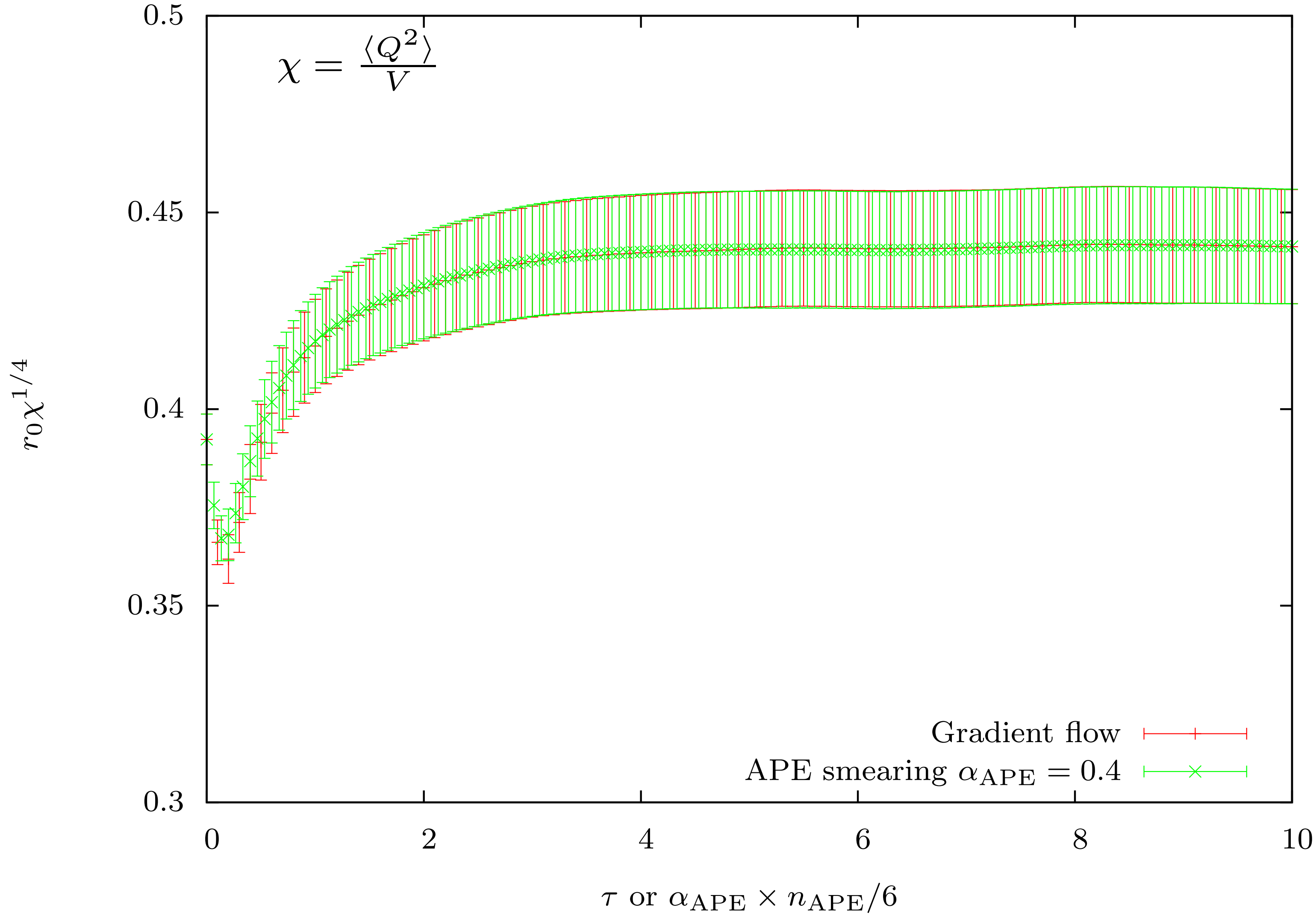
Topological Susceptibility - Cooling

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_c = 7.5$ cooling steps



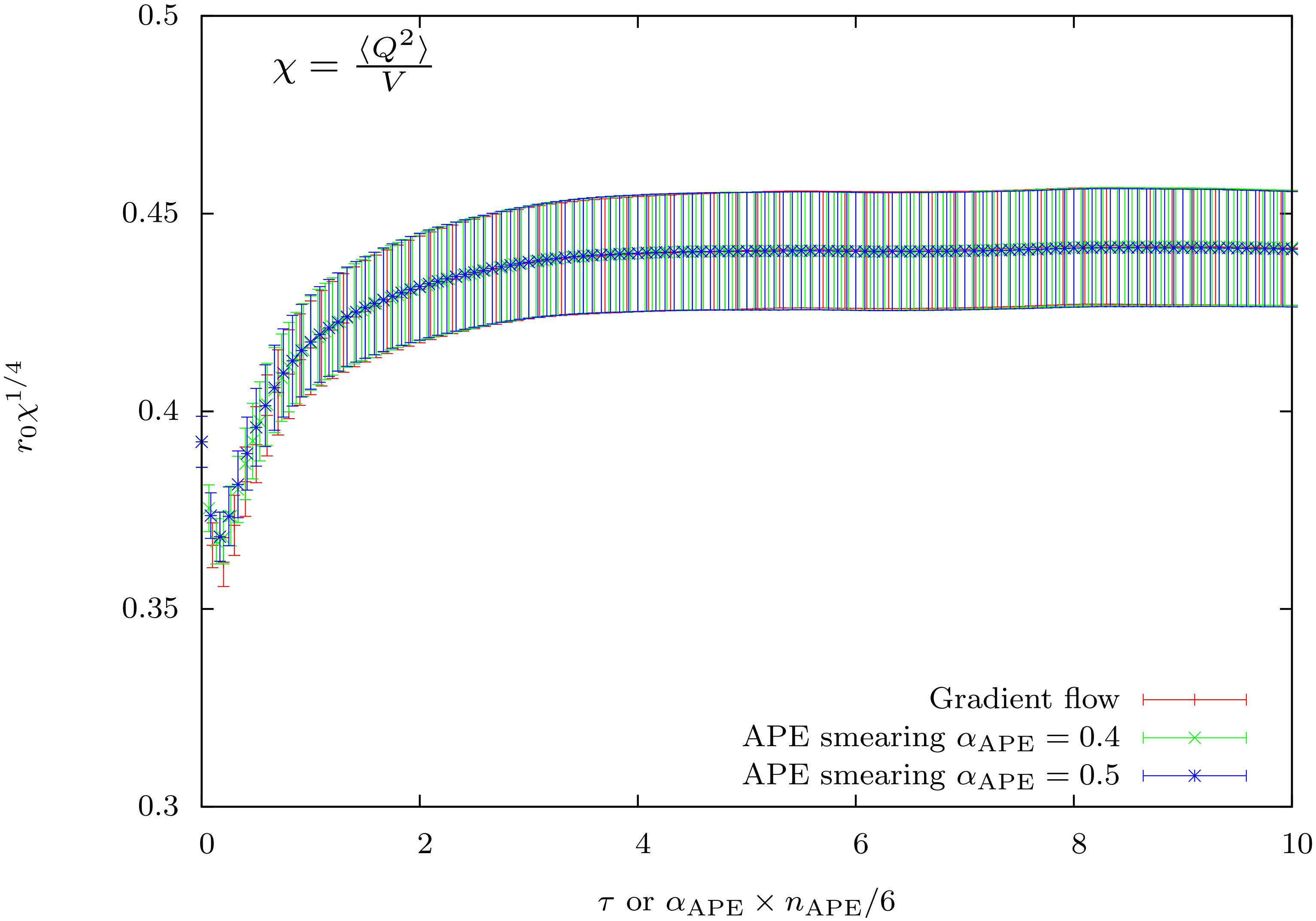
Topological Susceptibility - APE

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm APE} = 37.5$ APE smearing steps for $\alpha_{\rm APE} = 0.4$



