Borici-Creutz fermions on 2-dim lattice

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Outline

Introduction

- Simulation with dynamical chiral fermions in a lattice is a challenging task.
- The famous no-go theorem:Lattice fermion actions with,
  - locality
  - chiral symmetry
  - hermiticity

  must produce massless fermions in multiples of two in continuum limit.
- There exist lot of fermion prescriptions to avoid fermion doubling caused by the naive fermions.
- Every model has its own advantges and also individual shortcomings.

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- Borici Creutz action in 4 dimensional space is written as,

$$S_{BC} = \sum_n \Big[\frac{1}{2}\sum_\mu \bar{\psi}_n\gamma_\mu(\psi_{n+\mu} - \psi_{n-\mu}) - \frac{ir}{2}\sum_\mu \bar{\psi}_n(\Gamma - \gamma_\mu)(2\psi_n - \psi_{n+\mu} - \psi_{n-\mu}) + m\bar{\psi}_n\psi_n\Big]$$

- Then the Dirac operator in momentum space ($a = 1$),[1]

$$D_{BC}(p) = \underbrace{\sum_\mu \Big[i\gamma_\mu \sin p_\mu - i(\gamma'_\mu)(1 - cos(p_\mu)\Big]}_{\text{Two zeros at (0,0,0,0) and } (\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})}.$$

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More on Borici Creutz fermion

Important relations:

$$\gamma'_\mu = \sum_\mu \gamma_\mu \Gamma \gamma_\mu$$

$$\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)\; and\, \Gamma^2 = 1$$

$$\{\Gamma, \gamma_\mu\} = \{\Gamma, \gamma'_\mu\} = 1.$$

- Hypercubic symmetry is broken so we can introduce other dimension counter terms but as long as $M_f(p_\mu)$ is cubic symmetric, only three and four dimensional counterterms will be required.

More on BC fermion

- For BC action we here only discuss the dimension three counter term(c_3) and for simplicity tune the coefficient of 4 dimension counter term to zero,
- Adding that term the action looks like,

$$
\begin{aligned}
S_{BC} \quad = \quad & \sum_n \Big[\frac{1}{2} \sum_\mu \bar{\psi}_n \gamma_\mu (\psi_{n+\mu} - \psi_{n-\mu}) \\
& - \frac{ir}{2} \sum_\mu \bar{\psi}_n (\Gamma - \gamma_\mu)(2\psi_n - \psi_{n+\mu} - \psi_{n-\mu}) \\
& + ic_3 \bar{\psi}_n \Gamma \psi_n + m \bar{\psi}_n \psi_n \Big]
\end{aligned}
$$

- Other two types are in the family: KW and TO fermions.

More on BC fermion

- So its look like in momentum space($a = 1$)

$$D_{BC}(p) = \sum_{\mu} \left[i\gamma_\mu \sin p_\mu + i(\Gamma - \gamma_\mu) cos(p_\mu) \right] + i(c_3 - 2)\Gamma$$

- Now the term c_3 changes the number and postion of the zeros,

$$c_3 = \begin{cases} 0 & \text{two zeros (0,0,0,0) and } (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) \\ 4 & \text{two zeros } (\pi, \pi, \pi, \pi) \text{ and } (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) \\ 2 & \text{no zeros} \end{cases}$$

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Borici-Creutz fermions in 2D

Multipicity of the free Dirac operator In 2D

- The Borici-Creutz action has already been defined previously. In 2D, $\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2)$, $\{\Gamma, \gamma_\mu\} = 1$, and $\Gamma^2 = \frac{1}{2}$.[(2$\times$2) gamma matrices]
- The free Dirac operator in momentum space is written as,

$$D_{BC}(p) = \sum_\mu [i\gamma_\mu \sin p_\mu + i(\Gamma - \gamma_\mu) cos(p_\mu)] + i(c_3 - 2)\Gamma.$$

  - For $c_3 = 0$ and $c_3 = 4$ only one zero of the Dirac operator but dispersion becomes unphysical.
  - For $0 < c_3 < 0.59$ and $3.41 < c_3 < 4$ the Dirac operator has only two zeros i.e this is the region of minimal doubling.
  - And for the rest of the region i.e $0.59 < c_3 < 3.41$ the Dirac operator has four zeros. Out of those zeros, we get correct continuum limit of the Dirac operator only when $p_1 = p_2$.

