Determining $\alpha_s$ by using the gradient flow in the quenched theory

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Introduction - Motivation
Current state of $\alpha_s$ determinations

- Many attempts to estimate $\alpha_s$ and $\Lambda$ parameter in literature
- Summary and combined value by Flag Working Group [arXiv:1607.00299]
- Criteria:
  - Renormalization scale: all points must have $\alpha_{\text{eff}} < 0.2$
  - Perturbative behaviour: should be verified over a range of a factor 4 change in $\alpha_{\text{eff}}^{n_{1}}$ (or $\alpha_{\text{eff}} = 0.01$ is reached)
  - Continuum extrapolation: at $\alpha_{\text{eff}} = 0.3$ have three lattice spacing with $\mu a < 0.5$ for full $\mathcal{O}(a)$ improvement.
  - Finite-size effects: scale is determined in large enough volumes
  - Topology sampling
## Current State of $\alpha_s$ determination - quenched case

<table>
<thead>
<tr>
<th>Collaboration</th>
<th>Ren. scale</th>
<th>Pert. Behav.</th>
<th>Cont. Extrap.</th>
<th>$r_0 \Lambda_{\overline{MS}}$</th>
<th>Method</th>
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<tr>
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<td>★</td>
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<td>ALPHA 98</td>
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<td>★</td>
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<td>★</td>
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<tr>
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<td>★</td>
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<td>QCD vertices</td>
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<td>■</td>
<td>0.59(1)(^{+2}_{-1})</td>
<td>QCD vertices</td>
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<tr>
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<td>★</td>
<td>■</td>
<td>0.62(7)</td>
<td>QCD vertices</td>
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In this talk: $\alpha_s$ in the quenched case using the gradient flow
Gradient Flow - Setting the scale

- Gradient Flow has many applications (scale setting, operator relation, topology etc...) \textit{Lüscher (2010)}
- Simplest gauge invariant quantity: action density
  \[ E(t, x) = \frac{1}{4} G_a^{\mu\nu} G_a^{\mu\nu} \]
- Its expectation value \( \langle E(t, x) \rangle \) serves as a non-perturbative definition of a reference scale
- \( t_0 \) first introduced as a reference scale \textit{Lüscher (2010)}
  \[ t^2 \langle E(t) \rangle \bigg|_{t=t_0} = 0.3 \]
- \( w_0 \) can also be used as a reference scale \textit{BMW Collaboration (2012)}
  \[ t \frac{d}{dt} t^2 \langle E(t) \rangle \bigg|_{t=w_0^2} = 0.3 \]
Gradient Flow - Perturbative relation

Perturbative relation for its expectation value for QCD \((N_A = 8)\) in \(\overline{\text{MS}}\) scheme up to NNLO

\[
t^2 \langle E(t) \rangle = \frac{3\alpha_s}{4\pi} \left( 1 + \alpha_s k_1 + \alpha_s^2 k_2 + O(\alpha_s^3) \right)
\]

\(k_1 = 1.09778674\) \(\text{L"uscher (2010)}\)

\(k_2 = -0.9822456\) \(\text{Harlander and Neumann (2016)}\)
Brute-Force determination of $\alpha_5$
Procedure

Simulation details

- Use fine-lattices at $T = 0$
- Keep the physical volume constant $L^c T_c \simeq 2$
- Periodic Boundary Conditions
- Tree-level Symanzik action, Wilson flow, Clover-leaf definition of observable
- $Q = 0$ configurations selected

- $w_0^{Q=0}/w_0 = 0.992(4)$
- Use $w_1$ to set the scale: $t \frac{d}{dt} t^2 \langle E(t) \rangle \big|_{t=w_1^2} = 0.03$
- $w_1/r_0 = 0.115(2)$