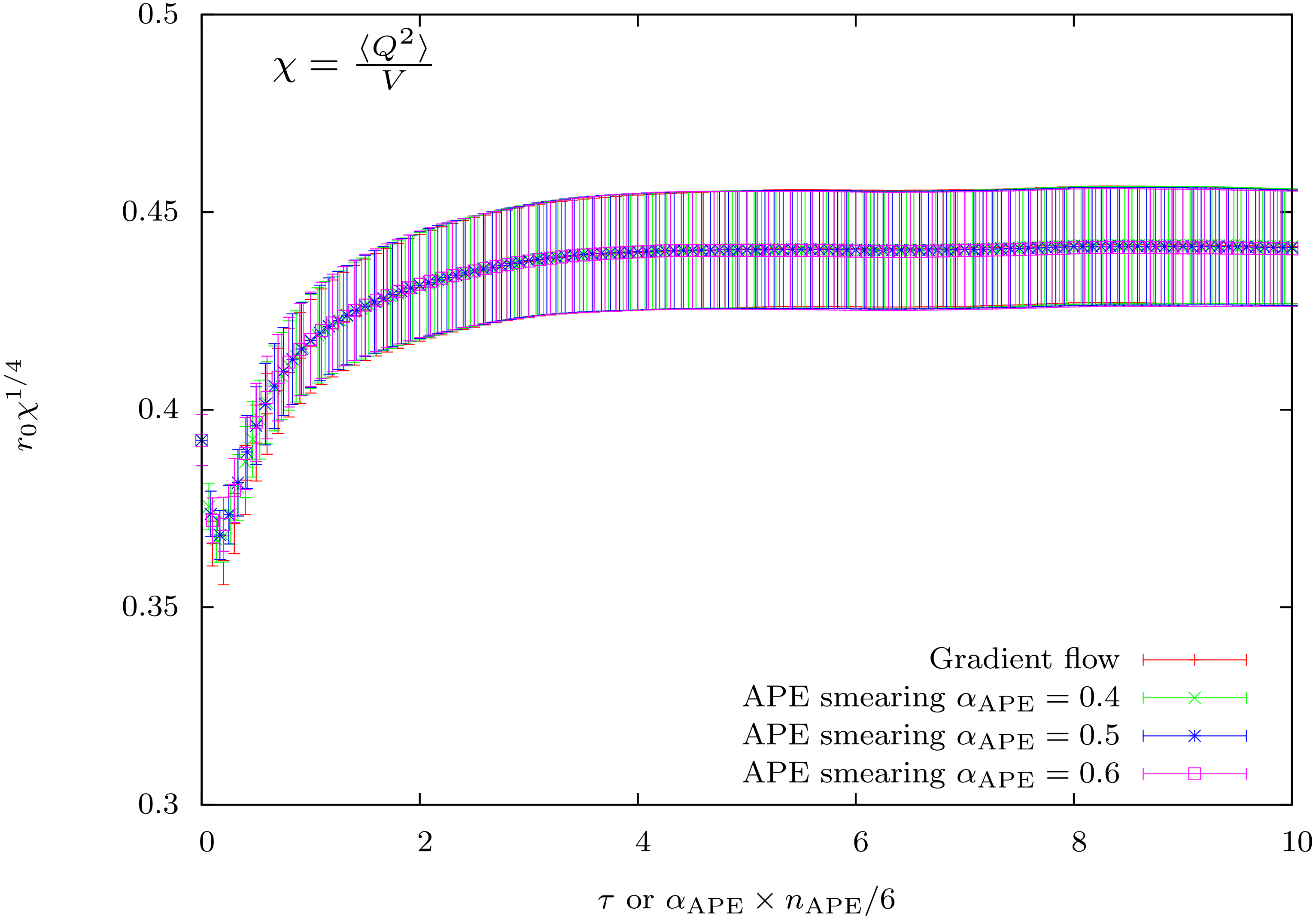
Topological Susceptibility - APE

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm APE} = 30$ APE smearing steps for $\alpha_{\rm APE} = 0.5$



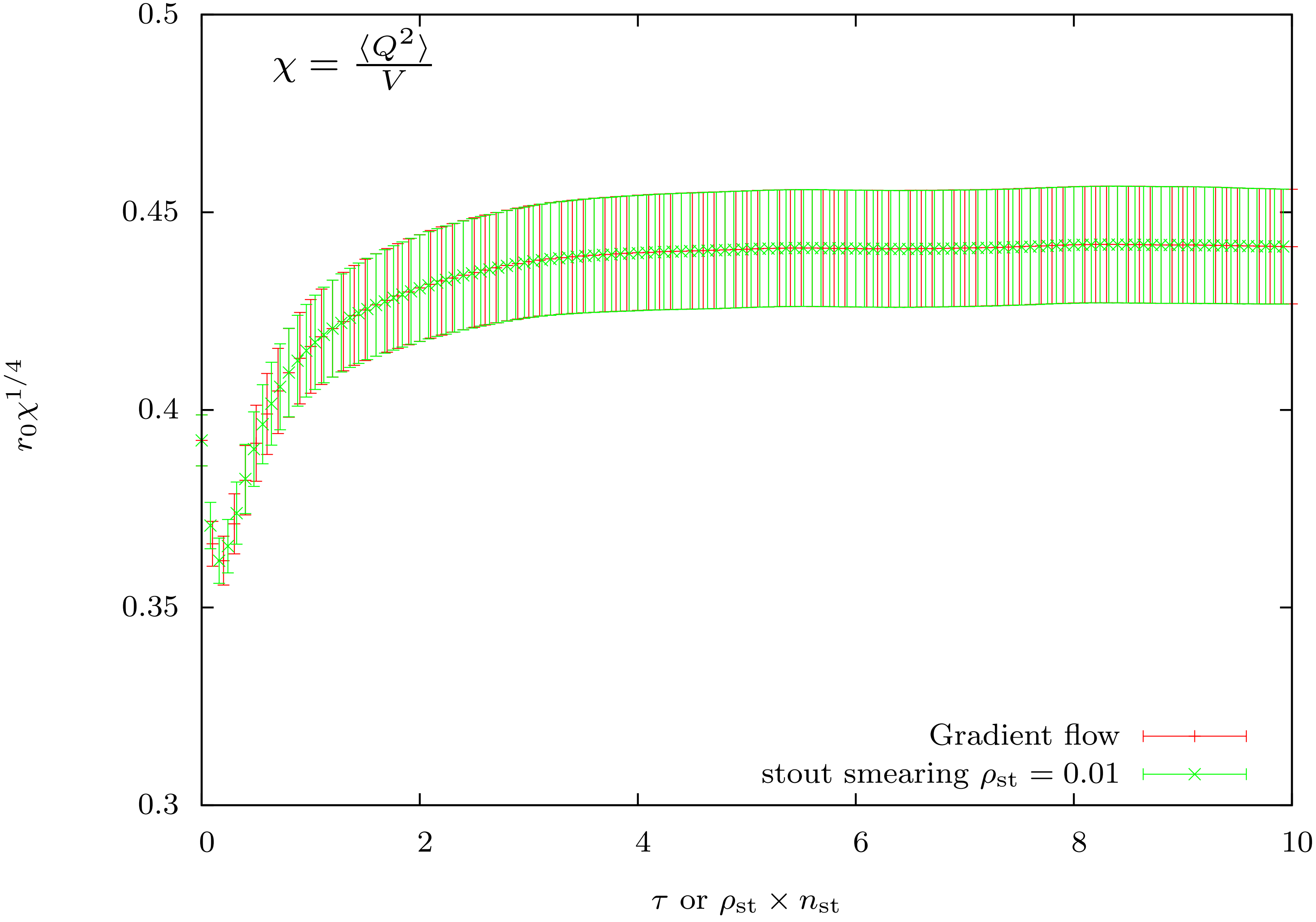
Topological Susceptibility - APE

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm APE} = 25$ APE smearing steps for $\alpha_{\rm APE} = 0.6$



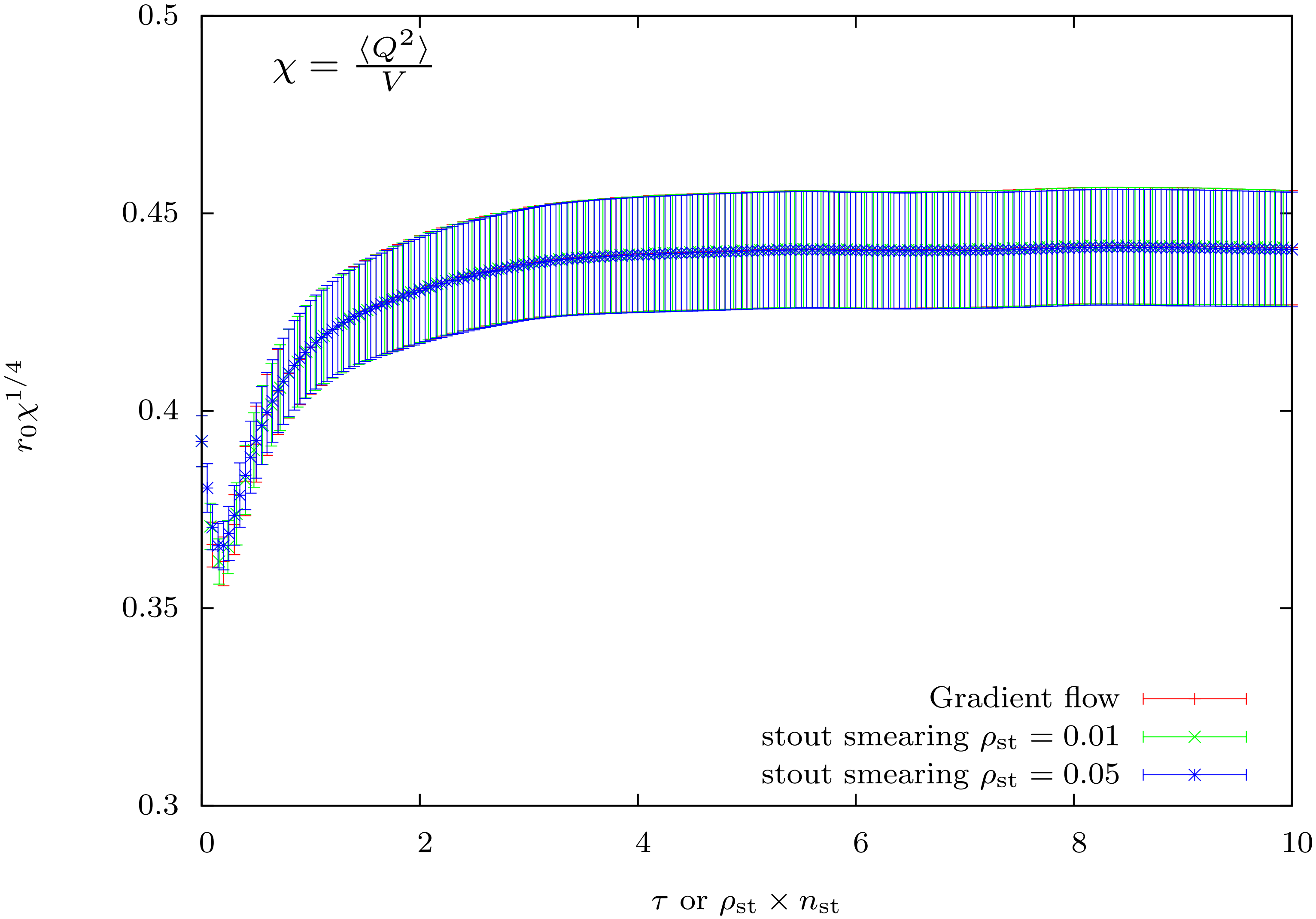
Topological Susceptibility - stout

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm st} = 250$ stout smearing steps for $\rho_{\rm st} = 0.01$



