$\mathbb{C}P(2)$ Model at Non-Zero Chemical Potential

via dimensional reduction of a quantum spin system

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Motivation

i) The (1+1)-d $\mathbb{C}P(2)$ model is a popular toy model for QCD, since it exhibits:

- Asymptotic freedom
- Dynamically generated mass gap
- Non-trivial topology

ii) Obtaining the $\mathbb{C}P(2)$ model from a (2+1)-d system of quantum spins provides experimentalists with a clear analogy to cold atoms in an optical lattice: C. Laflamme et al., Annals of Physics (2016), pp. 117-127

- The ultimate goal is to quantum simulate QCD at finite chemical potential

Formulation Overview

i) Microscopic (2+1)-d antiferromagnetic $SU(3)$ spin system:

$$H = -J \sum_{\substack{\langle ij \rangle \\ i \in A}} T_i^a T_j^{a*} - \mu^a \left(\sum_{i \in A} T_i^a - \sum_{j \in B} T_j^{a*} \right), \quad J > 0.$$

ii) Spontaneous symmetry breaking: $SU(3)/U(2) = \mathbb{C}\mathrm{P}(2)$.
Emergent Nambu-Goldstone bosons are governed by:

$$S[P] = \int_0^\beta dt \int_0^L dx \int_0^{L'} dy \, \mathrm{Tr} \left[\rho_s \partial_i P \partial_i P + \frac{\rho_s}{c^2} D_t P D_t P \right] - i\theta Q[P].$$

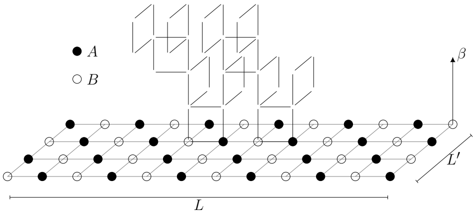
iii) Dimensional reduction: $\xi \propto \exp[4\pi L' \rho_s / 3c] \gg L'$

$$S[P] = \int_0^\beta dt \int_0^L dx \, \mathrm{Tr} \left[\rho_s L' \partial_x P \partial_x P + \frac{\rho_s L'}{c^2} D_t P D_t P \right] - i\theta Q[P].$$

Microscopic (2+1)-d Quantum Spin System

$$H = -J \sum_{\substack{\langle ij \rangle \\ i \in A}} T_i^a T_j^{a*} - \mu^a \left(\sum_{i \in A} T_i^a - \sum_{j \in B} T_j^{a*} \right)$$

- $\left[T_x^a, T_y^a \right] = i\delta_{xy} f_{abc} T_x^c, \quad \bar{T}_x^a = -T_x^{a*}.$
- Global $SU(3)$ symmetry at $\mu = 0$.
- Total spin conservation, $\left[H, \left(\sum_{x \in A} T_x^a - \sum_{y \in B} T_y^{a*} \right) \right] = 0.$
- $J > 0$ gives anti-ferromagnet.
- Spins live in the fundamental and antifundamental representations of SU(3) on sublattice A and B respectively.



Spontaneous Symmetry Breaking

- In the thermodynamic limit: β, L, $L' \to \infty$,

$$\text{Global} \;\; SU(3) \to U(2).$$

K. Harada, N. Kawashima, M. Troyer, PRL 90, 117203.

- Consequently 4 Nambu-Goldstone bosons emerge with degrees of freedom in the coset space:

$$\frac{SU(3)}{U(2)} = \mathbb{C}P(2).$$

- These massless fields can be described by complex 3×3 Hermitean projection matrices P using:

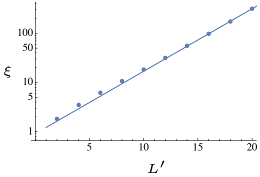
$$P(x) = P(x)^2, \quad \mathrm{Tr}\, P(x) = 1, \quad P(x) = P(x)^\dagger.$$

- Effective action for $\mathbb{C}$P(2) model,

$$S[P] = \int_0^\beta dt \int_0^L dx \int_0^{L'} dy \, \mathrm{Tr} \left[\rho_s \partial_i P \partial_i P + \frac{\rho_s}{c^2} D_t P D_t P \right] - i\theta Q[P],$$

$$i\theta Q[P] = \frac{1}{\pi} \int_0^\beta dt \int_0^L dx \int_0^{L'} dy \, \mathrm{Tr} \left[P \partial_x P \partial_t P \right], \quad D_t P = \partial_t P + [\mu_a N_a, P] \, .$$

