The gradient flow coupling from numerical stochastic perturbation theory

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Introduction

The Yang-Mills gradient flow

Gradient flow (GF):

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x), \quad B_\mu(0,x) = A_\mu(x)$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], \quad D_\mu = \partial_\mu + [B_\mu, \cdot\,]$$

- Gauge invariant observables are finite (Lüscher '10; Lüscher, Weisz '11)
- Allows for good statistical precision
- Simple to evaluate in simulations

Applications:

- small flow-time expansion
- **step-scaling** studies
- ...and many more! (Lüscher, Lattice '13)

Motivation: Perturbation theory (PTh) can provide useful, if not **essential**, information for these studies

Introduction

The Yang-Mills gradient flow in perturbation theory

(Lüscher '10; Lüscher, Weisz '11)

In PTh, GF obs. are quite non-trivial objects,

$$B_\mu = \sum_{k=1}^{\infty} g_0^k B_{\mu,k} \quad \xrightarrow{\text{GF eq.}} \quad \partial_t B_{\mu,k} - \partial_\nu \partial_\nu B_{\mu,k} = R_{\mu,k}$$

$$B_{\mu,k}|_{t=0} = \delta_{k1} A_\mu$$

where e.g.

$$\begin{aligned} R_{\mu,1} &= 0, \\ R_{\mu,2} &= 2[B_{\nu,1}, \partial_\nu B_{\mu,1}] - [B_{\nu,1}, \partial_\mu B_{\nu,1}], \\ R_{\mu,3} &= \text{complicated and not very illuminating expression} \\ &\qquad \text{almost as long as this whole sentence} \ldots \end{aligned}$$

Simple at lowest order

$$B_{\mu,1}(t,x) = \int \mathrm{d}^4 y \, K_t(x-y) A_\mu(y), \quad K_t(z) = \frac{e^{-z^2/4t}}{(4\pi t)^2}$$

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Simple at lowest order ...but quickly becomes **involved**,

$$B_{\mu,k}(t,x) = \int_0^t \mathrm{d}s \int \mathrm{d}^4 y \, K_{t-s}(x-y) R_{\mu,k}(s,y), \quad k = 2, 3, \ldots$$

Stochastic quantization

(Parisi, Wu '81; Zwanziger '81; Zinn-Justin '86; Zinn-Justin, Zwanziger '88)

Perturbation theory without gauge fixing

Langevin equation:

$$\partial_s A_\mu(s,x) = D_\nu F_{\nu\mu}(s,x) + g_0 \eta_\mu(s,x),$$

$$\langle \eta^a_\mu(s,x) \eta^b_\nu(r,y) \rangle_\eta = 2\delta^{ab}\delta_{\mu\nu}\delta(s-r)\delta(x-y)$$

Stochastic PTh:

(Parisi, Wu '81; Floratos, Iliopoulos '83; Grimus, Hüffel '83)

$$A_\mu = \sum_{k=1}^{\infty} g_0^k A_{\mu,k} \quad \xrightarrow{\text{Langevin eq.}} \quad \lim_{s\to\infty} \langle \mathcal{O}[A(s)] \rangle_\eta = \sum_{k=0}^{\infty} g_0^k \mathcal{O}_k$$

- Close analogy w/ PTh GF eqs.: $B_{\mu,k} \to A_{\mu,k}$, and $R_{\mu,1} \to R_{\mu,1} = \eta_\mu$
- If $\mathcal{O}$ is a GF obs.: $B_\mu(0,x) = A_\mu(x) \to B_{\mu,k}(t,s,x)|_{t=0} = A_{\mu,k}(s,x)$

Numerical stochastic perturbation theory (NSPT) (Di Renzo *et. al.* '94)

Provides a natural framework to solve the GF eqs. perturbatively! (MDB, Hesse '13)

Stochastic quantization in phase space (Horowitz '85; Lüscher, Schaefer '11)

NSPT based on stochastic molecular dynamics equations (MDB, Garofalo, Kennedy '15)

Kramers equation:

$$\partial_s U_s(x,\mu) = g_0 \pi_s(x,\mu) U_s(x,\mu),$$

$$\partial_s \pi_s(x,\mu) = -g_0 \nabla_{x,\mu} S_G(U_s) - 2\mu_0 \pi_s(x,\mu) + \eta_s(x,\mu)$$

$$\langle \eta_s^a(x,\mu) \eta_r^b(y,\nu) \rangle_\eta = 4\mu_0 \delta^{ab} \delta_{\mu\nu} \delta(s-r) a^{-4} \delta_{xy}$$

