Non-Local effective SU(2) Polyakov loop model from inverse Monte-Carlo methods

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Motivation

- QCD difficult to solve. Sign problem for finite $\mu$
- There are many ideas to deal with this, e.g. effective models
- Sign problem in Polyakov loop models are expected to be less severe
- Problem: How do we get the effective action from a known action?
Theory

Non-Local effective SU(2) Polyakov loop model from inverse Monte-Carlo methods
Inverse Monte-Carlo Method

Wilson action $S[U]$ \rightarrow \text{integrate out } U \rightarrow \text{effective action } S_{\text{eff}}[\chi]

\text{configurations } C(U) \rightarrow \text{calculate } \chi = \chi(U) \rightarrow \text{eff. configurations } C(\chi)

\text{MC} \rightarrow \text{IMC
Inverse Monte-Carlo Method II

- Take an effective action $S_{\text{eff}}(\lambda)$ with yet to find coupling constants $\lambda$

- Remember how DSEs are derived

$$\left\langle \frac{\delta S}{\delta \varphi}(\lambda) \right\rangle_{\text{eff}} = 0$$

- Demand that the effective theory approximates the full theory well, i.e.

$$\left\langle \frac{\delta S}{\delta \varphi}(\lambda) \right\rangle_{\text{full}} = 0$$

- Solve this equation numerically for $\lambda$
Geometric Ward-Identities and Geometric DSEs

Geometric Ward-Identities I

- Left invariance of the Haar measure yields the (mathematical) identity

\[ \int d\mu(g)(L_a f)(g) = 0, f \in L_2(G). \]

- For class functions \( F, \tilde{F} \) we obtain

\[ \int d\mu_{\text{red}} \mathbf{ar{L}} \cdot (F\mathbf{ar{L}}\tilde{F}) = \int d\mu_{\text{red}} (F\mathbf{ar{L}}^2\tilde{F} + \mathbf{ar{L}}F \cdot \mathbf{ar{L}}\tilde{F}) = 0 \]

class function
Geometric Ward-Identities II

- Use character expansion for class functions

\[ F(g) = F(\chi_1(g), \ldots, \chi_r(g)), \quad r = \text{rank}(G) \]

\[ L_a F(\chi) = \sum_q \frac{\partial F(\chi)}{\partial \chi_q(g)} L_a \chi_q(g) \]

- Set \( \tilde{F} = \chi_p \), with \( p \in \{1, \ldots, r\} \)
- Use

\[ \chi_\mu \chi_\nu = \sum_\lambda C^\lambda_{\mu\nu} \chi_\lambda, \quad \sum_a L_a^2 \chi_p(g) = -c_p \chi_p(g) \]

**Geometric Ward-Identity**

\[
0 = \int_G d\mu_{\text{red}} \left\{ \frac{1}{2} \sum_q \left[ (c_p + c_q) \chi_p \chi_q - \sum_\lambda C^\lambda_{\mu\nu} c_\lambda \chi_\lambda \right] \frac{\partial F(\chi)}{\partial \chi_q(g)} - c_p \chi_p(g) F \right\}
\]

\[ =: K_q \]
Geometric Ward-Identities and Geometric DSEs

Geometric DSEs

- Insert $\exp(-S_{\text{eff}})$ and take sum over all lattice points

$$V^{-1} \sum_{i \in L} \left\langle \frac{1}{2} \sum_{q} K_{q,i} \frac{\partial F_i}{\partial \chi_{q,i}} \exp(+S_{\text{eff}}) - c_p \chi_{p,i} F_i \exp(+S_{\text{eff}}) \right\rangle_{\text{eff}} = 0$$

- Take this DSE for IMC-method to calculate $\tilde{\lambda}$ via

$$V^{-1} \sum_{i \in L} \left\langle \frac{1}{2} \sum_{q} K_{q,i} \frac{\partial \tilde{F}_i}{\partial \chi_{q,i}} \exp(+S_{\text{eff}}(\tilde{\lambda})) - c_p \chi_{p,i} \tilde{F}_i \exp(+S_{\text{eff}}(\tilde{\lambda})) \right\rangle_{\text{full}} = 0$$