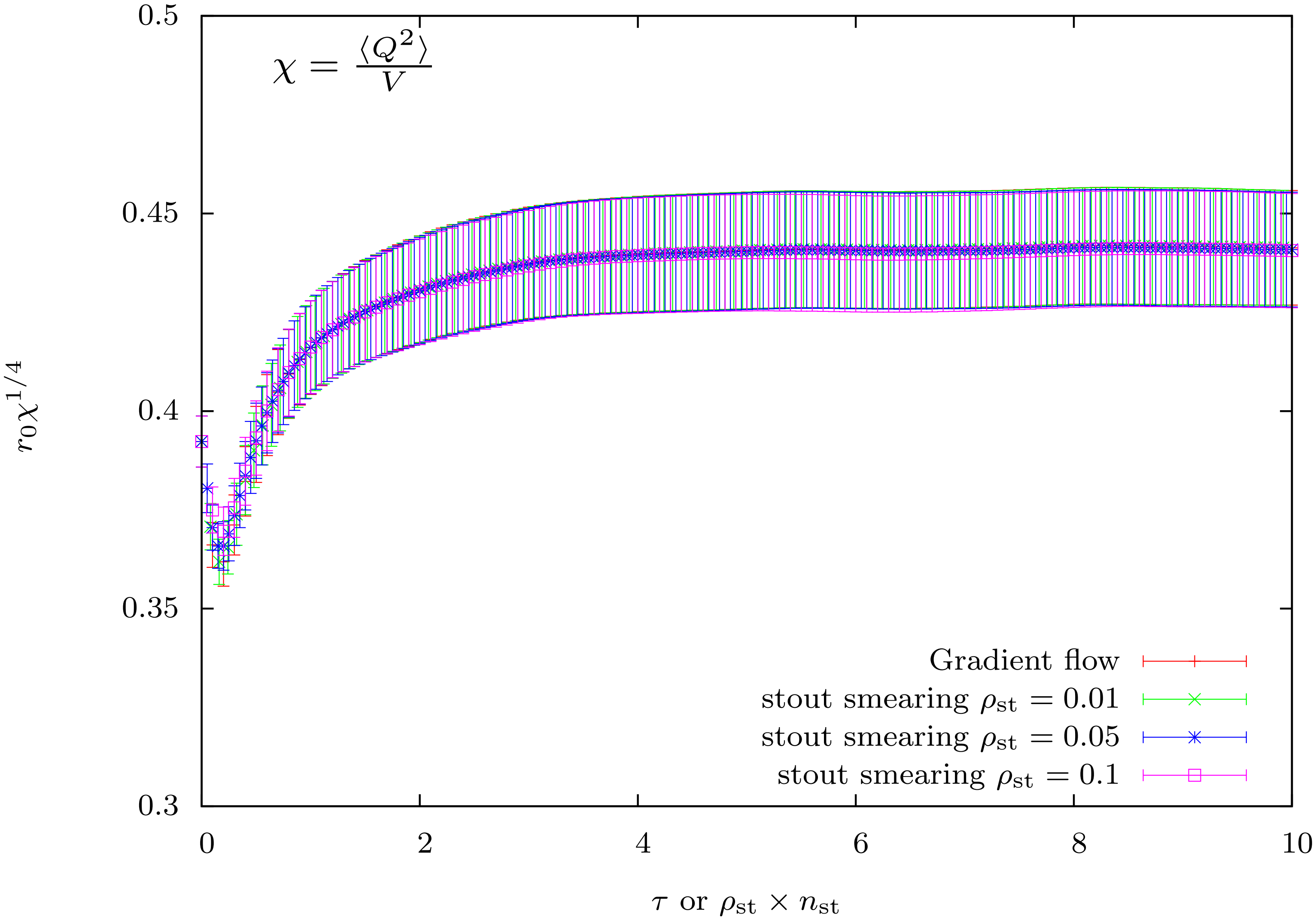
Topological Susceptibility - stout

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\mathrm{st}} = 50$ stout smearing steps for $\rho_{\mathrm{st}} = 0.05$



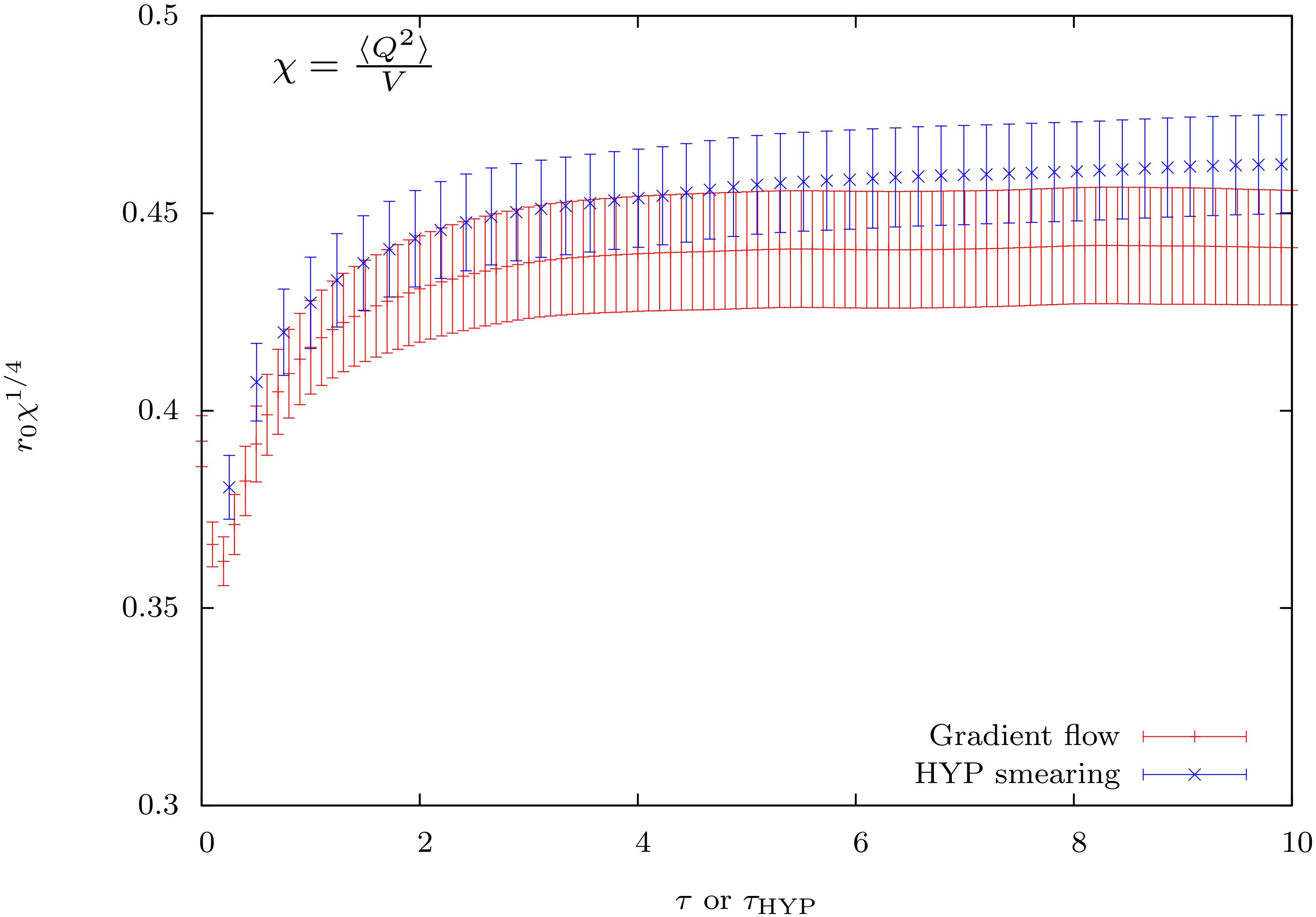
Topological Susceptibility - stout

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm st} = 25$ stout smearing steps for $\rho_{\rm st} = 0.1$



Topological Susceptibility - HYP

The Wilson flow time $t_0 \simeq 2.5a^2 \equiv n_{\rm st} = 10$ HYP smearing steps



Correlation between different smoothers

Let us have a look at the correlation coefficient

$$c_{\mathcal{Q}_1,\mathcal{Q}_2} = \frac{\left\langle \left(\mathcal{Q}_1 - \overline{\mathcal{Q}}_1\right)\left(\mathcal{Q}_2 - \overline{\mathcal{Q}}_2\right)\right\rangle}{\sqrt{\left\langle \left(\mathcal{Q}_1 - \overline{\mathcal{Q}}_1\right)^2\right\rangle \left\langle \left(\mathcal{Q}_2 - \overline{\mathcal{Q}}_2\right)^2\right\rangle}} \,.$$

	WF, t_0	cool, t_0	APE, t_0	stout, t_0	HYP, t_0
WF, t_0	1.00(0)	0.97(0)	1.00(0)	1.00(0)	0.97(0)
cool, t_0	0.97(0)	1.00(0)	0.97(0)	0.97(0)	0.94(0)
APE, t_0	1.00(0)	0.97(0)	1.00(0)	1.00(0)	0.97(0)
stout, t_0	1.00(0)	0.97(0)	1.00(0)	1.00(0)	0.97(0)
HYP, t_0	0.97(0)	0.94(0)	0.97(0)	0.97(0)	1.00(0)

Topological charges are highly correlated!