- $s \to s = na \cdot \delta\tau,\ n \in \mathbb{Z}$
- Numerically solved to $\mathrm{O}(g_0^N)$: $U = \mathbb{1} + \sum_{k=1}^{N} g_0^k U_k$, $\pi = \sum_{k=0}^{N-1} g_0^k \pi_k$

Remarks:

- $\delta\tau \to 0$ extrap. are **avoided** using a OMF4 integrator (Omelyan, Mryglod, Folk '03)
- $\gamma \equiv 2a\mu_0$ can be tuned to **minimize** $\mathrm{var}(\mathcal{O}_k) \times \tau_{\mathrm{int}}(\mathcal{O}_k)$
  - for γ fixed, $\tau_{\mathrm{int}}(\mathcal{O}_k) \propto a^{-2}$ while $\mathrm{var}(\mathcal{O}_k) \propto \ln(a)$ (**Langevin limit**) (Lüscher, Schaefer '11; Lüscher '15)
- stochastic gauge fixing
- **CPU time** $\propto N(N-1) \times (L/a)^{4+2+\frac{4}{p}}$, $p \equiv$ integrator order

The gradient flow coupling

(Lüscher '10; Fodor *et. al.* '12)

Definitions

We impose Schrödinger functional (SF) bc.'s, (Lüscher, Narayanan, Weisz, Wolff '92)

$$U(x,k)|_{x_0=0} = \mathbb{1} = U(x,k)|_{x_0=L}, \quad k=1,2,3$$

A **family** of finite-volume couplings can be defined as (Fritzsch, Ramos '13)

$$\alpha_{\rm GF}(\mu) \equiv \alpha_{{\rm GF},c}(\mu) \propto \langle t^2 E_{\rm sp}(t,x)\rangle\Big|^{\sqrt{8t}=cL}_{x_0=L/2}, \quad \mu = 1/\sqrt{8t}$$

where

$$E_{\rm sp}(t,x) \overset{a\to 0}{\sim} -\tfrac{1}{2}{\rm tr}\{G_{kl}(t,x)G_{kl}(t,x)\}$$

The **goal** is to determine in the pure $\mathrm{SU}(3)$ Yang-Mills theory

$$\boxed{\alpha_{\rm GF}(\mu) = \alpha_{\overline{\rm MS}}(\mu) + k_1\,\alpha^2_{\overline{\rm MS}}(\mu) + k_2\,\alpha^3_{\overline{\rm MS}}(\mu) + {\rm O}(\alpha^4)}$$

To this end we compute

$$\alpha_{\rm GF}(\mu) = \alpha_0(a) + c_1(a/L)\,\alpha_0^2(a) + c_2(a/L)\,\alpha_0^3(a) + {\rm O}(\alpha^4)$$

and use the known two-loop relation $\alpha_{\overline{\rm MS}} \leftrightarrow \alpha_0$ (Lüscher, Weisz '95)

The gradient flow coupling

Towards the continuum limit

We expect the continuum limit to be reached as (Symanzik '83)

$$k_1(a/L) - k_1 \overset{a/L\to 0}{\sim} (r_{1,1} + s_{1,1}\ln(L/a))\left(\frac{a}{L}\right) + (r_{1,2} + s_{1,2}\ln(L/a))\left(\frac{a}{L}\right)^2 + \mathrm{O}(a^3)$$

$$k_2(a/L) - k_2 \overset{a/L\to 0}{\sim} (r_{2,1} + s_{2,1}\ln(L/a) + t_{2,1}\ln(L/a)^2)\left(\frac{a}{L}\right) + (r_{2,2} + s_{2,2}\ln(L/a) + t_{2,2}\ln(L/a)^2)\left(\frac{a}{L}\right)^2 + \mathrm{O}(a^3)$$

Warning: SF bc.'s introduce $\mathrm{O}(a)$ effects even in the pure gauge theory

