Determining α_s by using the gradient flow in the quenched theory

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Introduction - Motivation

Current state of α_s determinations

- Many attempts to estimate α_s and Λ parameter in literature
- Summary and combined value by Flag Working Group [arXiv:1607.00299]
- Criteria:
  - Renormalization scale: all points must have $\alpha_{\mathrm{eff}} < 0.2$
  - Perturbative behaviour: should be verified over a range of a factor 4 change in $\alpha_{\mathrm{eff}}^{n_1}$ (or $\alpha_{\mathrm{eff}} = 0.01$ is reached)
  - Continuum extrapolation: at $\alpha_{\mathrm{eff}} = 0.3$ have three lattice spacing with $\mu a < 0.5$ for full $\mathcal{O}(a)$ improvement.
  - Finite-size effects: scale is determined in large enough volumes
  - Topology sampling

Current State of α_s determination - quenched case

Collaboration	Ren. scale	Pert. Behav.	Cont. Extrap.	$r_0\Lambda_{\overline{\rm MS}}$	Method
CP-PACS 04	★	★	○		Schrödinger Functional
ALPHA 98	★	★	○	0.602(48)	Schrödinger Functional
Lüscher 93	★	○	○	0.590(60)	Schrödinger Functional
Brambilla 10	○	★	○	$0.637(^{+32}_{-30})$	Heavy quark Potential
UKQCD 92	★	○	■	0.686(54)	Heavy quark Potential
Bali 92	★	○	■	0.661(27)	Heavy quark Potential
FlowQCD 15	★	★	★	0.618(11)	Lattice spacing scale
QCDSF/UKQCD 05	★	○	★	0.614(2)(5)	Lattice spacing scale
SESAM 99	★	■	■		Lattice spacing scale
Wingate 95	★	■	■		Lattice spacing scale
Davies 94	★	■	■		Lattice spacing scale
El-Khadra 92	★	■	○	0.560(24)	Lattice spacing scale
Sternbeck 10	★	★	■	0.62(1)	QCD vertices
Ilgenfritz 10	★	★	■		QCD vertices
Boucaud 08	○	★	■	$0.59(1)(^{+2}_{-1})$	QCD vertices
Boucaud 05	■	★	■	0.62(7)	QCD vertices

In this talk: α_s in the quenched case using the gradient flow

Gradient Flow - Setting the scale

- Gradient Flow has many applications (scale setting, operator relation, topology etc . . .) Lüscher (2010)
- Simplest gauge invariant quantity: action density

$$E(t,x) = \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu}$$

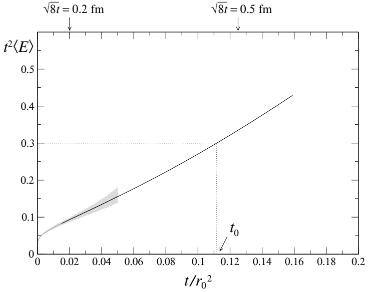
- Its expectation value $\langle E(t,x)\rangle$ serves as a non-perturbative definition of a reference scale
- t_0 first introduced as a reference scale Lüscher (2010)

$$t^2 \langle E(t)\rangle\big|_{t=t_0} = 0.3$$

- w_0 can also be used as a reference scale BMW Collaboration (2012)

$$t \frac{d}{dt} t^2 \langle E(t)\rangle\big|_{t=w_0^2} = 0.3$$

Gradient Flow - Perturbative relation



Courtesy Lüscher (2010)

Perturbative relation for its expectation value for QCD ($N_A = 8$) in $\overline{\mathrm{MS}}$ scheme up to NNLO

$$t^2\langle E(t)\rangle = \frac{3\alpha_s}{4\pi}\left(1 + \alpha_s k_1 + \alpha_s^2 k_2 + \mathcal{O}(\alpha_s^3)\right)$$

$k_1 = 1.09778674$ Lüscher (2010)

$k_2 = -0.9822456$ Harlander and Neumann (2016)