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Gross Neveu Model in 2 dimensions

- The free action (with $r = 1$) is

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- After including the four fermi interactions,

$$S_{BCGN} = \sum_n \Big[S_{BC} - \frac{g^2}{2N}[(\bar{\psi}_n \psi_n)^2 + (\bar{\psi}_n i\Gamma \psi_n)^2]\Big].$$

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$$

HMC of the model

- For numerical simulation we take $c_3 = 0.1$.The value of c_3 is taken where the BC action describes two flavor fermions with Lorentz invariant dispersion relation.
- The lattice version of the action is written as,

$$S = \bar{\psi}_i M_{ij} \psi_j + \frac{N}{2g^2}(\sigma^2 + \pi_\Gamma^2),$$

- where the auxilary fields are defines as,

$$\sigma = -\frac{g^2}{N}(\bar{\psi}\psi),$$
$$\pi_\Gamma = -\frac{g^2}{N}(\bar{\psi} i\Gamma \psi)$$

Simulation details

- With pseudofermions the action becomes,

$$S = \phi^\dagger (M^\dagger M)^{-1} \phi + \frac{1}{g^2}(\sigma^2 + \pi_\Gamma^2).$$

- Hybrid Monte Carlo(HMC) details:
  The configurations are generated by considering step-size($\triangle t$)=0.1 in the leapfrog method and ten steps per trajectory in the molecular dynamics chain. We do not use any preconditioning during the simulation. First 1000 ensembles are rejected for thermalization and analysis is performed over the next 8000 ensembles.
- And bare mass is 0.3 and $\beta = 0.7$.

Correlators

For meson mass spectrum calculation, we need to evaluate the correlators

$$C_{ij}(t) = \langle O_i(t)\, O_j^\dagger(0) \rangle.$$

Here we list some of the parity odd interpolators for the GN model which we expect to couple to ground state as well as excited states:

$$\begin{aligned} O_1(t) &= \overline{\psi}(x,t)\gamma_5\psi(x,t) \\ O_i(t) &= \frac{1}{4}\big((\overline{\psi}(x+m,t) - \overline{\psi}(x-m,t)\big)\gamma_5\big(\psi(x+n,t) - \psi(x-n,t)\big) \\ O_j(t) &= \frac{1}{4}\big((\overline{\psi}(x+m,t) - \overline{\psi}(x-m,t)\big)\gamma_1\big(\psi(x+n,t) - \psi(x-n,t)\big) \end{aligned}$$

last two are derivative sources useful for extracting excited states.

- O_2, O_3 when, $m = n = 3$ & $m = 5, n = 3$ in 2nd correlator.
- O_4, O_5 when, $m = 4, n = 3$ & $m = 5, n = 3$ in 3rd correlator.

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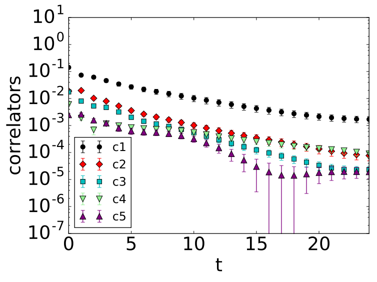
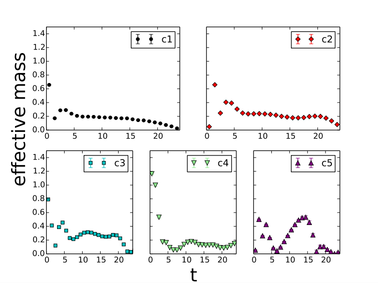
(a)


(b)


Diagonal correlators (in the plots $c1 \equiv C_{11}$, etc.) and effective mass of meson in GN model for 16×48 lattice

The effective masses are extracted from the correlators at different time slices by the formula

$$M_{eff} = \ln\left(\frac{c(t)}{c(t+1)}\right).$$

Only useful for ground state, but not for excited states.

variational method for extracting states

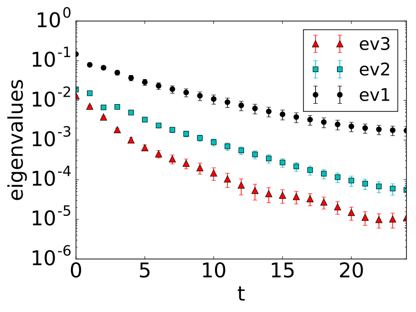
To get the mass spectrum from eigenvalues one solves the generalized eigenvalue problem defined by,

