Computing the nucleon Dirac radius directly at $Q^2 = 0$

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Introduction

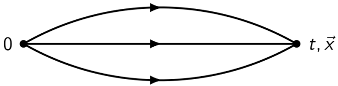
- Proton radius puzzle: 7σ discrepancy between Muonic hydrogen Lamb shift ($r_E^p = 0.84087(39)$ fm) and atomic hydrogen and scattering experiments using electrons ($r_E^p = 0.8775(51)$ fm)
- Model dependent fit
- $Q^2_{min} < 0.005$ GeV^2 from scattering experiment (e.g. from Mainz, Phys.Rev.Lett. 105 (2010) 242001) whereas on the lattice $Q^2_{min} = 0.05$ GeV^2 for $V = (5.8 \text{ fm})^3$
- Controversy about finding the radius from fitting scattering data.
  "Consistency of electron scattering data with a small proton radius", **Phys.Lett. B737 (2014) 57-59, Phys.Rev. C93 (2016) no.6, 065207, Phys.Rev. C93 (2016) no.5, 055207**
  "Solution of the proton radius puzzle? Low momentum transfer electron scattering data are not enough", **arXiv:1511.00479, Phys.Rev. D92 (2015) no.1, 013013, arXiv:1606.02159**
- This motivates the need for a direct calculation of the radius without fitting to form factors.

Momentum derivative (without smearing)

Nucleon two-point function:

$$C_2(\vec{p},t)_{\alpha\beta} = \sum_{\vec{x}} e^{-i\vec{p}\vec{x}} \langle N_\alpha(\vec{x},t)\bar{N}_\beta(0)\rangle =$$

$$\sum_{\vec{x}} \epsilon^{abc}\epsilon^{def} f_{\alpha\gamma\delta\epsilon} f_{\beta\zeta\eta\theta} \Big\langle \underbrace{e^{-i\vec{p}\vec{x}} G^{cd}_{\epsilon\zeta}(x,0)}_{G^{cd}_{\epsilon\zeta}(x,0;\vec{p})} \Big(G^{af}_{\gamma\theta}(x,0) G^{be}_{\delta\eta}(x,0) - G^{ae}_{\gamma\eta}(x,0) G^{bf}_{\delta\theta}(x,0) \Big) \Big\rangle$$



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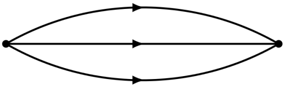
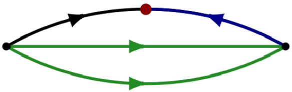


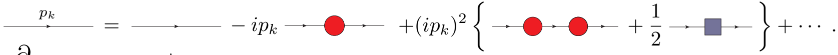
Nucleon three-point function ($\vec{p}' = 0$):

$$C_3(\vec{p},\tau,T)_{\alpha\beta} = \sum_{\vec{x},\vec{y}} e^{-i\vec{p}\vec{y}} \big\langle N_\alpha(\vec{x},T)\ O_\Gamma(\vec{y},\tau)\ \bar{N}_\beta(0) \big\rangle \sim \sum_{\vec{y}} \big\langle G_S(y)\ \Gamma\ G(y,0;\vec{p}) \big\rangle$$

$G_S(y)$ is the sequential back-ward propagator



Rome method: (Phys. Lett. B 718, 589 (2012) [arXiv:1208.5914])



$$\frac{\partial}{\partial p_k} G(x,y;\vec{p})\Big|_{\vec{p}=0} = -i \sum_z G(x,z)\, \Gamma_V^k \, G(z,y)$$

$$\frac{\partial^2 G(x,y;\vec{p})}{(\partial p_k)^2}\Big|_{\vec{p}=0} = -2 \sum_{z,z'} G(x,z)\, \Gamma_V^k \, G(z,z')\, \Gamma_V^k \, G(z',y) - \sum_z G(x,z)\, \Gamma_T^k \, G(z,y)$$