Dimensional Reduction

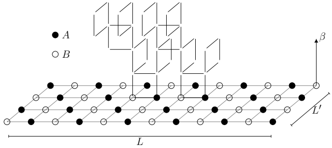


- (1+1)-d $\mathbb{C}$P(2) model is asymptotically free:

$$\xi \propto \exp\left[\frac{4\pi L' \rho_s}{3c}\right].$$

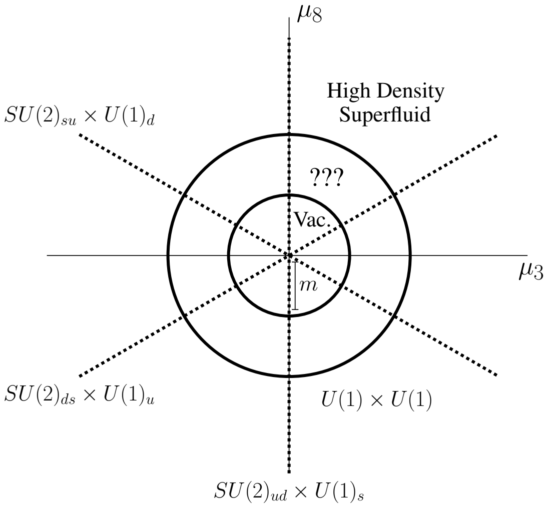
- For finite L', Mermin-Wagner theorem states Nambu-Goldstone bosons must pick up a mass $m = 1/\xi$.
- If $\xi \gg L'$ then system undergoes dimensional reduction in the L' extent.
- Effective action after dimensional reduction,

$$S[P] = \frac{c}{g^2}\int_0^\beta dt \int_0^L dx\, \mathrm{Tr}\left[\partial_x P \partial_x P + \frac{1}{c^2} D_t P D_t P\right] - i\theta Q[P],$$



$$g^2 = \frac{c}{\rho_s L'}.$$

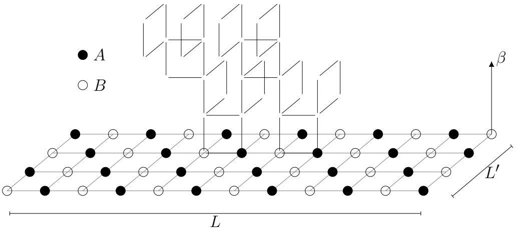
(1+1)-d $\mathbb{C}P(2)$ Phase Diagram



$\lambda_8 \pm \frac{3}{\sqrt{3}}\lambda_3$ has $SU(2) \times U(1)$ symmetry.

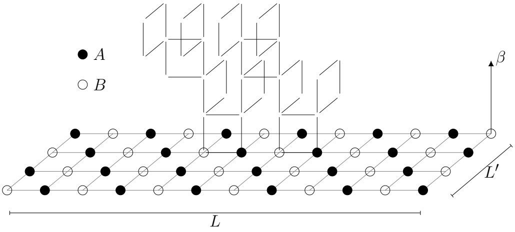
Simulation

- The (2+1)-d spin system is quantized in the T_3-T_8 basis.
- We have u, d, or s on sublattice A and $\bar{u}$, $\bar{d}$, or $\bar{s}$ on sublattice B.
- The time extent is discretized with the Trotter decomposition to give four time-slices for each Euclidean time step ϵ.
- The system is updated using a worm algorithm. Typically 10^5 thermalization updates and 10^6 measurement updates.



Worm Algorithm

- Randomly pick lattice site, we note the spin color at this site.
- Randomly choose one of the two spin colors not at starting site.
- Flip start site spin to the color chosen, creates the worm head or defect.
- Worm head propagates according to the rules derived from the Hamiltonian, flipping spins as it goes.
- Worm closes once the worm head returns to the starting site where the worm tail remained.
- Defect is eliminated and system has been updated.



Observables

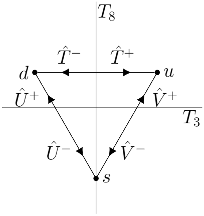
- Particle Number Density

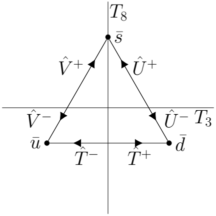
$$\langle n \rangle = \frac{\langle M_3 \rangle}{L},$$

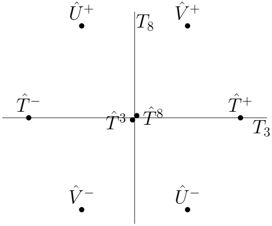
$$M_3 = \sum_{x \in A} T_x^a - \sum_{y \in B} T_y^{a*},$$

M_3 is calculated in a particular time-slice.