In the continuum all numbers become 1.00

Fixing the smoothing scale

- One can fix a physical flow time:

$$\lambda_S \simeq \sqrt{8t}\,.$$

- Similarly for cooling:

$$\lambda_S \simeq a\sqrt{\frac{8n_c}{3}}\,.$$

- For the APE smearing:

$$\lambda_S \simeq a\sqrt{\frac{4\alpha_{\mathrm{APE}}n_{\mathrm{APE}}}{3}}.$$

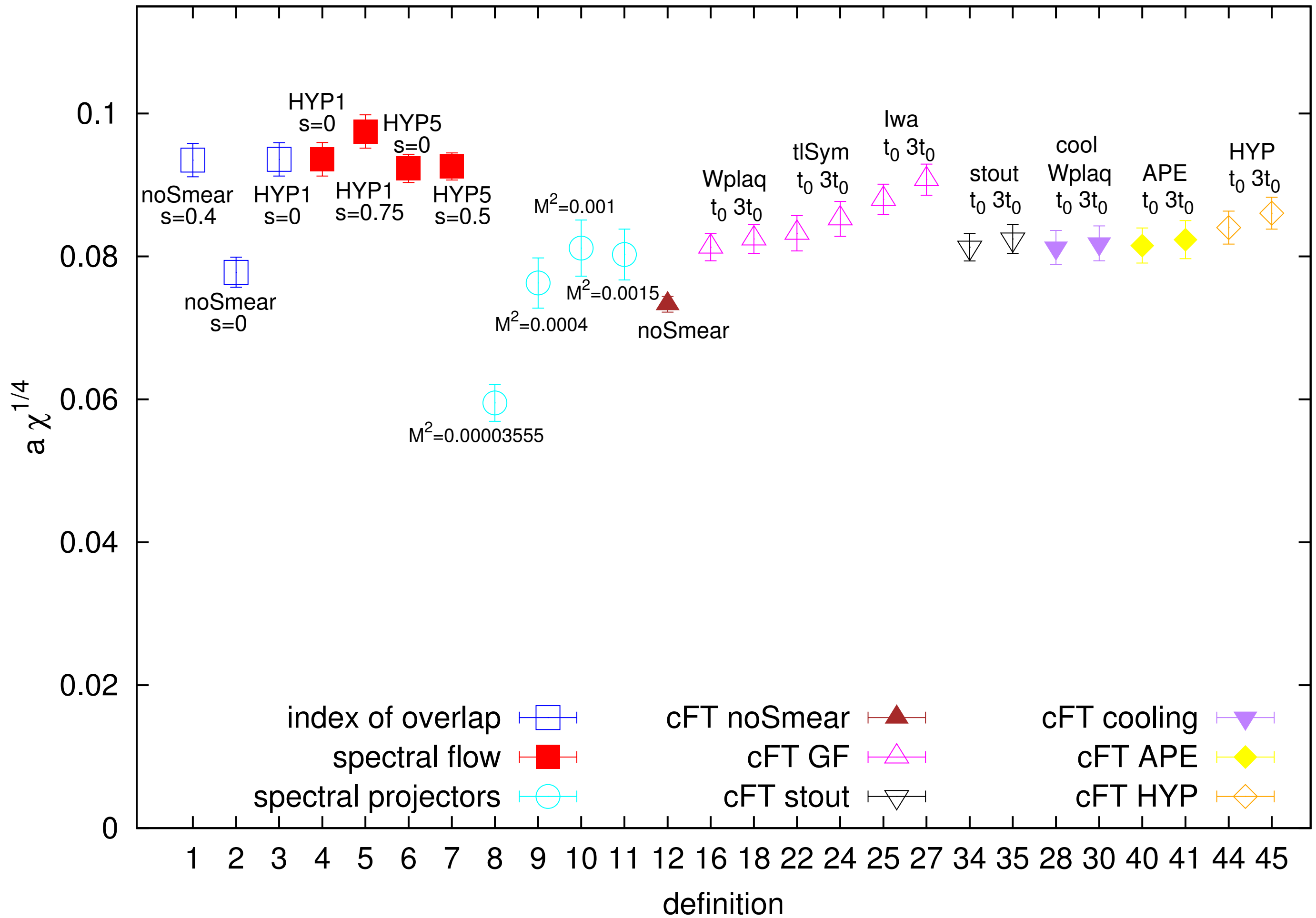
- For the stout smearing

$$\lambda_S \simeq a\sqrt{8\rho_{\mathrm{st}}n_{\mathrm{st}}}\,.$$

- Similar procedure can be applied to t_0?

General comparison: Topological Susceptibility

Comparison of results for the topological susceptibility.

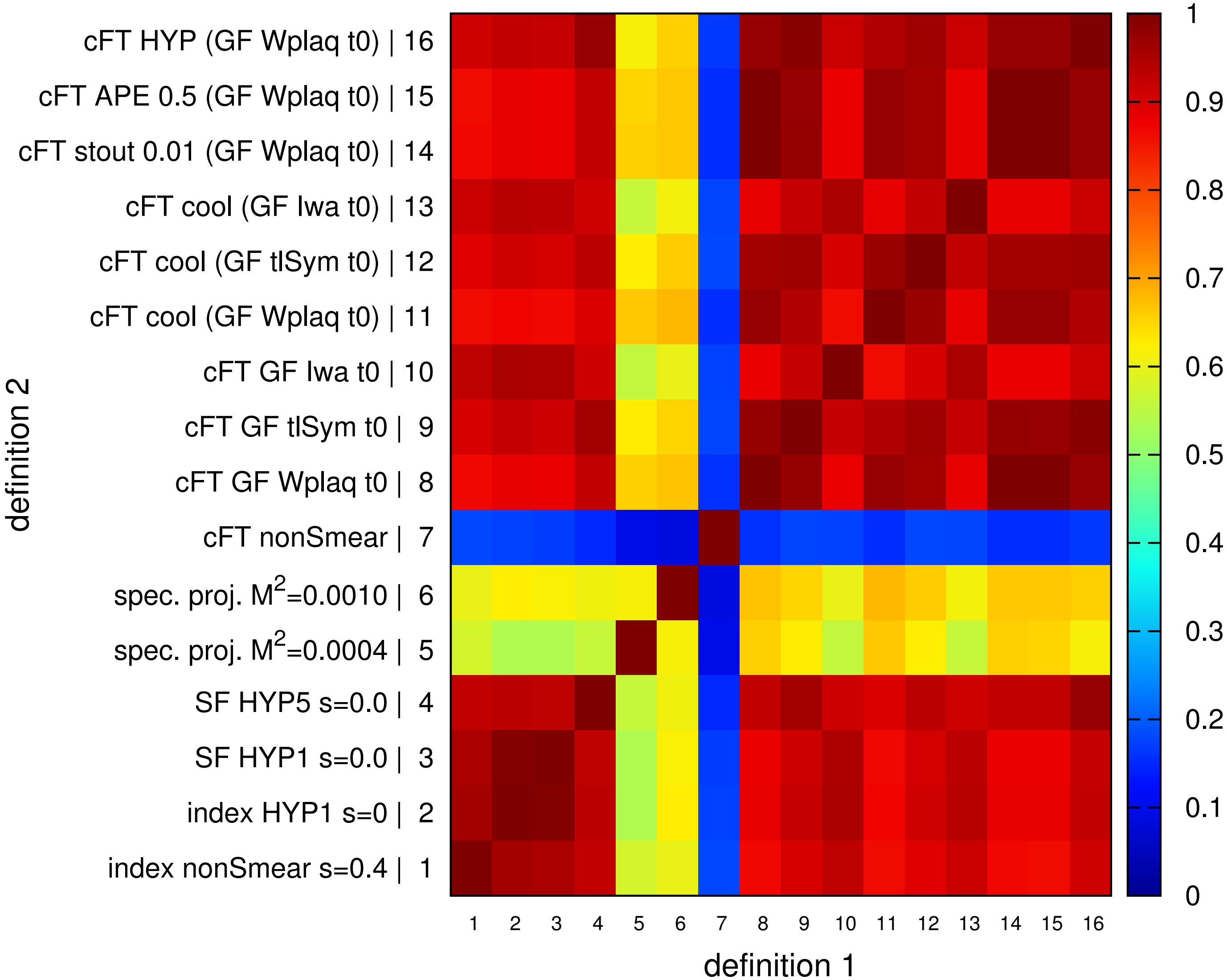


Using $N_f = 2$ twisted mass configuration with:

$\beta = 3.90$, $a \simeq 0.085\text{fm}$, $r_0/a = 5.35(4)$, $m_\pi \simeq 340$ MeV, $m_\pi L = 2.5$, $L/a = 16$