For clover fermions:

$$\Gamma^\mu_{V/T} G(z,y) \equiv U_\mu^\dagger(z-\hat{\mu})\, \tfrac{1+\gamma^\mu}{2}\, G(z-\hat{\mu},y) \mp U_\mu(z)\, \tfrac{1-\gamma^\mu}{2}\, G(z+\hat{\mu},y)$$

Momentum derivative (with smearing)

Smeared-source smeared sink propagator:

$$\tilde{G}(x,y;\vec{p}) = e^{-i\vec{p}(\vec{x}-\vec{y})} \sum_{x',y'} K(x,x')\,G(x',y')\,K(y',y)$$

$$= \sum_{x',y'} \underbrace{e^{-i\vec{p}(\vec{x}-\vec{x}')}\,K(x,x')}_{K(x,x';\vec{p})}\;\underbrace{e^{-i\vec{p}(\vec{x}'-\vec{y}')}\,G(x',y')}_{G(x',y';\vec{p})}\;\underbrace{e^{-i\vec{p}(\vec{y}'-\vec{y})}\,K(y',y)}_{K(y',y;\vec{p})}$$

$$(K\,G\,K)' = K'\,G\,K + K\,(GK)'$$

$$(K\,G\,K)'' = K''\,G\,K + 2\,K'\,(GK)' + K\,(GK)''$$

Smeared source point-sink propagator:

$$(G\,K)' = G\,[-i\Gamma_V\,G\,K + K']$$

$$(G\,K)'' = G\,[-2i\Gamma_V\,(GK)' - \Gamma_T\,G\,K + K'']$$

Wuppertal smearing:

$$K(x,y;\vec{p}) = \sum_{x',x'',\ldots} \underbrace{K_0(x,x';\vec{p})\; K_0(x',x'';\vec{p})\;\ldots\; K_0(x'^{\cdots\prime},y;\vec{p})}_{N_W}$$

$$K_0(x,y;\vec{p}) = e^{-i\vec{p}(\vec{x}-\vec{y})}\frac{1}{1+6\alpha}\left(\delta_{x,y} + \alpha\sum_{j=1}^{3}\left[U_j(x)\delta_{x+\hat{j},y} + U_j^\dagger(x-\hat{j})\delta_{x-\hat{j},y}\right]\right)$$

$$\begin{aligned} K_0^{(m)}(x,y) &\equiv \left(\frac{\partial}{\partial p^j}\right)^m K_0(x,y;\vec{p})\Big|_{\vec{p}=0} \\ &= \frac{\alpha}{1+6\alpha}\left[i^m U_j(x)\delta_{x+\hat{j},y} + (-i)^m U_j^\dagger(x-\hat{j})\delta_{x-\hat{j},y}\right] \end{aligned}$$

Iteratively for $K = K_0^{N_W}$:

$$(K_0^N)' = K_0'\; K_0^{N-1} + K_0\; (K_0^{N-1})'$$

$$(K_0^N)'' = K_0''\; K_0^{N-1} + 2K_0'\; (K_0^{N-1})' + K_0\; (K_0^{N-1})''$$

Ground-state contributions ($\vec{p}' = 0$):

$$C_2(\vec{p}, t) = e^{-E(\vec{p})t} \langle \Omega | N_\alpha | p \rangle \langle p | \bar{N}_\beta | \Omega \rangle$$

$$C_3(\vec{p}, \tau, T) = e^{-m(T-\tau)} e^{-E(\vec{p})\tau} \langle \Omega | N | 0 \rangle \langle 0 | \bar{q} \gamma^\mu q | p \rangle \langle p | \bar{N} | \Omega \rangle$$

$$\langle p' | V_q^\mu | p \rangle = \bar{u}(p') \underbrace{\mathscr{F}(\Gamma, \vec{p}', \vec{p})}_{F_1^q \gamma^\mu + F_2^q \frac{i\sigma^{\mu\nu} q_\nu}{2m}} u(p), \quad \text{and} \qquad \langle \Omega | N | p \rangle = Z(\vec{p}) [\Gamma u(p)]$$