These can in principle be removed by adding a counterterm to the action

$$\overset{a\to 0}{\sim} c_{\mathrm{t}}(g_0) F_{0k}F_{0k}|_{x_0=0,T}, \quad c_{\mathrm{t}}(g_0) = 1 + c_{\mathrm{t}}^{(1)} g_0^2 + c_{\mathrm{t}}^{(2)} g_0^4 + \mathrm{O}(g_0^6)$$

(Lüscher, Narayanan, Weisz, Wolff '92; Lüscher, Sommer, Weisz, Wolff '93; Bode, Weisz, Wolff '99)

The gradient flow coupling

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$$k_1(a/L) - k_1 \overset{a/L\to 0}{\sim} \cancel{(r_{1,1} + s_{1,1}\ln(L/a))\left(\frac{a}{L}\right)}$$
$$+(r_{1,2} + s_{1,2}\ln(L/a))\left(\frac{a}{L}\right)^2 + \mathrm{O}(a^3)$$

$$k_2(a/L) - k_2 \overset{a/L\to 0}{\sim} (r_{2,1} + s_{2,1}\ln(L/a) + \cancel{t_{2,1}\ln(L/a)^2})\left(\frac{a}{L}\right)$$
$$+(r_{2,2} + s_{2,2}\ln(L/a) + t_{2,2}\ln(L/a)^2)\left(\frac{a}{L}\right)^2 + \mathrm{O}(a^3)$$

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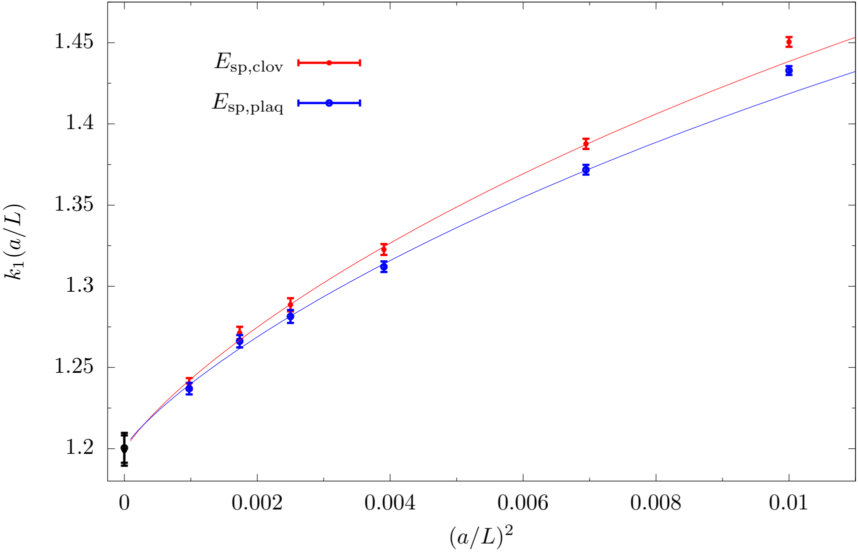
At present we simply consider $c_{\mathrm{t}} = 1$, and subtract the know $r_{1,1}$ term