Brute-Force determination of α_s

Procedure

Simulation details

- Use fine-lattices at $T = 0$
- Keep the physical volume constant $LT_c \simeq 2$
- Periodic Boundary Conditions
- Tree-level Symanzik action, Wilson flow, Clover-leaf definition of observable
- $Q = 0$ configurations selected

Lattices used

β	N	a (in r_0)	# cfgs
5.3570	48	0.05651	529
5.3669	48	0.05583	4680
5.4500	56	0.05011	215
5.4700	56	0.04911	222
5.5000	56	0.04690	197
5.5830	64	0.04178	161
5.6000	64	0.04115	200
5.8000	80	0.03229	400
5.9500	96	0.02699	234
6.0500	112	0.02395	100
6.1500	128	0.02138	50
6.3600	160	0.01648	103

- $w_0^{Q=0}/w_0 = 0.992(4)$
- Use w_1 to set the scale: $t\frac{d}{dt}t^2\langle E(t)\rangle\big|_{t=w_1^2} = 0.03$
- $w_1/r_0 = 0.115(2)$

Improvement

- Discretization correction terms at tree-level Fodor et al. (2014)

$$t^2\langle E(t)\rangle = \frac{3\alpha_s}{4\pi}\left(C(a^2/t) + \mathcal{O}(\alpha_s)\right)$$

  where

$$C(a^2/t) = 1 + \sum_{m=1}^{\infty} C_{2m}\frac{a^{2m}}{t^m}$$

  Coefficients known up to $\mathcal{O}(a^8)$

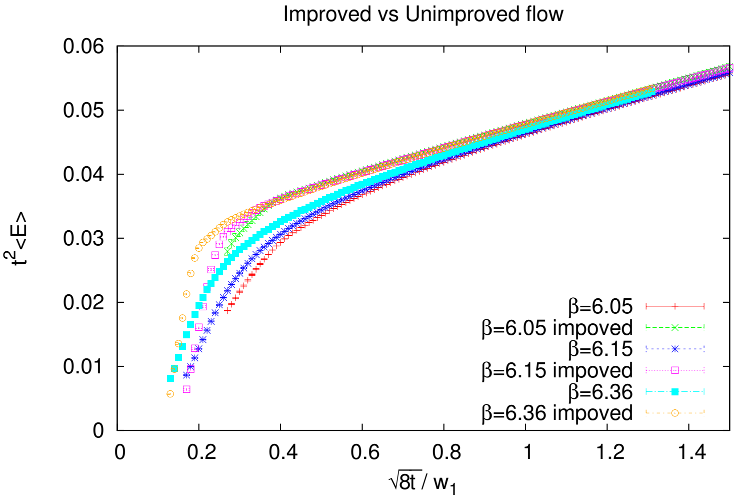
- Finite-Volume correction Fodor et al. (2012)