$$R(\vec{p}) = \frac{C_3(\vec{p}, \tau)_{\alpha\beta}}{\sqrt{C_2(0, T)\, C_2(\vec{p}, T)}} \sqrt{\frac{C_2(\vec{p}, T-\tau)\, C_2(0, \tau)}{C_2(0, T-\tau)\, C_2(\vec{p}, \tau)}}$$

$$= \frac{\left[(1+\gamma^0)\, \mathscr{F}(\Gamma, 0, \vec{p})\, (m + \not{p})\, (1+\gamma^0)\right]_{\alpha\beta}}{8\sqrt{2E(\vec{p})(E(\vec{p}) + m)}} \text{ (positive parity)}$$

Compute $\left.\frac{\partial R}{\partial p_i}\right|_{\vec{p}=0}$ and $\left.\frac{\partial^2 R}{\partial {p_i}^2}\right|_{\vec{p}=0}$ for $\mu = 0, 1, 2$ vector components in x, y and z directions to find for $\Gamma_{\text{pol}} = (1 + \gamma^3\gamma_5)\frac{1+\gamma^0}{2}$:

Anomalous magnetic moment and Dirac radius

$$\kappa = -2\,m\ \ \text{Im}(\text{Tr}[R'(\mu = 2)\ \Gamma_{pol}]) - \text{Tr}[R(\mu = 0)\ \Gamma_{pol}]$$

$$r_1^2 = \frac{12\,m\,\text{Im}[R'(\mu = 2)\ \Gamma_{pol}] + 3\,\text{Tr}[R(\mu = 0)\ \Gamma_{pol}] - 12\,m^2\,\text{Tr}[R''(\mu = 0)\ \Gamma_{pol}]}{4\,m^2\,\text{Tr}[R(\mu = 0)\ \Gamma_{pol}]}$$

Where $r_1^2 = \frac{-6}{F_1}\left.\frac{dF_1}{dQ^2}\right|_{Q^2=0}$

Average over equivalent vector components and directions:

$$\text{Tr}[R'(\mu = 2)\Gamma_{pol}] = \frac{1}{2}(\text{Tr}[\partial_1 R(\mu = 2)\ \Gamma_{pol}] - \text{Tr}[\partial_2 R(\mu = 1)\ \Gamma_{pol}])$$

$$\text{Tr}[R''(\mu = 0)\Gamma_{pol}] = \frac{1}{3}(\text{Tr}[\partial_1^2 R(\mu = 0)\ \Gamma_{pol}] + \text{Tr}[\partial_2^2 R(\mu = 0)\ \Gamma_{pol}] + \text{Tr}[\partial_3^2 R(\mu = 0)\ \Gamma_{pol}])$$

Removal of excited states

- Ratio-plateau method: compute ratio

$$R(T,\tau) = c_{00} + c_{10}e^{\Delta E_{10}(\vec{p})\tau} + c_{01}e^{\Delta E_{10}(\vec{p}')(T-\tau)}$$

  where c_{00} is the desired ground-state matrix element. Then average a fixed number of points around $\tau = T/2$.

  Asymptotic errors:

$$R,\ \frac{\partial}{\partial p_i}R\Big|_{\vec{p}=0} \sim e^{-\Delta E_{10}\ T/2}, \qquad \frac{\partial^2}{\partial {p_i}^2}R\Big|_{\vec{p}=0} \sim T\ e^{-\Delta E_{10}T/2}$$