(Need dim($\tilde{F}$) = dim($\tilde{\lambda}$) different class-functions $F_i$ to solve the equations for $\tilde{\lambda}$)
Integrating out all spatial links and applying the strong coupling expansion yields

The linear Polyakov model

\[ S = \sum_p \sum_r \sum_{\langle i,j \rangle = r} \lambda_{p,r} \chi_{p,i} \chi_{p,j}, \]

Expanding the action term and applying a resummation of higher order terms yields

The logarithmic Polyakov model

\[ S = -\sum_p \sum_r \sum_{\langle i,j \rangle = r} \log \left( 1 + g_{p,r} \chi_{p,i} \chi_{p,j} \right) \]

[J. Langelage, S. Lottini, O. Philipsen 2010]
Neglecting terms with $r \geq r_{\text{max}}$, and representations with $p \geq p_{\text{max}}$ yields

$$e^{-S} = \prod_{p=1}^{p_{\text{max}}} \prod_{r=1}^{r_{\text{max}}} \prod_{<i,j>=r} \exp\left(\lambda_p r \chi_p i \chi_p j\right),$$

Insert into geometric DSE and set $\vec{F}_i = \vec{f}_i \exp(-S_{\text{eff}})$, with $\vec{f}_i = \{f_{p,r,i}\}$

$$f_{p,r,i} = \frac{1}{g_{p,r}} \frac{\partial(e^{-S})_{p,r,i}}{\partial \chi_{1,i}}$$

Now match the effective model to the full theory
NUMERIC RESULTS
Local Polyakov Models: Linear vs. Logarithmic

Lattice: $20^3 \times 4$, Configs: 10,000

Lin. model improves if we add representations

Non-Local effective SU(2) Polyakov loop model from inverse Monte-Carlo method
Log. resummation seems to work quite well. No higher representations needed.

For small $\beta$ the log. resummation is expected to improve results.

Still far from the full theory for large $\beta$. → Try non-local models.
Adding larger distances for the interaction improves the result.

We “overshoot” when we include too large distances.
Larger Lattice seems to fix overshooting.
But: Higher representations change the result. We overshoot again.
Logarithmic resummation not doing well for large $\beta$ (many non-local terms)
Linear model does not overshoot, even on the smaller lattice.
Motivation

Non-local Linear Polyakov Model

Non-local Linear Polyakov Models

We get close to the full theory with 2 representations.
Adding more rep. seems not to spoil the result. Still close to full theory.

Approaches the full theory very slowly near $\beta_c$ (large correlation length)
Non-local Linear Polyakov Models

- Same result for larger lattice. No overshooting. Close to full theory with 2 rep.
Higher representations seem not to spoil the result.

Adding non-local terms for large $\beta$ works much better than for log. model

Approaches the full theory very slowly around $\beta_c$
Look at long-distance behaviours of couplings to make model predictable and compare to analytical models.
Compare to Greensite’s and Langfeld’s analytical model

\[ S = c_0 \sum_x P_x - \frac{1}{2} c_1 \sum_x P_x^2 - 2c_2 \sum_{x,y} P_x Q(x - y)P_y, \]

\[ Q(x - y) = \begin{cases} (\sqrt{-\nabla^2})_{xy} & |x - y| \leq r_{\text{max}} \\ 0 & |x - y| > r_{\text{max}} \end{cases} \]

\[ \beta = 2.22 \]

- Shape agrees quite well.
- Fitted coupling \( c_2 \approx 0.227(53) \) does not agree with prediction of \( 0.491(1) \)
- Maybe if we insert additional terms (linear P-term)?
Fitting linear part and extracting “correlation length” yields peak around $\beta_c \approx 2.29$

- Dependence seems not to scale with the volume
- Might suggest model becomes local again in the continuum
Conclusion

- IMC-method works well to fix theories
- Logarithmic resummation does not work well for $\beta \gtrapprox \beta_c$
- Non-local linear Polyakov model seems to work well for $\beta > \beta_c$.
- Difficult around $\beta_c$. Need more non-local terms.
- Model might become local in the continuum limit

Outlook

- Improvements around $\beta_c$
- Maybe add additional terms in our ansatz (linear polyakov term)
- Check larger lattices and continuum limit
- Add fermions
- Other gauge groups ($SU(3), G_2$)
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