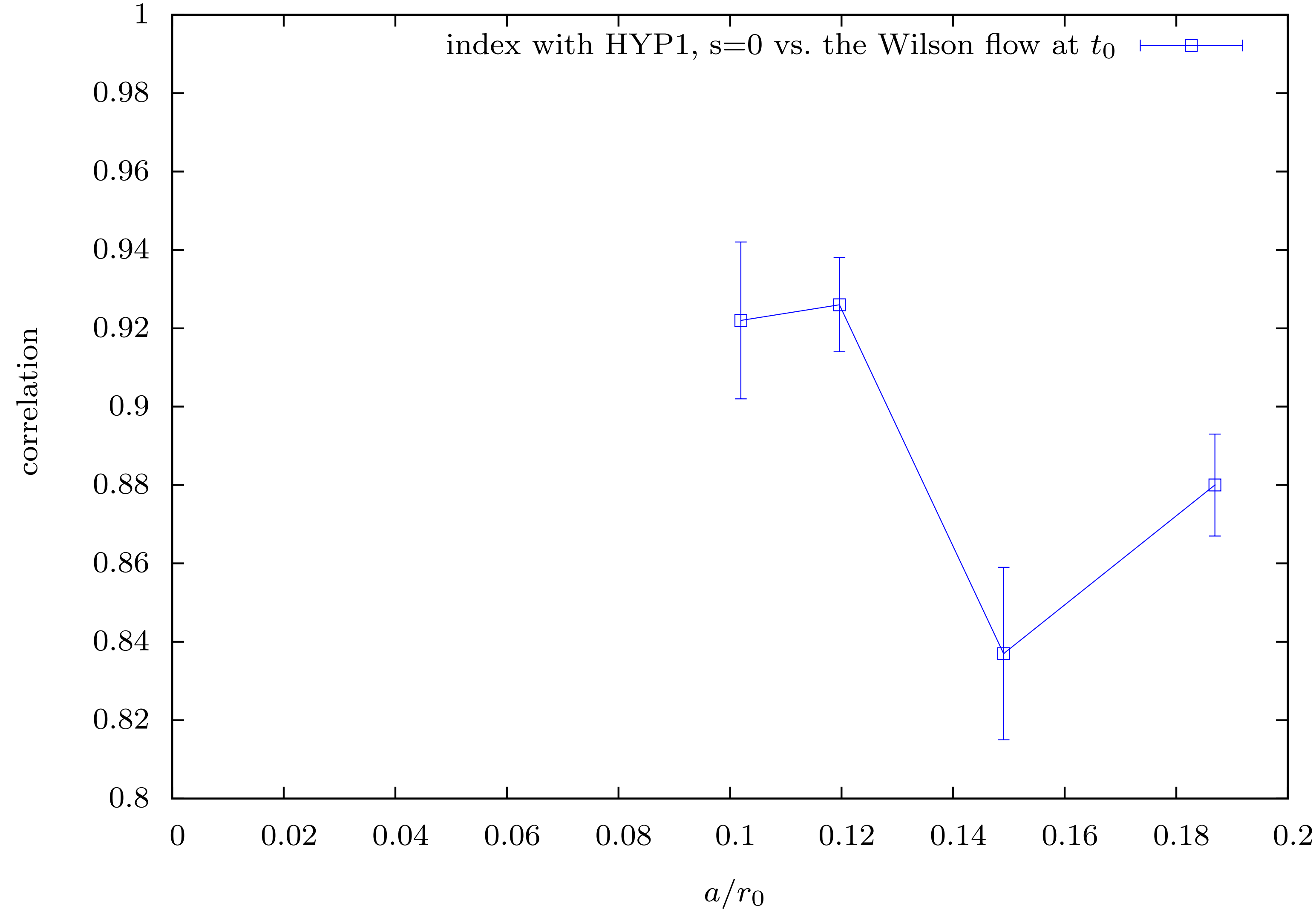
General comparison: Correlation Coefficient

Comparison of the correlation coefficient between fermionic and gluonic definitions.



Continuum limit

Correlation for a fermionic and gluonic definitions as we approach the continuum limit.



Conclusions

☞ Topological susceptibilities are in the same ballpark: $a\chi^{1/4} \in [0.08, 0.09]$.

☞ Correlation coefficient increases towards to 1 as $a \to 0$.

☞ Different definitions influenced by different lattice artifacts.

☞ Most correlation coefficients are above 80 %.

☞ Cooling, APE smearing, stout smearing are numerically equivalent if matched:

$$\tau \simeq \frac{n_c}{3}, \qquad \tau \simeq \alpha_{\rm APE}\frac{n_{\rm APE}}{6} \qquad \tau \simeq \rho_{\rm st} n_{\rm st} \,.$$

Conclusions

☞ Topological susceptibilities are in the same ballpark: $a\chi^{1/4} \in [0.08, 0.09]$.

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☞ Different definitions influenced by different lattice artifacts.

☞ Most correlation coefficients are above 80 %.

☞ Cooling, APE smearing, stout smearing are numerically equivalent if matched:

$$\tau \simeq \frac{n_c}{3}, \qquad \tau \simeq \alpha_{\mathrm{APE}} \frac{n_{\mathrm{APE}}}{6} \qquad \tau \simeq \rho_{\mathrm{st}} n_{\mathrm{st}} \,.$$
$$\sim 120 \times \text{faster}, \qquad \sim 20 \times \text{faster} \qquad \sim 30 \times \text{faster} \,.$$

Conclusions

THANK YOU!

Appendix: General comparison: Correlation Coefficient

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Comparison of the correlation coefficient between fermionic and gluonic definitions.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1.00(0)	0.96(0)	0.95(0)	0.92(1)	0.58(4)	0.60(3)	0.18(6)	0.86(1)	0.90(1)	0.93(0)	0.86(1)	0.89(1)	0.91(0)	0.86(1)	0.86(1)	0.91(1)
2	0.96(0)	1.00(0)	0.99(0)	0.93(0)	0.54(4)	0.62(3)	0.17(4)	0.88(1)	0.92(0)	0.95(0)	0.87(1)	0.91(1)	0.94(0)	0.88(1)	0.88(1)	0.92(0)
3	0.95(0)	0.99(0)	1.00(0)	0.93(0)	0.54(4)	0.62(3)	0.17(4)	0.88(1)	0.91(0)	0.95(0)	0.86(1)	0.90(0)	0.93(0)	0.88(1)	0.88(1)	0.92(0)
4	0.92(1)	0.93(0)	0.93(0)	1.00(0)	0.56(4)	0.61(3)	0.15(4)	0.92(0)	0.96(0)	0.91(0)	0.90(1)	0.93(0)	0.91(0)	0.92(0)	0.92(0)	0.97(0)
5	0.58(4)	0.54(4)	0.54(4)	0.56(4)	1.00(0)	0.62(4)	0.10(3)	0.66(3)	0.63(3)	0.56(4)	0.66(3)	0.62(3)	0.56(4)	0.65(3)	0.65(3)	0.62(3)
6	0.60(3)	0.62(3)	0.62(3)	0.61(3)	0.62(4)	1.00(0)	0.09(4)	0.67(3)	0.65(3)	0.60(4)	0.68(3)	0.66(3)	0.61(4)	0.66(3)	0.66(3)	0.65(3)
7	0.18(6)	0.17(4)	0.17(4)	0.15(4)	0.10(3)	0.09(4)	1.00(0)	0.16(4)	0.18(4)	0.17(4)	0.15(4)	0.18(4)	0.18(4)	0.16(4)	0.16(4)	0.17(4)
8	0.86(1)	0.88(1)	0.88(1)	0.92(0)	0.66(3)	0.67(3)	0.16(4)	1.00(0)	0.97(0)	0.88(1)	0.97(0)	0.96(0)	0.88(1)	1.00(0)	1.00(0)	0.97(0)
9	0.90(1)	0.92(0)	0.91(0)	0.96(0)	0.63(3)	0.65(3)	0.18(4)	0.97(0)	1.00(0)	0.92(0)	0.94(0)	0.96(0)	0.92(0)	0.97(0)	0.97(0)	0.99(0)
10	0.93(0)	0.95(0)	0.95(0)	0.91(0)	0.56(4)	0.60(4)	0.17(4)	0.88(1)	0.92(0)	1.00(0)	0.86(1)	0.90(1)	0.95(0)	0.88(1)	0.88(1)	0.91(0)
11	0.86(1)	0.87(1)	0.86(1)	0.90(1)	0.66(3)	0.68(3)	0.15(4)	0.97(0)	0.94(0)	0.86(1)	1.00(0)	0.97(0)	0.88(1)	0.97(0)	0.97(0)	0.94(0)
12	0.89(1)	0.91(1)	0.90(0)	0.93(0)	0.62(3)	0.66(3)	0.18(4)	0.96(0)	0.96(0)	0.90(1)	0.97(0)	1.00(0)	0.92(0)	0.96(0)	0.96(0)	0.96(0)
13	0.91(0)	0.94(0)	0.93(0)	0.91(0)	0.56(4)	0.61(4)	0.18(4)	0.88(1)	0.92(0)	0.95(0)	0.88(1)	0.92(0)	1.00(0)	0.88(1)	0.88(1)	0.91(0)
14	0.86(1)	0.88(1)	0.88(1)	0.92(0)	0.65(3)	0.66(3)	0.16(4)	1.00(0)	0.97(0)	0.88(1)	0.97(0)	0.96(0)	0.88(1)	1.00(0)	1.00(0)	0.97(0)
15	0.86(1)	0.88(1)	0.88(1)	0.92(0)	0.65(3)	0.66(3)	0.16(4)	1.00(0)	0.97(0)	0.88(1)	0.97(0)	0.96(0)	0.88(1)	1.00(0)	1.00(0)	0.97(0)
16	0.91(1)	0.92(0)	0.92(0)	0.97(0)	0.62(3)	0.65(3)	0.17(4)	0.97(0)	0.99(0)	0.91(0)	0.94(0)	0.96(0)	0.91(0)	0.97(0)	0.97(0)	1.00(0)