- Shift Operator Correlators



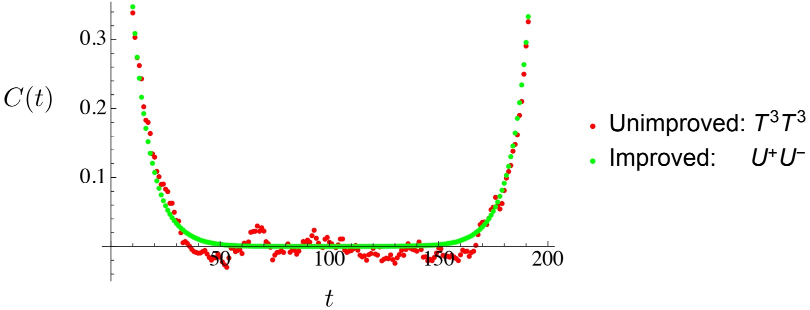




$\langle U_0^+ U_x^- + U_0^- U_x^+ \rangle$ $\langle T_0^+ T_x^- + T_0^- T_x^+ \rangle$ $\langle V_0^+ V_x^- + V_0^- V_x^+ \rangle$

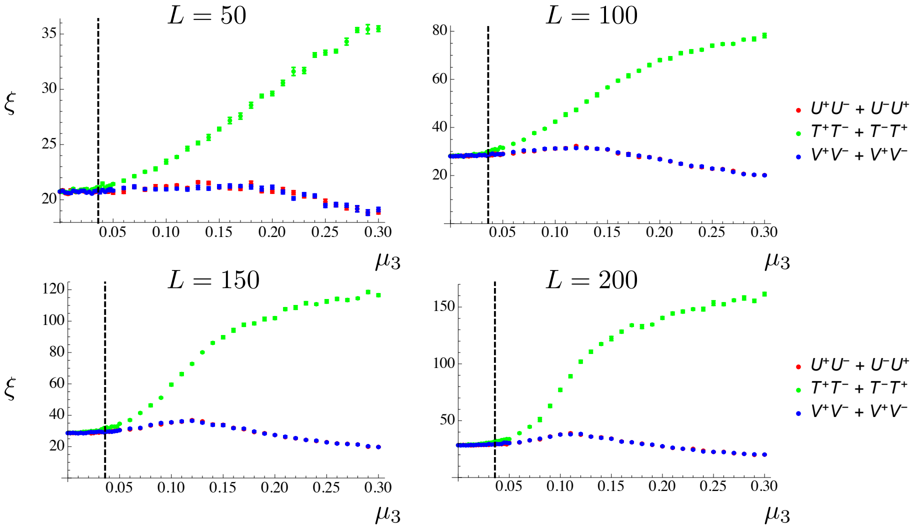
Improved vs. Unimproved Correlator Estimator

$$\beta = 10, \quad L = 10, \quad L' = 2.$$



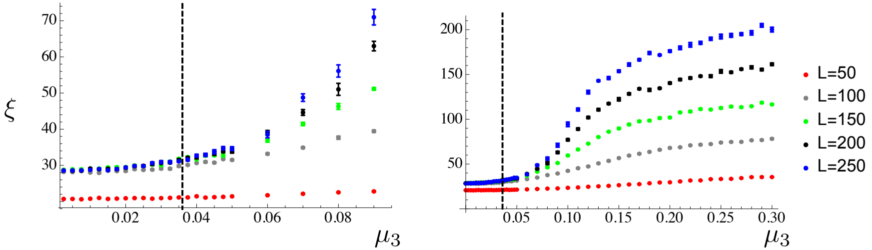
Correlation Length vs. Chemical Potential

$\beta c \approx L, \quad L' = 10, \quad \xi = 28.5(2)$ at $\mu = 0$.