$$t^2\langle E\rangle = \frac{3\alpha_s}{4\pi}\left(1 + \delta(t/L^2)\right)$$

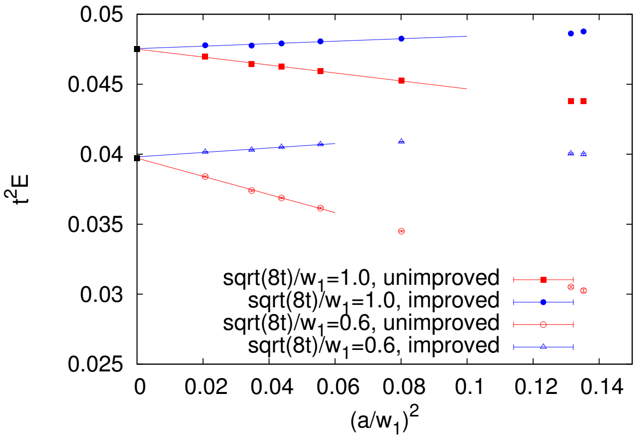
  where

$$\delta = 1 - \frac{64t^2\pi}{3L^2} + 8e^{-L^2/8t} + 24e^{-L^4/4t} + \ldots$$

Improvement



Improvement

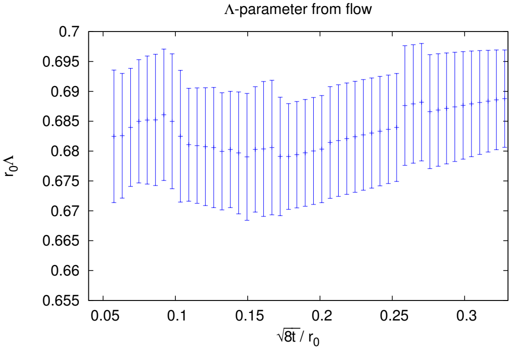


Λ parameter

1. $t^2\langle E(t)\rangle \Rightarrow \alpha_s$ from perturbative relation
2. Use 4-loop β-function in the $\overline{\mathrm{MS}}$-scheme to run α_s at a high scale

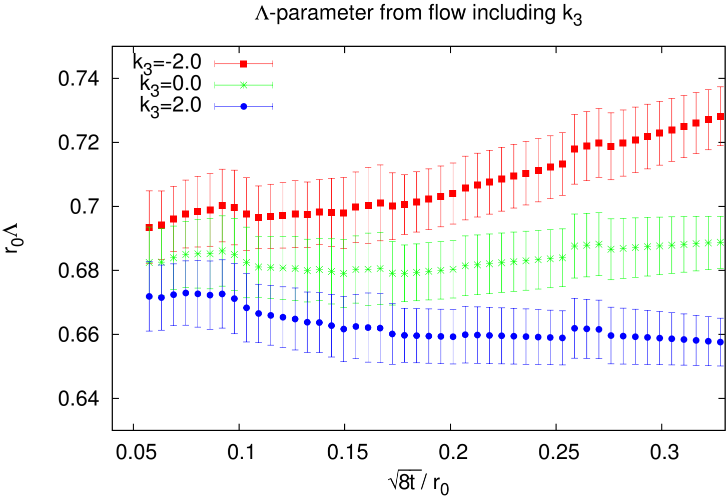
   Ritbergen, Vermaseren, Larin (1997)

$$t^2\langle E(t)\rangle = \frac{3\alpha_s}{4\pi}\left(1+\alpha_s k_1+\alpha_s^2 k_2+\mathcal{O}(\alpha_s^3)\right)$$



Λ parameter

$$t^2\langle E(t)\rangle = \frac{3\alpha_s}{4\pi}\left(1+\alpha_s k_1 + \alpha_s^2 k_2 + \alpha_s^3 k_3 + \mathcal{O}(\alpha_s^4)\right)$$