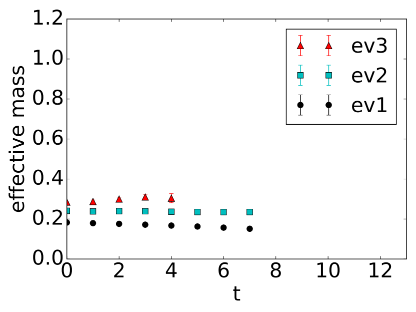
$$C(t)\vec{v}^{(n)} = \lambda^{(n)}(t)\,C(t_0)\vec{v}^{(n)}$$

where $C(t)$ is the $N \times N$ correlation matrix constructed from N interpolators $O_i,\ (i = 1, 2 \cdots, N)$. The n-th eigenvalue behaves as

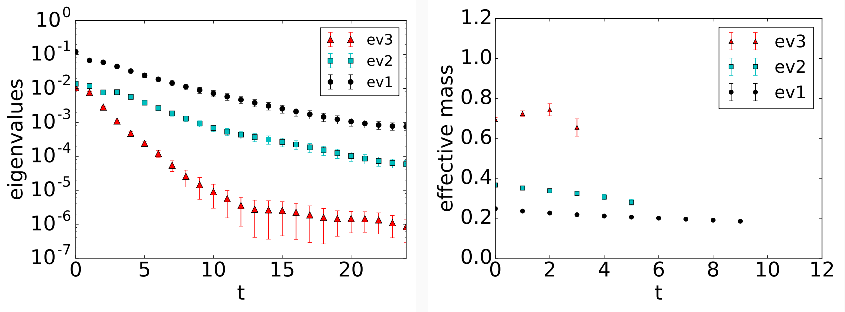
$$\lambda^{(n)}(t) = e^{-(t-t_0)E_n}\Big[1 + \mathcal{O}(e^{-(t-t_0)\Delta_n})\Big],$$

After tried with many combinations of correlators we take O_1, O_2 and O_3 in our correlation matrix basis.

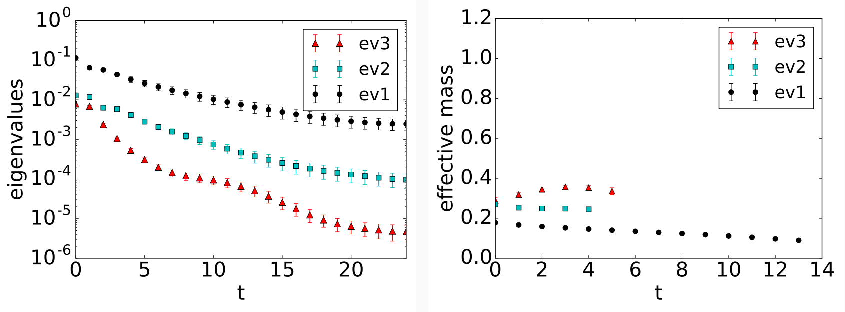




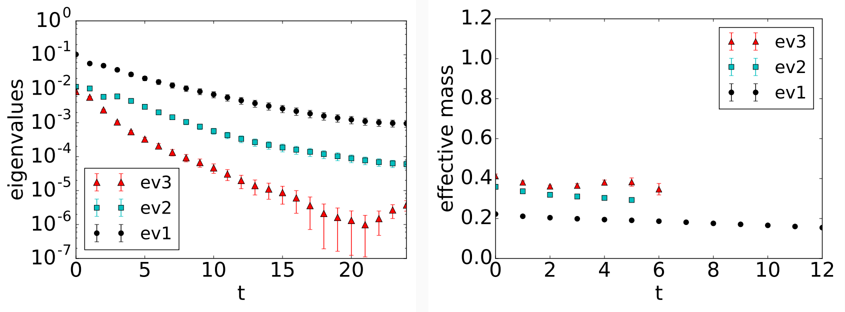
Eigenvalues of correlation matrix and effective mass of meson in GN model for 16×48 lattice



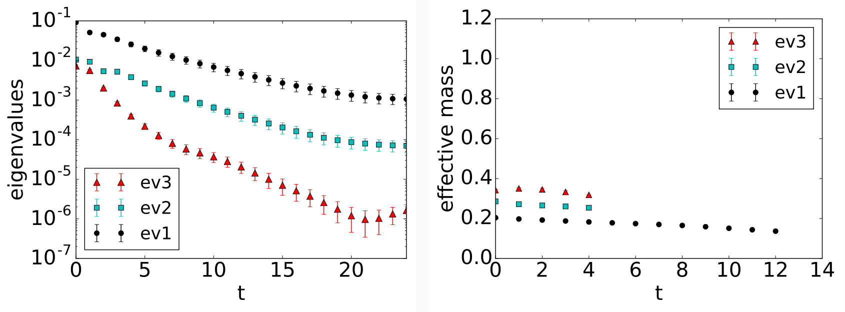
Eigenvalues and effective mass for 18×48 lattice