Results

Continuum limit of $k_1(a/L)$ for $c = 0.3$



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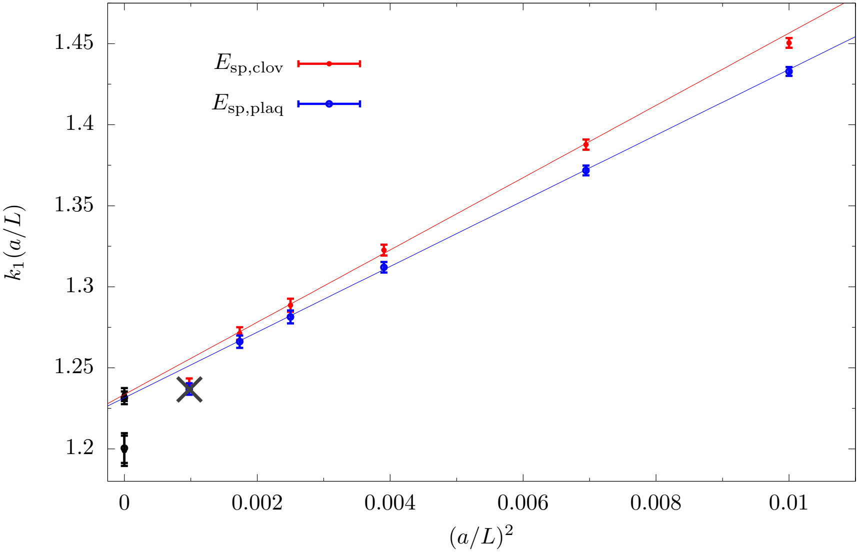
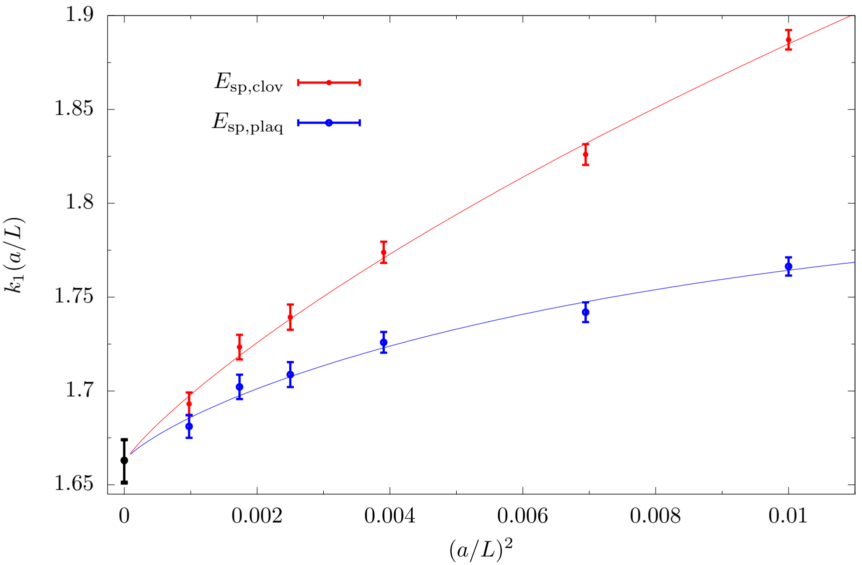


Results

Continuum limit of $k_1(a/L)$ **for** $c = 0.4$



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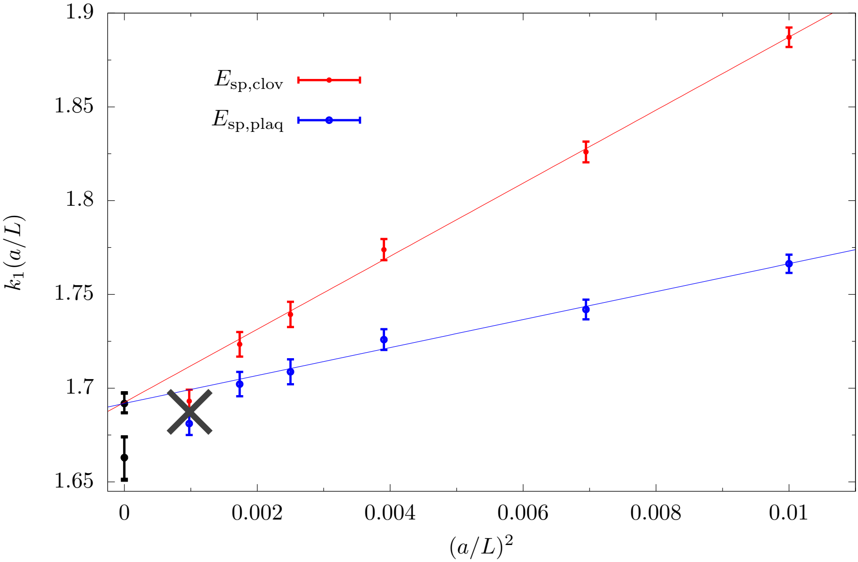
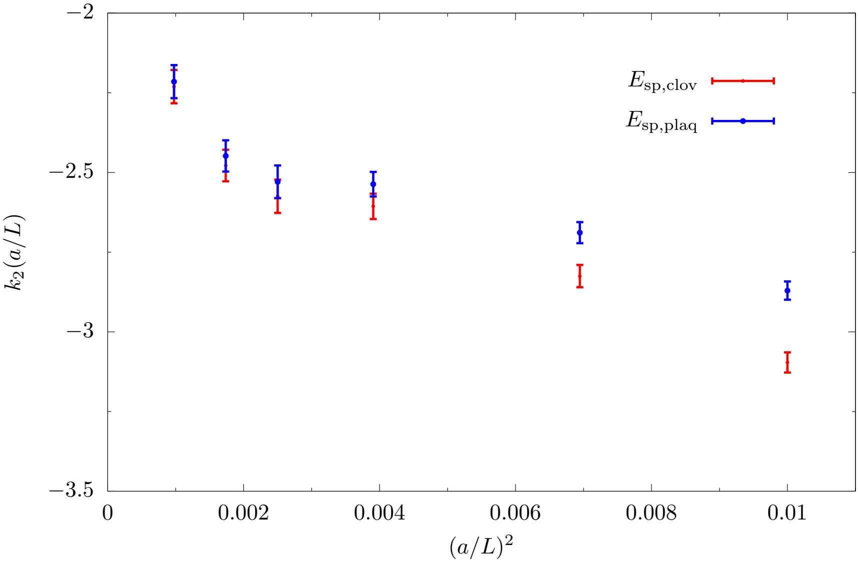