Λ parameter

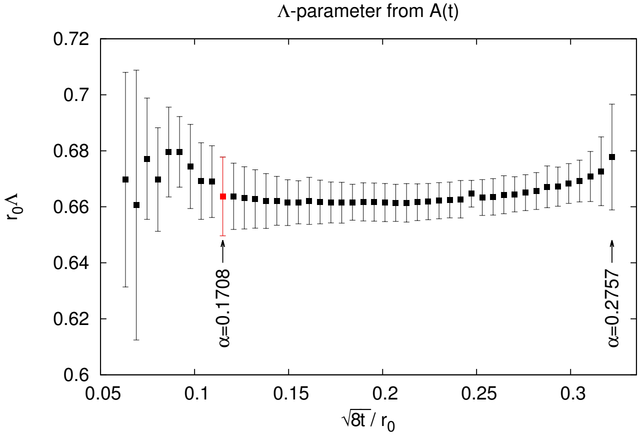
We want to eliminate k_3 contribution

$$
\begin{aligned}
A(t) &\equiv (t^2\langle E\rangle)^2 + C\Big(t\frac{dt^2\langle E\rangle}{dt}\Big) \\
&= \alpha_s^2\Big(\frac{9}{(4\pi)^2} + \frac{3\beta_0 C}{(4\pi)^2}\Big) + \alpha_s^3\left(\frac{18k_1}{(4\pi)^2} + C\Big(\frac{3\beta_1}{(4\pi)^3} + \frac{6k_1\beta_0}{(4\pi)^2}\Big)\right) \\
&+ \alpha_s^4\left(\frac{9(k_1^2+2k_2)}{(4\pi)^2} + C\Big(\frac{3\beta_2}{(4\pi)^4} + \frac{6k_1\beta_1}{(4\pi)^3} + \frac{9k_2\beta_0}{(4\pi)^2}\Big)\right) \\
&+ \alpha_s^5\left(\frac{9(2k_1k_2+2k_3)}{(4\pi)^2} + C\Big(\frac{3\beta_3}{(4\pi)^5} + \frac{6k_1\beta_2}{(4\pi)^4} + \frac{9k_2\beta_1}{(4\pi)^3} + \frac{12k_3\beta_0}{(4\pi)^2}\Big)\right)
\end{aligned}
$$

By requiring combination of k_3-terms to be zero $\Rightarrow$ $C = -0.13636364$

Λ parameter

We follow the same procedure as previously but now α_s determined via $A(t)$ function



$r_0\Lambda$=0.664(14)

Step-scaling

Step-Scaling procedure

- Lattice sizes 14,16,20,24,28,32,40,48
- Choose c-value (0.1,0.12) $\Rightarrow t = (cN)^2$
- For each β of pairs $(N, 2N)$ find the difference

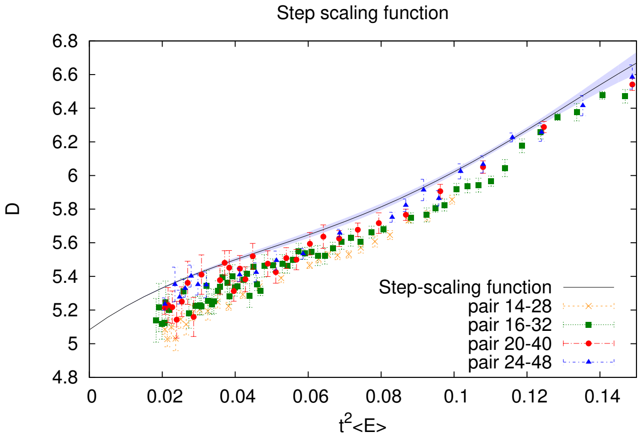
$$D(t^2\langle E\rangle\big|_{\mu}) = \frac{1}{t^2\langle E\rangle}\Big|_{2\mu} - \frac{1}{t^2\langle E\rangle}\Big|_{\mu}$$

- Find the function D in the continuum

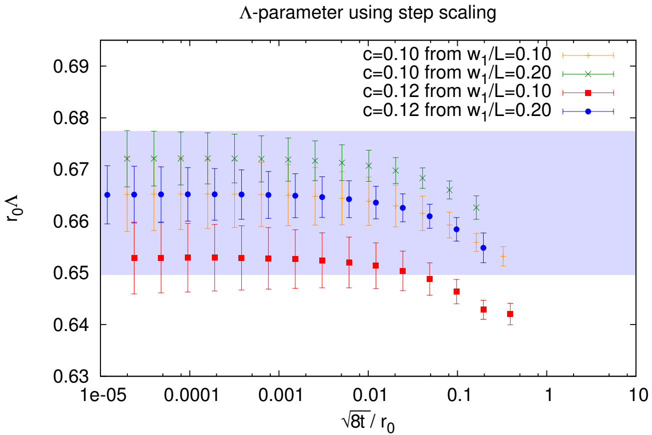
$$\frac{1}{t^2\langle E\rangle}\Big|_{2\mu} - \frac{1}{t^2\langle E\rangle}\Big|_{\mu} = \frac{4\pi}{3}\Big(\frac{1}{\alpha_s(2\mu)} - \frac{1}{\alpha_s(\mu)} + (2k_1^2 - k_2)\big(\alpha(2\mu) - \alpha(\mu)\big)\Big)$$
$$= \frac{4\pi}{3}\Big[\frac{2\beta_0}{\pi}\ln 2 + \frac{2\beta_1}{\pi^2}\ln 2\frac{4\pi}{3}\mathrm{t}^2\langle \mathrm{E}\rangle\Big|_{\mu} + \dots\Big]$$