Appendix: General comparison: Correlation Coefficient

Comparison of the correlation coefficient between fermionic and gluonic definitions.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1.00(0)	0.96(0)	0.95(0)	0.92(1)	0.58(4)	0.60(3)	0.18(6)	0.86(1)	0.90(1)	0.93(0)	0.86(1)	0.89(1)	0.91(0)	0.86(1)	0.86(1)	0.91(1)
2	0.96(0)	1.00(0)	0.99(0)	0.93(0)	0.54(4)	0.62(3)	0.17(4)	0.88(1)	0.92(0)	0.95(0)	0.87(1)	0.91(1)	0.94(0)	0.88(1)	0.88(1)	0.92(0)
3	0.95(0)	0.99(0)	1.00(0)	0.93(0)	0.54(4)	0.62(3)	0.17(4)	0.88(1)	0.91(0)	0.95(0)	0.86(1)	0.90(0)	0.93(0)	0.88(1)	0.88(1)	0.92(0)
4	0.92(1)	0.93(0)	0.93(0)	1.00(0)	0.56(4)	0.61(3)	0.15(4)	0.92(0)	0.96(0)	0.91(0)	0.90(1)	0.93(0)	0.91(0)	0.92(0)	0.92(0)	0.97(0)
5	0.58(4)	0.54(4)	0.54(4)	0.56(4)	1.00(0)	0.62(4)	0.10(3)	0.66(3)	0.63(3)	0.56(4)	0.66(3)	0.62(3)	0.56(4)	0.65(3)	0.65(3)	0.62(3)
6	0.60(3)	0.62(3)	0.62(3)	0.61(3)	0.62(4)	1.00(0)	0.09(4)	0.67(3)	0.65(3)	0.60(4)	0.68(3)	0.66(3)	0.61(4)	0.66(3)	0.66(3)	0.65(3)
7	0.18(6)	0.17(4)	0.17(4)	0.15(4)	0.10(3)	0.09(4)	1.00(0)	0.16(4)	0.18(4)	0.17(4)	0.15(4)	0.18(4)	0.18(4)	0.16(4)	0.16(4)	0.17(4)
8	0.86(1)	0.88(1)	0.88(1)	0.92(0)	0.66(3)	0.67(3)	0.16(4)	1.00(0)	0.97(0)	0.88(1)	0.97(0)	0.96(0)	0.88(1)	1.00(0)	1.00(0)	0.97(0)
9	0.90(1)	0.92(0)	0.91(0)	0.96(0)	0.63(3)	0.65(3)	0.18(4)	0.97(0)	1.00(0)	0.92(0)	0.94(0)	0.96(0)	0.92(0)	0.97(0)	0.97(0)	0.99(0)
10	0.93(0)	0.95(0)	0.95(0)	0.91(0)	0.56(4)	0.60(4)	0.17(4)	0.88(1)	0.92(0)	1.00(0)	0.86(1)	0.90(1)	0.95(0)	0.88(1)	0.88(1)	0.91(0)
11	0.86(1)	0.87(1)	0.86(1)	0.90(1)	0.66(3)	0.68(3)	0.15(4)	0.97(0)	0.94(0)	0.86(1)	1.00(0)	0.97(0)	0.88(1)	0.97(0)	0.97(0)	0.94(0)
12	0.89(1)	0.91(1)	0.90(0)	0.93(0)	0.62(3)	0.66(3)	0.18(4)	0.96(0)	0.96(0)	0.90(1)	0.97(0)	1.00(0)	0.92(0)	0.96(0)	0.96(0)	0.96(0)
13	0.91(0)	0.94(0)	0.93(0)	0.91(0)	0.56(4)	0.61(4)	0.18(4)	0.88(1)	0.92(0)	0.95(0)	0.88(1)	0.92(0)	1.00(0)	0.88(1)	0.88(1)	0.91(0)
14	0.86(1)	0.88(1)	0.88(1)	0.92(0)	0.65(3)	0.66(3)	0.16(4)	1.00(0)	0.97(0)	0.88(1)	0.97(0)	0.96(0)	0.88(1)	1.00(0)	1.00(0)	0.97(0)
15	0.86(1)	0.88(1)	0.88(1)	0.92(0)	0.65(3)	0.66(3)	0.16(4)	1.00(0)	0.97(0)	0.88(1)	0.97(0)	0.96(0)	0.88(1)	1.00(0)	1.00(0)	0.97(0)
16	0.91(1)	0.92(0)	0.92(0)	0.97(0)	0.62(3)	0.65(3)	0.17(4)	0.97(0)	0.99(0)	0.91(0)	0.94(0)	0.96(0)	0.91(0)	0.97(0)	0.97(0)	1.00(0)

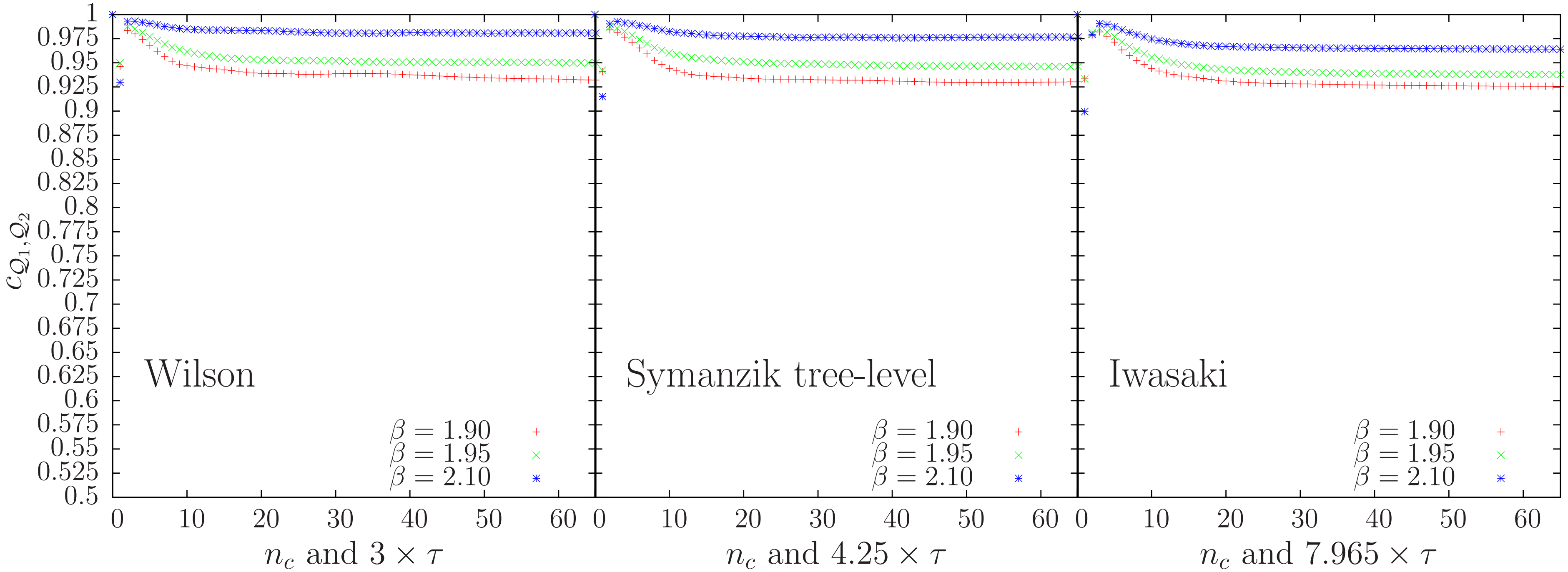
Appendix: General comparison: Correlation Coefficient