Results

Continuum limit of $k_2(a/L)$ for $c = 0.3$



Results

How precise do we need to be?

An **accurate** determination of $\alpha_{\overline{\mathrm{MS}}}(M_Z)$ requires

$$\begin{aligned}\sigma(k_1) &= 10^{-2} \\ \sigma(k_2) &= 10^{-1}\end{aligned} \quad \Rightarrow \quad \frac{\sigma(\alpha_{\overline{\mathrm{MS}}}(M_Z))}{\alpha_{\overline{\mathrm{MS}}}(M_Z)} \sim 0.2\%$$

Λ-parameter: $r_c \equiv \Lambda_{\mathrm{GF,c}}/\Lambda_{\overline{\mathrm{MS}}}$ (Lüscher '10)

$$r_0 = 1.872\ldots, \quad r_{0.3} = 1.983(11), \quad r_{0.4} = 2.584(17)$$

β-function:

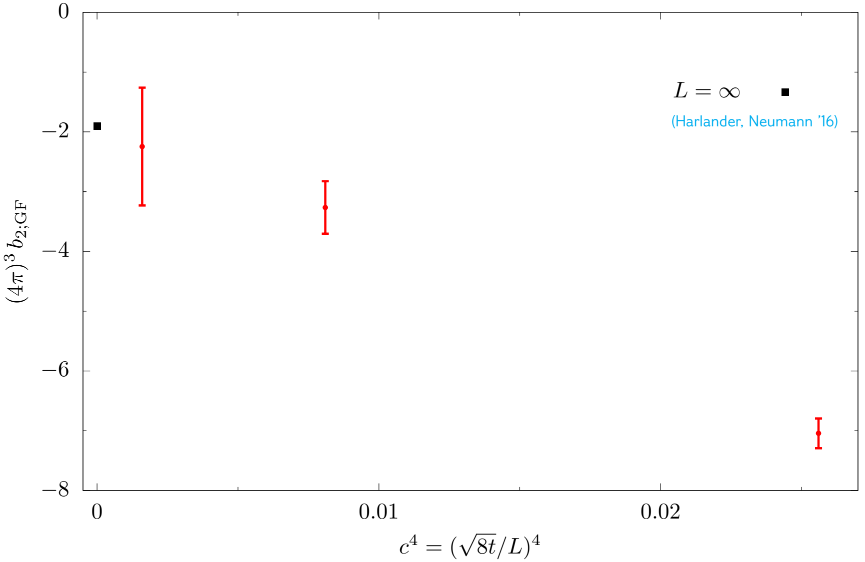
$$\beta_{\mathrm{GF}}(g) = -g^3\left(b_0 + b_1\, g^2 + b_{2;\mathrm{GF}}\, g^4 + \mathrm{O}(g^6)\right)$$

$$b_0 = \frac{11}{(4\pi)^2}, \quad b_1 = \frac{102}{(4\pi)^4}$$

Results

Infinite volume limit of $b_{2;\mathrm{GF}}$ taking $k_2 = k_2(L/a = 32) \pm 5 \cdot 10^{-1} \times (0.3/c)^2$

(Lüscher '14; MDB, Fritzsch, Korzec, Ramos, Sint, Sommer '16)