Lattices used

<table>
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<tr>
<th>$\beta$</th>
<th>N</th>
<th>a (in $r_0$)</th>
<th># cfgs</th>
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<td>5.3570</td>
<td>48</td>
<td>0.05651</td>
<td>529</td>
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<tr>
<td>6.1500</td>
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<tr>
<td>6.3600</td>
<td>160</td>
<td>0.01648</td>
<td>103</td>
</tr>
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</table>
• Discretization correction terms at tree-level Fodor et al. (2014)

\[ t^2 \langle E(t) \rangle = \frac{3 \alpha_s}{4\pi} \left( C(a^2/t) + \mathcal{O}(\alpha_s) \right) \]

where

\[ C(a^2/t) = 1 + \sum_{m=1}^{\infty} C_{2m} \frac{a^{2m}}{t^m} \]

Coefficients known up to \( \mathcal{O}(a^8) \)

• Finite-Volume correction Fodor et al. (2012)

\[ t^2 \langle E \rangle = \frac{3 \alpha_s}{4\pi} \left( 1 + \delta(t/L^2) \right) \]

where

\[ \delta = 1 - \frac{64t^2\pi}{3L^2} + 8e^{-L^2/8t} + 24e^{-L^4/4t} + \ldots \]
Improved vs Unimproved flow

\[ \beta = 6.05 \]
\[ \beta = 6.05 \text{ improved} \]
\[ \beta = 6.15 \]
\[ \beta = 6.15 \text{ improved} \]
\[ \beta = 6.36 \]
\[ \beta = 6.36 \text{ improved} \]
Improvement

\[ t^2 E \]

\( \sqrt{8t}/w_1 = 1.0, \text{ unimproved} \)

\( \sqrt{8t}/w_1 = 1.0, \text{ improved} \)

\( \sqrt{8t}/w_1 = 0.6, \text{ unimproved} \)

\( \sqrt{8t}/w_1 = 0.6, \text{ improved} \)
1. $t^2 \langle E(t) \rangle \Rightarrow \alpha_s$ from perturbative relation
2. Use 4-loop $\beta$-function in the $\overline{\text{MS}}$-scheme to run $\alpha_s$ at a high scale

Ritbergen, Vermaseren, Larin (1997)

$$t^2 \langle E(t) \rangle = \frac{3\alpha_s}{4\pi} \left( 1 + \alpha_s k_1 + \alpha_s^2 k_2 + \mathcal{O}(\alpha_s^3) \right)$$
\[ t^2 \langle E(t) \rangle = \frac{3\alpha_s}{4\pi} (1 + \alpha_s k_1 + \alpha_s^2 k_2 + \alpha_s^3 k_3 + O(\alpha_s^4)) \]

\( \Lambda \)-parameter from flow including \( k_3 \)

- \( k_3 = -2.0 \)
- \( k_3 = 0.0 \)
- \( k_3 = 2.0 \)
We want to eliminate $k_3$ contribution

$$A(t) \equiv (t^2\langle E \rangle)^2 + C \left( t \frac{dt^2\langle E \rangle}{dt} \right)$$

$$= \alpha_s^2 \left( \frac{9}{(4\pi)^2} + \frac{3\beta_0 C}{(4\pi)^2} \right) + \alpha_s^3 \left( \frac{18k_1}{(4\pi)^2} + C \left( \frac{3\beta_1}{(4\pi)^3} + \frac{6k_1\beta_0}{(4\pi)^2} \right) \right) + \alpha_s^4 \left( \frac{9(k_1^2 + 2k_2)}{(4\pi)^2} + C \left( \frac{3\beta_2}{(4\pi)^4} + \frac{6k_1\beta_1}{(4\pi)^3} + \frac{9k_2\beta_0}{(4\pi)^2} \right) \right) + \alpha_s^5 \left( \frac{9(2k_1k_2 + 2k_3)}{(4\pi)^2} + C \left( \frac{3\beta_3}{(4\pi)^5} + \frac{6k_1\beta_2}{(4\pi)^4} + \frac{9k_2\beta_1}{(4\pi)^3} + \frac{12k_3\beta_0}{(4\pi)^2} \right) \right)$$