- Keep w_1/L fixed and use 14,16,20,24 to do step scaling

Step-scaling function D for $c = 0.1$



Λ parameter from $T = 0$ and step-scaling



- c=0.10 from $w_1/L = 0.10 \Rightarrow t^2\langle E\rangle = 0.0107$
- c=0.10 from $w_1/L = 0.20 \Rightarrow t^2\langle E\rangle = 0.0102$
- c=0.12 from $w_1/L = 0.10 \Rightarrow t^2\langle E\rangle = 0.0109$
- c=0.12 from $w_1/L = 0.20 \Rightarrow t^2\langle E\rangle = 0.0103$

$\Rightarrow$ For all $\alpha_s < 0.01$

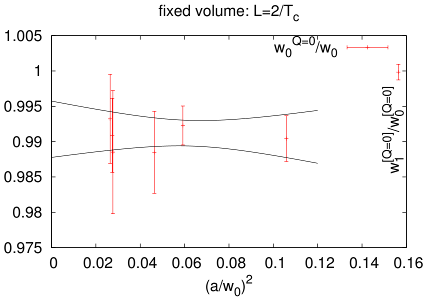
Conclusions

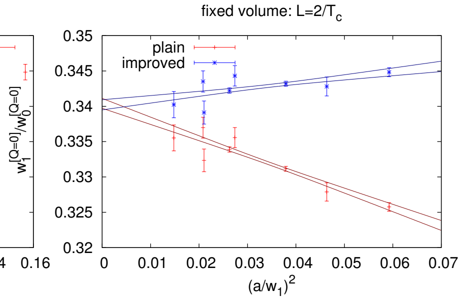
- We present a way of determining the Λ-parameter (and α_s) using the gradient flow
- By brute-force elements we find a good plateau for the Λ-parameter that meets the criteria of a good α_s determination
  $\Rightarrow r_0\Lambda = 0.664(14)$
- Step-scaling results are also in a good agreement with those from $T = 0$ simulations
- Still work in progress so... be patient for final results soon

Thank you for your attention!

w_1 to w_0 and r_0 relations

- $w_1/w_0^{Q=0} = 0.340(7)$
- $w_0/r_0 = 0.341(2)$ Sommer et al. (2014)





Continuum function D-Sketch of analytic derivation up to NLO

From $d\alpha_s/d\ln\mu$ we can find

$$\frac{d(1/\alpha_s)}{d\ln\mu} = -\frac{1}{\alpha_s^2}\frac{d\alpha_s}{d\ln\mu} = \frac{2\beta_0}{\pi} + \frac{2\beta_1}{\pi^2}\alpha_s + \frac{2\beta_2}{\pi^3}\alpha_s^2$$

Then we can find the difference

$$\frac{1}{\alpha_s(2\mu)} - \frac{1}{\alpha_s(\mu)} = \int_{\ln\mu}^{\ln 2\mu} \frac{d(1/\alpha_s)}{d\ln(\mu'/\mu)} d\ln(\mu'/\mu)$$

$$= \frac{2\beta_0}{\pi}\ln\mu + \frac{2\beta_1}{\pi^2}\int_0^{\ln 2} \alpha_\mathrm{s}(\mu')\mathrm{d}\ln(\mu'/\mu) + \frac{2\beta_2}{\pi^3}\int_0^{\ln 2} \alpha_\mathrm{s}^2(\mu')\mathrm{d}\ln(\mu'/\mu)$$

Using similar procedure we find $\alpha_s(2\mu) - \alpha_s(\mu)$

Then we fit using

$$5.0831 + 15.71x + ax^2 + bx^3 + cx^4 + dx^2\frac{1}{N^2} + ex^3\frac{1}{N^2} + fx^4\frac{1}{N^2} + g(5.0831 + 15.71x)\frac{1}{N^2}$$