Outlook & Conclusions

Conclusions:

- NSPT is a potentially **powerful** tool to obtain precise perturbative results for GF obs. (not only the coupling!)
- Careful work is however needed to **control** continuum extrapolations

Outlook:

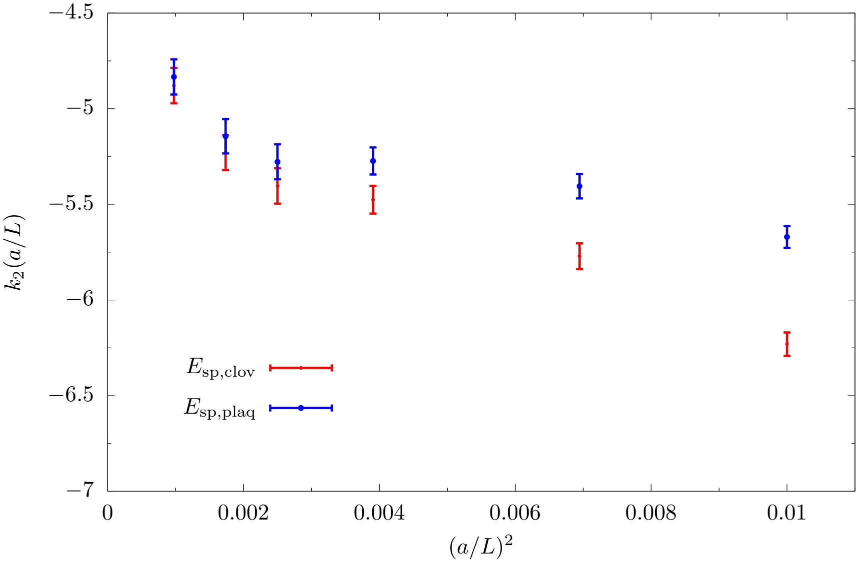
- We plan to add finer lattices: $L/a = 36, 40, 48(?)$
- After implementing full O(a) improvement we expect **precise** results for k_2 to be **possible**
- Results at finite L/a may be used to improve continuum extrapolations of non-perturbative step-scaling functions (Lüscher, Sommer, Weisz, Wolff '93)
- The inclusion of fermions is **straightforward** within Kramers and **efficient** integrators can be employed
- Including quarks is then expected to cost a **modest** factor as only the **free** Dirac equation needs to be solved (Di Renzo, Scorzato '04)


Thanks

BACKUP

Results

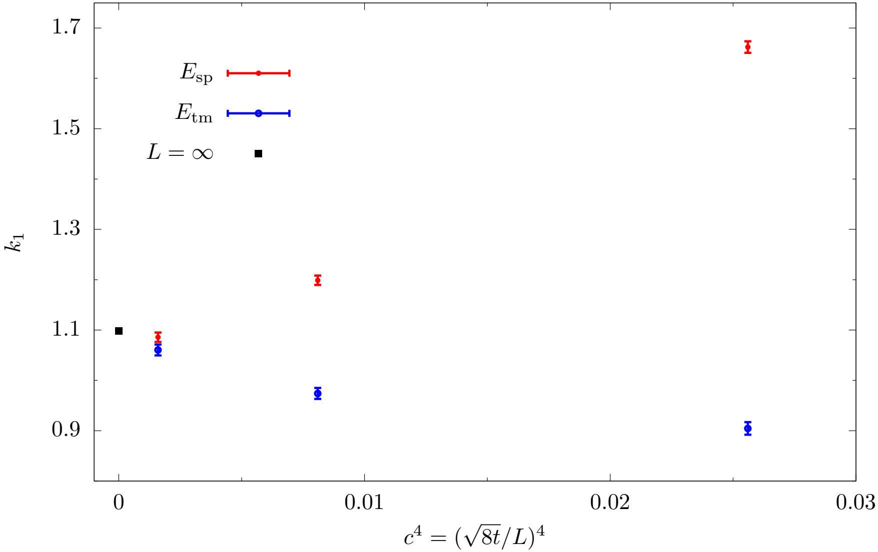
Continuum limit of $k_2(a/L)$ for $c = 0.4$



Results

Infinite volume limit of k_1

(Lüscher '10; Lüscher '14)



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