By requiring combination of $k_3$-terms to be zero $\Rightarrow C = -0.13636364$
We follow the same procedure as previously but now $\alpha_s$ determined via $A(t)$ function

\[ r_0 \Lambda = 0.664(14) \]
Step-scaling
Step-Scaling procedure

- Lattice sizes 14, 16, 20, 24, 28, 32, 40, 48
- Choose $c$-value $(0.1, 0.12) \Rightarrow t = (cN)^2$
- For each $\beta$ of pairs $(N, 2N)$ find the difference

$$D(t^2\langle E \rangle|_\mu) = \frac{1}{t^2\langle E \rangle|_{2\mu}} - \frac{1}{t^2\langle E \rangle|_\mu}$$

- Find the function $D$ in the continuum

$$\frac{1}{t^2\langle E \rangle|_{2\mu}} - \frac{1}{t^2\langle E \rangle|_\mu} = \frac{4\pi}{3} \left( \frac{1}{\alpha_s(2\mu)} - \frac{1}{\alpha_s(\mu)} + (2k_1^2 - k_2)(\alpha(2\mu) - \alpha(\mu)) \right)$$

$$= \frac{4\pi}{3} \left[ \frac{2\beta_0}{\pi} \ln 2 + \frac{2\beta_1}{\pi^2} \ln 2 \frac{4\pi}{3} t^2\langle E \rangle|_\mu + \ldots \right]$$

- Keep $w_1/L$ fixed and use 14, 16, 20, 24 to do step scaling
Step-scaling function $D$ for $c = 0.1$
\( \Lambda \) parameter from \( T = 0 \) and step-scaling

\[ \Lambda = \text{parameter using step scaling} \]

- \( \Lambda = 0.10 \) from \( w_1/L = 0.10 \)
- \( \Lambda = 0.10 \) from \( w_1/L = 0.20 \)
- \( \Lambda = 0.12 \) from \( w_1/L = 0.10 \)
- \( \Lambda = 0.12 \) from \( w_1/L = 0.20 \)

\[ \langle E \rangle = 0.0107 \] for \( \alpha_s < 0.01 \)

\[ \langle E \rangle = 0.0102 \]

\[ \langle E \rangle = 0.0109 \]

\[ \langle E \rangle = 0.0103 \]
Conclusions

- We present a way of determining the Λ-parameter (and \( \alpha_s \)) using the gradient flow.
- By brute-force elements we find a good plateau for the Λ-parameter that meets the criteria of a good \( \alpha_s \) determination.
  \[ r_0 \Lambda = 0.664(14) \]
- Step-scaling results are also in a good agreement with those from \( T = 0 \) simulations.
- Still work in progress so... be patient for final results soon.
Thank you for your attention!
$w_1$ to $w_0$ and $r_0$ relations

- $w_1/w_0^{Q=0} = 0.340(7)$
- $w_0/r_0 = 0.341(2)$  Sommer et al. (2014)
Continuum function $D$-Sketch of analytic derivation up to NLO

From $d\alpha_s/d\ln\mu$ we can find

$$\frac{d(1/\alpha_s)}{d\ln\mu} = -\frac{1}{\alpha_s^2} \frac{d\alpha_s}{d\ln\mu} = \frac{2\beta_0}{\pi} + \frac{2\beta_1}{\pi^2}\alpha_s + \frac{2\beta_2}{\pi^3}\alpha_s^2$$

Then we can find the difference

$$\frac{1}{\alpha_s(2\mu)} - \frac{1}{\alpha_s(\mu)} = \int_{\ln\mu}^{\ln2\mu} \frac{d(1/\alpha_s)}{d\ln(\mu'/\mu)} d\ln(\mu'/\mu)$$

$$= \frac{2\beta_0}{\pi} \ln\mu + \frac{2\beta_1}{\pi^2} \int_0^{\ln2} \alpha_s(\mu')d\ln(\mu'/\mu) + \frac{2\beta_2}{\pi^3} \int_0^{\ln2} \alpha_s^2(\mu')d\ln(\mu'/\mu)$$

Using similar procedure we find $\alpha_s(2\mu) - \alpha_s(\mu)$

Then we fit using

$$5.0831 + 15.71x + ax^2 + bx^3 + cx^4 + dx^2 \frac{1}{N^2} + ex^3 \frac{1}{N^2} + fx^4 \frac{1}{N^2} + g(5.0831 + 15.71x) \frac{1}{N^2}$$