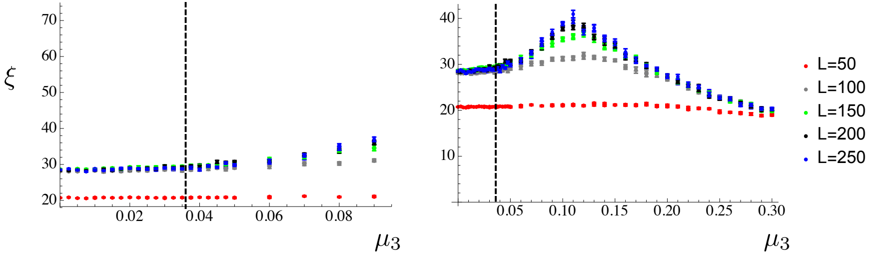


Correlation Length vs. Chemical Potential

$\langle T_0^+ T_x^- + T_0^- T_x^+ \rangle, \quad \beta c \approx L, \quad L' = 10.$

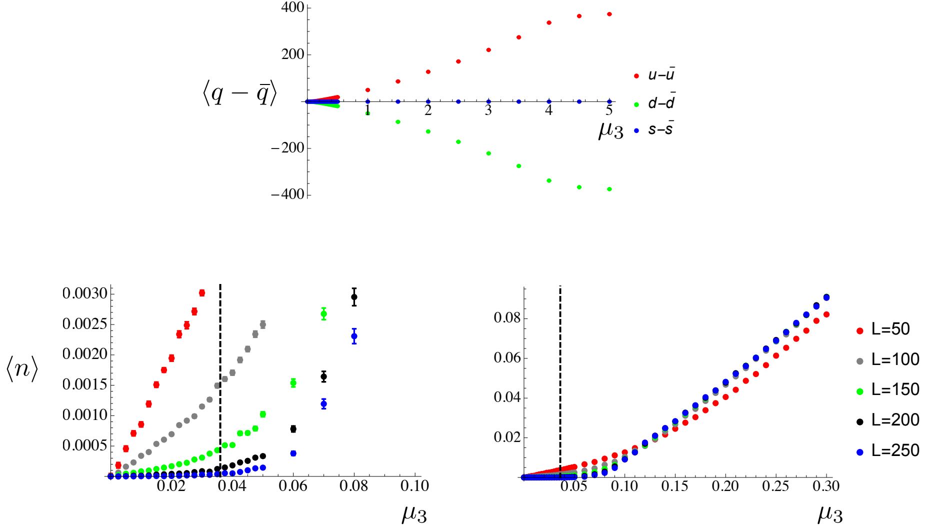


$\langle U_0^+ U_x^- + U_0^- U_x^+ \rangle$, and $\langle V_0^+ V_x^- + V_0^- V_x^+ \rangle$.



Magnetization vs. Chemical Potential

$\beta c \approx L, \quad L' = 10.$



Conclusions and Outlook

- Behaviour of the magnetization at $\mu_3 \neq 0$ implies dimensional reduction occurs.

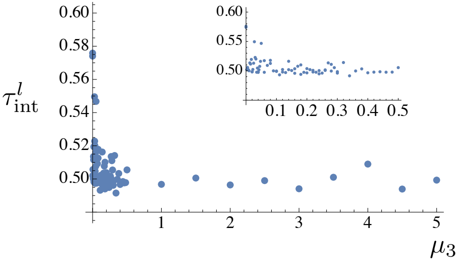
- At $\mu_8 = 0$, $\mu_3 \neq 0$ the behaviour of the correlation functions suggests $U(1) \times U(1) \to U(1)$.

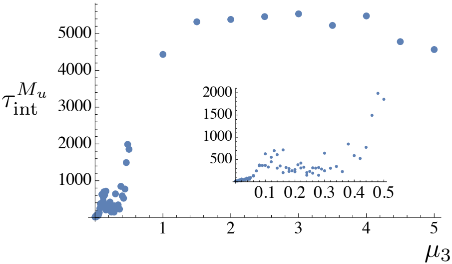
- Study $\mu_8 \neq 0$, and $\mu_3, \mu_8 \neq 0$.

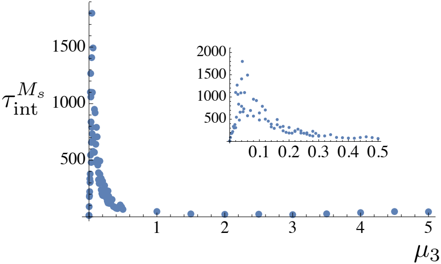
- Cold atom experiments.

Autocorrelation vs. Chemical Potential

$L = 150$
$\beta c \approx L$
$L' = 10$







Correlation Length vs. Chemical Potential

$$\langle T_0^+ T_x^- + T_0^- T_x^+ \rangle, \quad \beta c \approx L, \quad L' = 10.$$