Chiral transition, eigenmode localisation and Anderson-like models

Matteo Giordano, Tamás G. Kovács and Ferenc Pittler

Eötvös Loránd University (ELTE), Budapest

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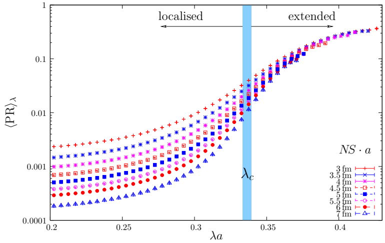
Localisation in the Dirac Spectrum

QCD transition: change in both confining and chiral properties, $\sim$ same T_c

Above T_c lowest-lying Dirac eigenmodes are localised for $\lambda < \lambda_c(T)$

[García-García, Osborn (2007), Kovács, Pittler (2012), Cossu, Hashimoto (2016)]



$$\mathrm{IPR} = \sum_x |\psi(x)|^4$$

$$\mathrm{PR} = \mathrm{IPR}^{-1}/V_4$$

(fraction of 4-vol occupied by a mode)

$$T \simeq 2.6\, T_c$$

- Onset of localisation at the chiral crossover
  [García-García, Osborn (2007), Kovács, Pittler (2012)]
- Observed also with unimproved staggered fermions, $N_T = 4$: localised modes appear at the 1st-order PT [MG, Kovács, Katz, Pittler (2014)]

Understanding localisation might help to understand how chiral symmetry restoration and deconfinement are related

Anderson Model vs. High-Temperature QCD

Tight-binding Hamiltonian for "dirty" conductors [Anderson (1958)]

$$H^{\mathrm{AM}}_{\vec{x},\vec{y}} = \varepsilon_{\vec{x}}\delta_{\vec{x},\vec{y}} + \sum_{\mu=1}^{3}(\delta_{\vec{x}+\hat{\mu},\vec{y}} + \delta_{\vec{x}-\hat{\mu},\vec{y}})$$

Random potential $|\varepsilon_{\vec{x}}| \leq \frac{W}{2}$

Anderson transition: eigenstates localised for $E > E_c(W)$ (mobility edge)
Second-order phase transition with divergent $\xi \sim |E - E_c|^{-\nu}$

Localisation/delocalisation transition in the staggered QCD spectrum at $\lambda_c(T)$: same universality class of the 3D Unitary Anderson Model

- Correlation-length critical exponents match

$$\nu_{\mathrm{UAM}} = 1.43(4) \qquad \nu_{\mathrm{QCD}} = 1.43(6)$$

[Slevin, Ohtsuki (1999)] [MG, Kovács, Pittler (2014)]

- Multifractal eigenfunctions near localisation/delocalisation transition, multifractal exponents match [Ujfalusi, MG, Pittler, Kovács, Varga (2015)]

Anderson Model vs. High-Temperature QCD

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Random potential $|\varepsilon_{\vec{x}}| \leq \frac{W}{2}$, random phases $\phi_{\vec{y},\vec{x}} = -\phi_{\vec{x},\vec{y}}$

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Dimensional Reduction

Why same universality class? Same RMT symmetry class (unitary), but

- UAM: 3D, diagonal noise controlled by W (mainly)
- QCD: 4D, off-diagonal noise controlled by T (as this controls λ_c)

QCD above T_c is effectively 3D: time slices strongly correlated, quark eigenfunctions look qualitatively the same at all t

- Temporal gauge: $U_4(t,\vec{x}) = \mathbf{1}$, $0 \le t < N_T - 1$, $U_4(N_T - 1, \vec{x}) = P(\vec{x})$
- Wave functions obey effective boundary conditions involving the Polyakov line

$$\psi(N_T, \vec{x}) = -P(\vec{x})\psi(0, \vec{x})$$

- $P(\vec{x})$ fluctuates in space, providing effective 3D diagonal disorder

3D noise present also below T_c, but boundary conditions are ineffective and QCD is effectively 4D there: why so?

Dirac-Anderson Hamiltonian

"Hamiltonian" $H = -iD_{\rm stag}$ split into "free" + "interaction", $H = H_0 + H_I$

$$(H_0)_{xx'} = \tfrac{\eta_4(\vec{x})}{2i} \left[U_4(t,\vec{x})\delta_{t+1,t'} - U_{-4}(t,\vec{x})\delta_{t-1,t'} \right] \delta_{\vec{x},\vec{x}'} \qquad \text{(temporal hoppings)}$$

$$(H_I)_{xx'} = \textstyle\sum_{j=1}^{3} \tfrac{\eta_j(\vec{x})}{2i} \left[U_j(t,\vec{x})\delta_{\vec{x}+\hat{\jmath},\vec{x}'} - U_{-j}(t,x)\delta_{\vec{x}-\hat{\jmath},\vec{x}'} \right] \delta_{t,t'} \qquad \text{(spatial hoppings)}$$