Eigenvalues and effective mass for 20×48 lattice

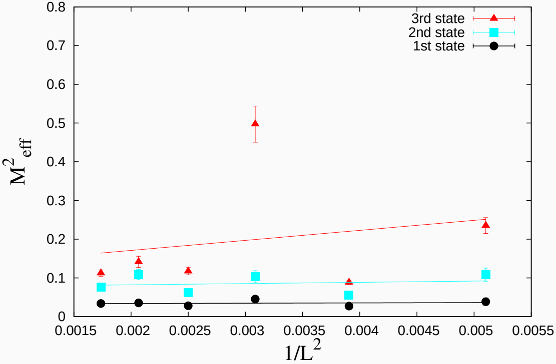


Eigenvalues and effective mass for 22×48 lattice



Eigenvalues and effective mass for 24×48 lattice

Volume dependence of the effective mass



Volume dependence of the effective mass

volume dependence of the effective mass

The ground state and the first excited state show no volume dependence and hence can be considered as bound states

Second excited state shows volume dependence. Specially, for 18×48 lattice size, we get an large mass for the second excited state.

But looking at the fit of the points we expect 2nd excited state to be a scattering state.

conclusive??

The fit for the second excited state shown in Figure includes that point. In general, scattering states show strong volume dependence and increase linearly with $1/L^2$, the volume dependency of the second excited state in our case is not very conclusive.

Simulation details for chiral condensate

- We simulate our model by hybrid monte carlo (HMC) method and evaluate the order parameter for the chiral phase transition $\langle\sigma\rangle$ as a function of coupling constant. We use point sources to estimate the condensate.

$$\begin{aligned} \langle\bar{\psi}\psi\rangle &= -\langle TrM^{-1}\rangle \\ \langle\sigma\rangle &= -\beta\langle\bar{\psi}\psi\rangle \\ \text{where } \beta &= \frac{1}{g^2}. \end{aligned}$$

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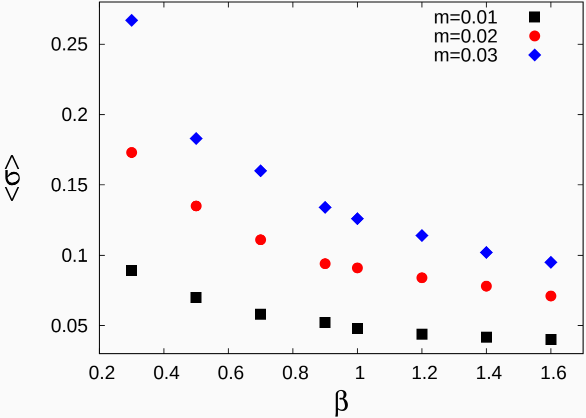
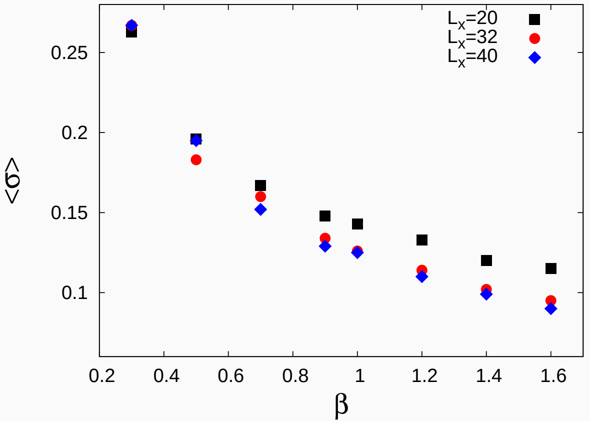
Chiral Condensate

$\langle\sigma\rangle$ vs β of m=0.01,0.02 & 0.03 for Gross-Neveu model with BC fermions in a 32$\times$32 lattice



Finite volume effects of Chiral condensate

Finite volume effects of $\langle\sigma\rangle$ vs β for m=0.03 of three different lattice sizes $20 \times 20,\ 32 \times 32$, and 40×40



BC fermion in a 2D U(1) gauge theory

The lattice action with BC fermion reads,

$$S = \beta \sum_p [1 - \frac{1}{2}(U_p + U_p^\dagger)] + \phi^\dagger (D^\dagger D)^{-1} \phi.$$