Comparison of the correlation coefficient between fermionic and gluonic definitions.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1.00(0)	0.96(0)	0.95(0)	0.92(1)	0.58(4)	0.60(3)	0.18(6)	0.86(1)	0.90(1)	0.93(0)	0.86(1)	0.89(1)	0.91(0)	0.86(1)	0.86(1)	0.91(1)
2	0.96(0)	1.00(0)	0.99(0)	0.93(0)	0.54(4)	0.62(3)	0.17(4)	0.88(1)	0.92(0)	0.95(0)	0.87(1)	0.91(1)	0.94(0)	0.88(1)	0.88(1)	0.92(0)
3	0.95(0)	0.99(0)	1.00(0)	0.93(0)	0.54(4)	0.62(3)	0.17(4)	0.88(1)	0.91(0)	0.95(0)	0.86(1)	0.90(0)	0.93(0)	0.88(1)	0.88(1)	0.92(0)
4	0.92(1)	0.93(0)	0.93(0)	1.00(0)	0.56(4)	0.61(3)	0.15(4)	0.92(0)	0.96(0)	0.91(0)	0.90(1)	0.93(0)	0.91(0)	0.92(0)	0.92(0)	0.97(0)
5	0.58(4)	0.54(4)	0.54(4)	0.56(4)	1.00(0)	0.62(4)	0.10(3)	0.66(3)	0.63(3)	0.56(4)	0.66(3)	0.62(3)	0.56(4)	0.65(3)	0.65(3)	0.62(3)
6	0.60(3)	0.62(3)	0.62(3)	0.61(3)	0.62(4)	1.00(0)	0.09(4)	0.67(3)	0.65(3)	0.60(4)	0.68(3)	0.66(3)	0.61(4)	0.66(3)	0.66(3)	0.65(3)
7	0.18(6)	0.17(4)	0.17(4)	0.15(4)	0.10(3)	0.09(4)	1.00(0)	0.16(4)	0.18(4)	0.17(4)	0.15(4)	0.18(4)	0.18(4)	0.16(4)	0.16(4)	0.17(4)
8	0.86(1)	0.88(1)	0.88(1)	0.92(0)	0.66(3)	0.67(3)	0.16(4)	1.00(0)	0.97(0)	0.88(1)	0.97(0)	0.96(0)	0.88(1)	1.00(0)	1.00(0)	0.97(0)
9	0.90(1)	0.92(0)	0.91(0)	0.96(0)	0.63(3)	0.65(3)	0.18(4)	0.97(0)	1.00(0)	0.92(0)	0.94(0)	0.96(0)	0.92(0)	0.97(0)	0.97(0)	0.99(0)
10	0.93(0)	0.95(0)	0.95(0)	0.91(0)	0.56(4)	0.60(4)	0.17(4)	0.88(1)	0.92(0)	1.00(0)	0.86(1)	0.90(1)	0.95(0)	0.88(1)	0.88(1)	0.91(0)
11	0.86(1)	0.87(1)	0.86(1)	0.90(1)	0.66(3)	0.68(3)	0.15(4)	0.97(0)	0.94(0)	0.86(1)	1.00(0)	0.97(0)	0.88(1)	0.97(0)	0.97(0)	0.94(0)
12	0.89(1)	0.91(1)	0.90(0)	0.93(0)	0.62(3)	0.66(3)	0.18(4)	0.96(0)	0.96(0)	0.90(1)	0.97(0)	1.00(0)	0.92(0)	0.96(0)	0.96(0)	0.96(0)
13	0.91(0)	0.94(0)	0.93(0)	0.91(0)	0.56(4)	0.61(4)	0.18(4)	0.88(1)	0.92(0)	0.95(0)	0.88(1)	0.92(0)	1.00(0)	0.88(1)	0.88(1)	0.91(0)
14	0.86(1)	0.88(1)	0.88(1)	0.92(0)	0.65(3)	0.66(3)	0.16(4)	1.00(0)	0.97(0)	0.88(1)	0.97(0)	0.96(0)	0.88(1)	1.00(0)	1.00(0)	0.97(0)
15	0.86(1)	0.88(1)	0.88(1)	0.92(0)	0.65(3)	0.66(3)	0.16(4)	1.00(0)	0.97(0)	0.88(1)	0.97(0)	0.96(0)	0.88(1)	1.00(0)	1.00(0)	0.97(0)
16	0.91(1)	0.92(0)	0.92(0)	0.97(0)	0.62(3)	0.65(3)	0.17(4)	0.97(0)	0.99(0)	0.91(0)	0.94(0)	0.96(0)	0.91(0)	0.97(0)	0.97(0)	1.00(0)

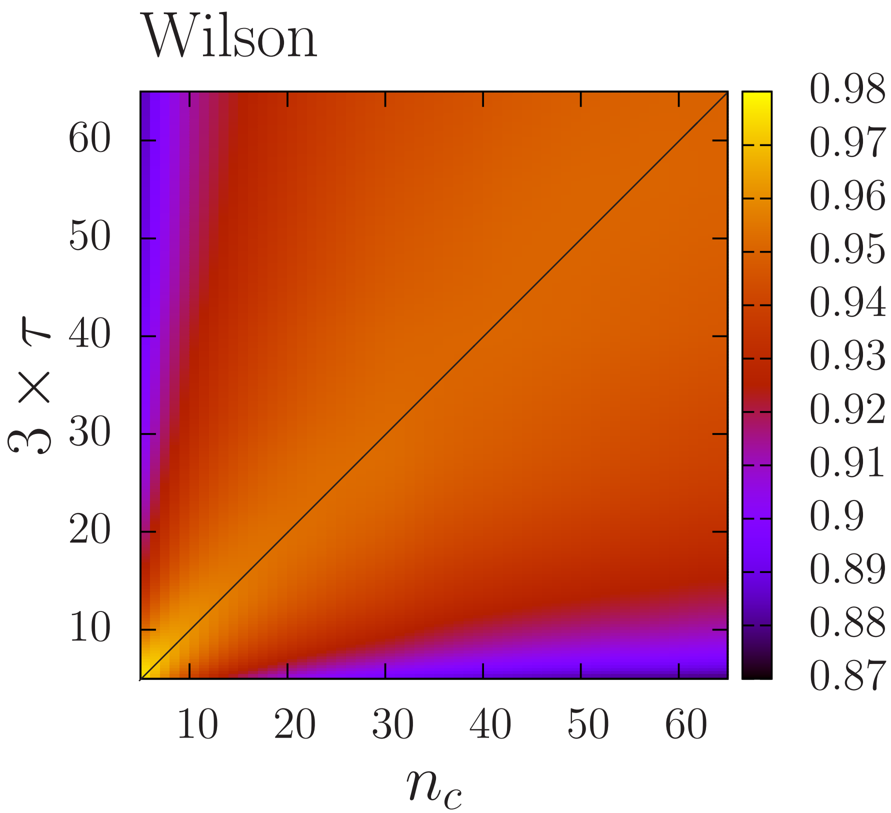
Appendix: General comparison: Correlation Coefficient

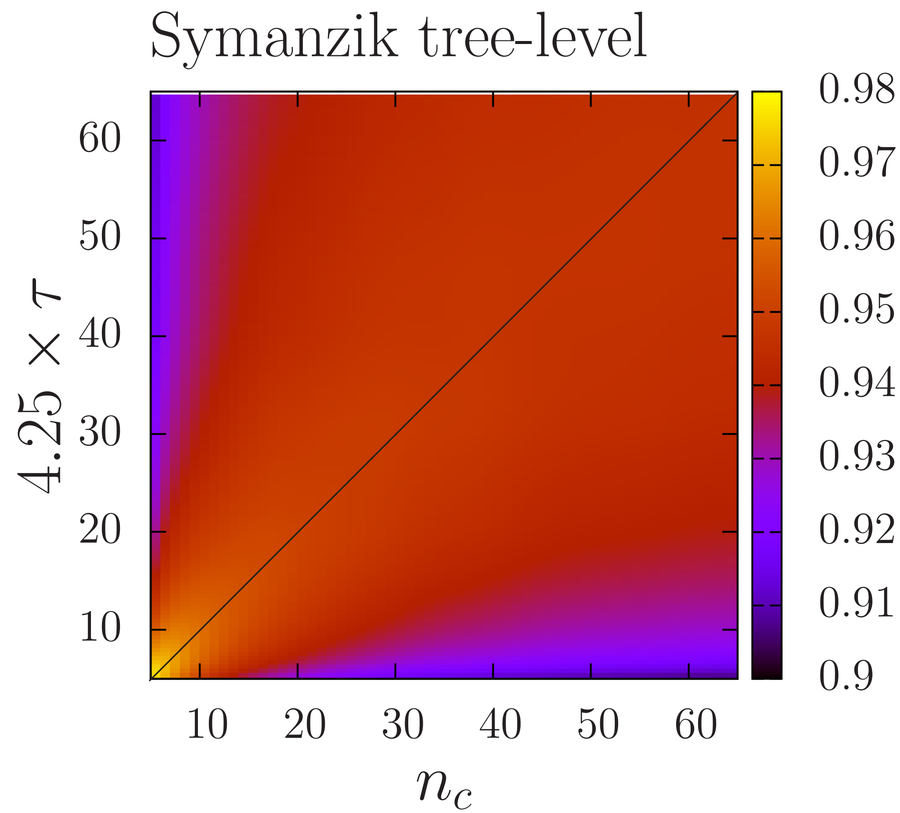
Comparison of correlation coefficient between cooling and the gradient flow.