- Summation method: compute sums

$$S(T) = \sum_\tau R(T,\tau) = b + c_{00}T + dTe^{-\Delta ET} + \ldots$$

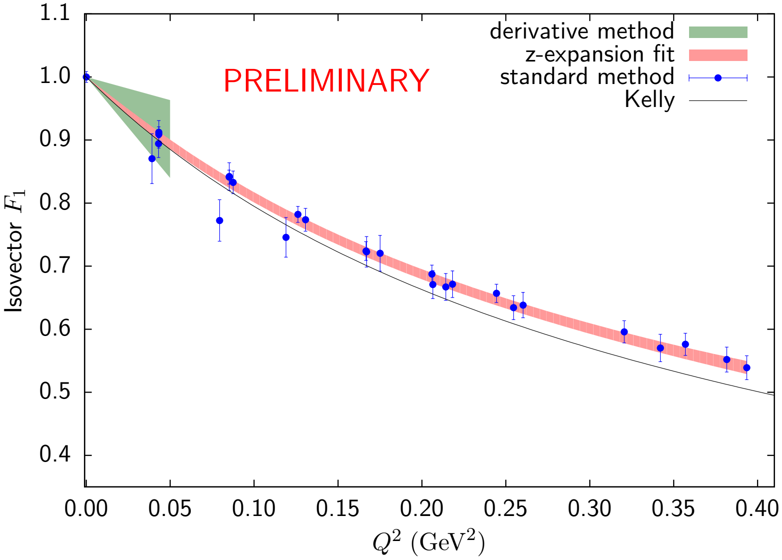
  then find their slope, which gives c_{00}.

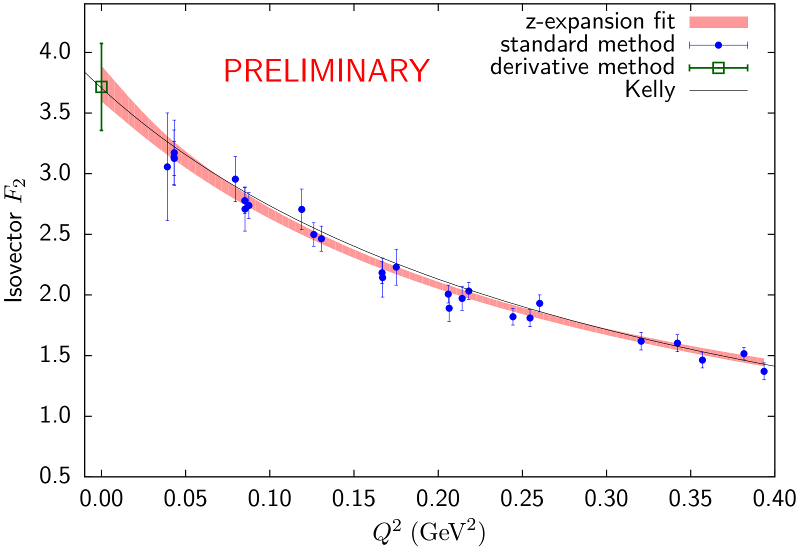
  Asymptotic errors:

$$S,\ \frac{\partial}{\partial p_i}S\Big|_{\vec{p}=0} \sim T\ e^{-\Delta E_{10}T}, \qquad \frac{\partial^2}{\partial {p_i}^2}S\Big|_{\vec{p}=0} \sim T^2\ e^{-\Delta E_{10}T}$$

Ensemble

- BMW $N_f = 2 + 1$ 2HEX-clover, $L_s = L_t = 64$.
- $\beta = 3.5$, $a^{-1} = 2.131(13)$ GeV, $a = 0.093$ fm .
- Physical pion mass, $m_\pi L = 4$.
- Three source-sink separations $T/a \in \{10, 13, 16\}$, which corresponds to T between 0.9 and 1.5 fm.
- 442 gauge configurations analyzed
- AMA, using 64 sources per configuration with approximate propagators and one source for the bias correction.


derivative method
z-expansion fit
standard method
Kelly
PRELIMINARY
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
Isovector F_1
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
Q^2 (GeV^2)


z-expansion fit
standard method
derivative method
Kelly
PRELIMINARY
Isovector F_2
Q^2 (GeV^2)
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40

Numerical results:

Quantity	Traditional method	Derivative method
κ^v	3.74(14)	3.71(35)
$[r_1^v]^2$ [fm]^2	0.547(91)	0.45(28)

Summary

- Our approach is based on the Rome method and is model independent, .
- Our results on κ^v and $[r_1^v]^2$ are consistent with fitting to form factors.
- The statistical errors are higher on quantities containing second derivative of the ratio.