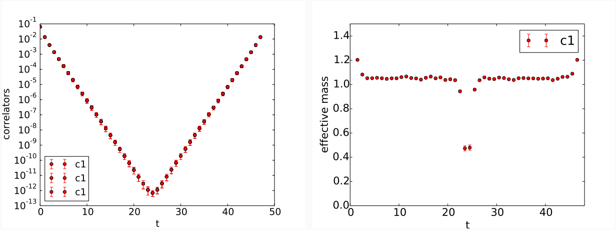
where U_p is the Wilson Plaquette action with

$$U_p = U_{i,\mu} U_{i+\mu,\nu} U^\dagger_{i+\nu,\nu} U^\dagger_{i,\nu}.$$

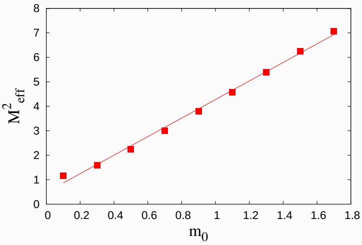
where, i is the site index and μ, ν are the directions and D is the BC Dirac operator defined earlier. After including gauge fields we get,

$$\begin{aligned} D_{mn} &= \frac{1}{2}(\gamma_\mu + i(\Gamma - \gamma_\mu))\, U_\mu(n-\mu)\delta_{n,m+\mu} - \\ &\quad \frac{1}{2}(\gamma_\mu - i(\Gamma - \gamma_\mu))\, U^\dagger_\mu(n)\delta_{n,m-\mu} - ((2-c_3)i\Gamma - m_0)\delta_{m,n}. \end{aligned}$$

The correlator with operator $O_1(t)$ couples to the ground state and provides the mass for the lowest state.



Effective mass of meson in 2D QED for $m_0 = 0.05$ and $\beta = 0.3$.



Fermion mass dependence of m^2_{eff} in QED$_2$ for a fixed $\beta = 0.3$.

- In first two figures we have shown the correlator at different time slices and the effective meson mass in 2D QED.
- The results are presented for the fermion (electron) mass $m_0 = 0.05$ and $\beta = 0.7$.
- The ground state mass $m_{eff} \approx 1.0$ and is much larger than $(2m_0)$.
- The square of meson mass (m_{eff}^2) shows a linear dependence on the fermion mass as illustrated in the last figure.
- For light fermions, the meson mass is much larger than twice the bare fermion mass $2m_0$. As the mass increases, the available phase space decreases and the contribution to the meson mass from interaction diminishes so the difference $(m_{eff} - 2m_0)$ becomes smaller.

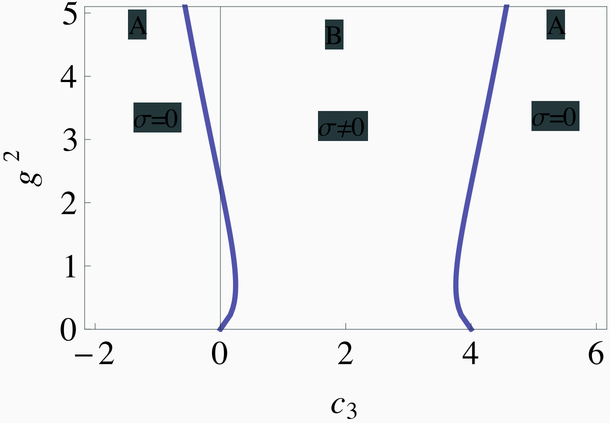
Summary

- We have studied the Gross-Neveu model with minimally doubled fermion action which has been proposed by Creutz and Borici.
- we have studied the model with HMC algorithm.
- We have calculate the mass spectrum of GN model to find ground and excited states of meson spectrum using that formulation.
- The order parameter $\langle\sigma\rangle$ is plotted against $\beta = 1/g^2$ shows chiral symmetry breaking.
- Issues(4 D),
  Counter terms ?? Renormaization ?? operator mixing issues ??

Thank you!

Back up slides

Phase Diagram in parameter space



Chiral boundaries in the parametric space i.e. c_3 vs g^2 for BC fermions

As can be seen from figure, for heavy fermions, the meson mass becomes less than $2m_0$.This can be explained from the fact that for heavy fermions, the quantum corrections to the effective mass become small as explained above, but the binding energy due to strong coupling is still large, so in combination of these two, the effective mass become less than the sum of individual particles as one observes for atomic or nuclear mass where the mass of the atom/nucleus is less than the sum of the individual constituent masses.