Appendix: General comparison: Correlation Coefficient

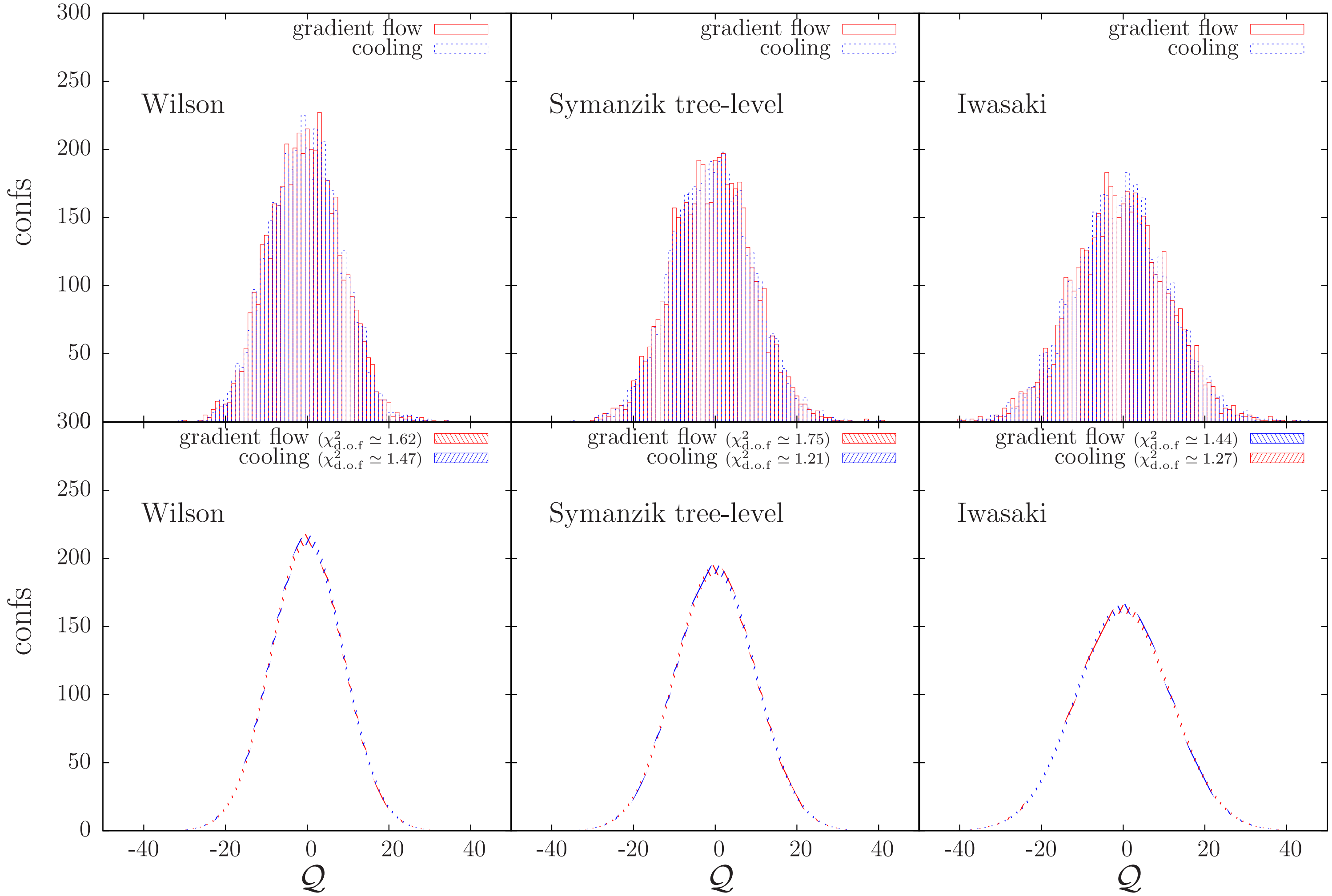
Comparison of correlation coefficient between cooling and the gradient flow.





Appendix: General comparison: Distribution

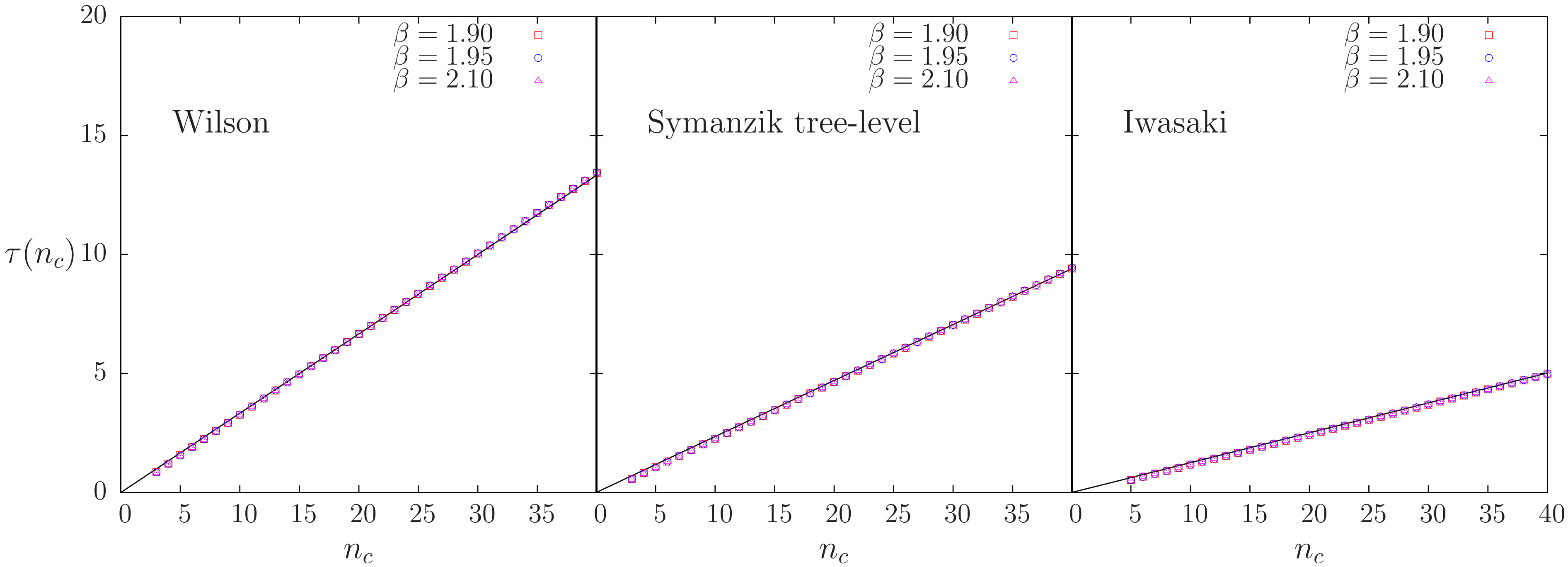
Comparison of distributions between cooling and the gradient flow.



Appendix: General comparison: $\tau(n_c)$

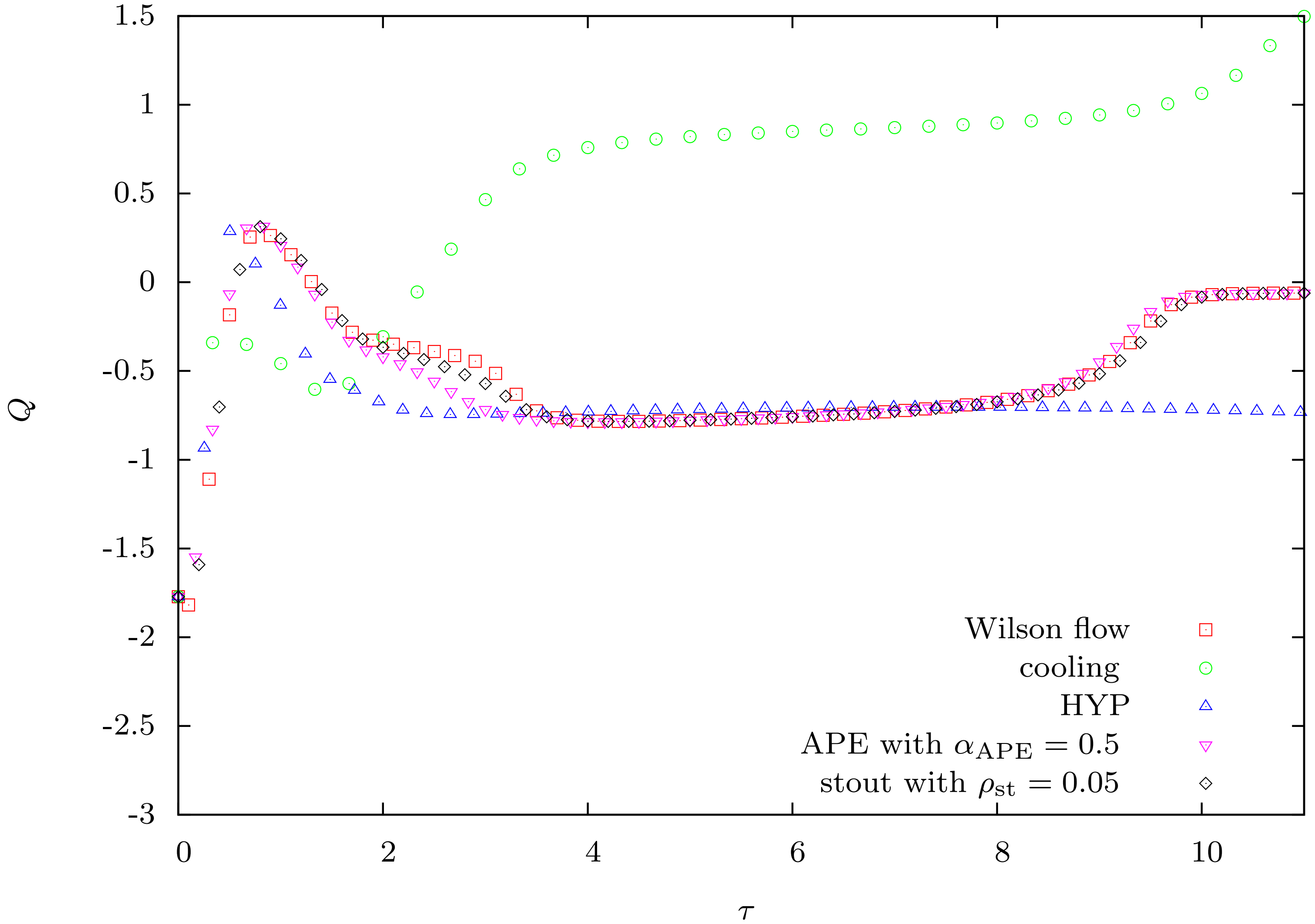
Comparison of $\tau(n_c)$ for different smoothing actions.

Prediction $\tau \simeq \frac{n_c}{3-15b_1}$.



Appendix: Topological Charge: level of agreement

Why topological susceptibility has such a high level of agreement?



THANK YOU!