Temporal diagonal (td) gauge: $U_4(N_T - 1, \vec{x}) = P(\vec{x}) = \mathrm{diag}(e^{i\phi_a(\vec{x})})$

Work in the basis of the "unperturbed" eigenvectors of H_0

Eigenvectors of H_0: $H_0 \psi_0^{\vec{x}\,a\,k} = \eta_4(\vec{x}) \sin \omega_{ak}(\vec{x}) \psi_0^{\vec{x}\,a\,k}$

Effective Matsubara frequencies: $\omega_{ak}(\vec{x}) = \frac{1}{N_T}(\pi + \phi_a(\vec{x}) + 2\pi k)$

Indices: spatial site $\vec{x}$, colour $a = 1, \ldots, N_c$, temporal momentum $k = 0, \ldots, N_T - 1$

Dirac-Anderson Hamiltonian II

In the basis of the "unperturbed" eigenvectors of H_0

$$H_{\vec{x},\vec{y}} = \delta_{\vec{x},\vec{y}} D(\vec{x}) + \sum_{j=1}^{3} \frac{\eta_j(\vec{x})}{2i} \left[\delta_{\vec{x}+\hat{\jmath},\vec{y}} V_{+j}(\vec{x}) - \delta_{\vec{x}-\hat{\jmath},\vec{y}} V_{-j}(\vec{x})\right]$$

3D Anderson-type Hamiltonian with internal degrees of freedom:
colour $a = 1, \ldots, N_c$, temporal momentum $k = 0, \ldots, N_T - 1$

Diagonal noise (random on-site potential)

$$[D(\vec{x})]_{ak,bl} = \eta_4(\vec{x}) \sin \omega_{ak}(\vec{x}) \delta_{ab} \delta_{kl} \,,$$

Off-diagonal noise (random hoppings)

$$[V_{\pm j}(\vec{x})]_{ak,bl} = \frac{1}{N_T} \sum_{t=0}^{N_T-1} e^{i\frac{2\pi t}{N_T}(l-k)} e^{i\frac{t}{N_T}[\phi_b(\vec{x}\pm\hat{\jmath}) - \phi_a(\vec{x})]} \left[U_{\pm j}^{(\mathrm{td})}(t,\vec{x})\right]_{ab}$$

Unlike the AM, disorder strength bounded ($|\sin \omega_{ak}| \leq 1$, $V_{\pm j}$ unitary), but type of disorder different in the confined and deconfined phase

Polyakov Lines, Localisation and Effective Dimensionality

Above T_c, $P(\vec{x})$ gets ordered along $\mathbf{1}$ with "islands" of "wrong" $P(\vec{x}) \neq \mathbf{1}$

- $\sin \omega_{a\,k=0}|_{\phi_a=0} = \sin \frac{\pi}{N_T}$ provides an effective gap in the spectrum
- "wrong" $P(\vec{x})$ allows for smaller $\lambda \Rightarrow$ localising "trap" for eigenmodes

[Bruckmann, Kovács, Schierenberg (2011), MG, Kovács, Pittler (2015)]

Ordering of PL induces correlation across time slices $\to$ reduced mixing of temporal momentum components: if $\phi_a(\vec{x}) = 0$ and $U^{(\mathrm{td})}_{\pm j}(t, \vec{x}) = \bar{U}^{(\mathrm{td})}_{\pm j}(\vec{x})$

$$[V_{\pm j}(\vec{x})]_{ak,bl} = \left[\bar{U}^{(\mathrm{td})}_{\pm j}(\vec{x})\right]_{ab} \frac{1}{N_T} \sum_{t=0}^{N_T-1} e^{i\frac{2\pi t}{N_T}(l-k)} = \left[\bar{U}^{(\mathrm{td})}_{\pm j}(\vec{x})\right]_{ab} \delta_{kl}$$

Strong mixing of t-mom components	$\to$	single effectively 4D system
Decoupling of t-mom components	$\to$	N_T effectively 3D systems

Should be relevant also to χSB: below/above T_c

- high/low density of small "unperturbed" eigenvalues
- strong/weak "push" towards the origin due to mixing

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Toy Model

Relevant features for χSB and localisation should be ordering of the diagonal noise and correlation of spatial links across time slices: check in a toy model

Hamiltonian with same structure of the Dirac-Anderson Hamiltonian

$$\mathcal{H}^{\text{toy}}_{\vec{x},\vec{y}} = d(\vec{x})\delta_{\vec{x},\vec{y}} + \sum_{j=1}^{3} \frac{\eta_j(\vec{x})}{2i} \left(v_{+j}(\vec{x})\delta_{\vec{x}+\hat{\jmath},\vec{y}} - v_{-j}(\vec{x})\delta_{\vec{x}-\hat{\jmath},\vec{y}} \right)$$

Replace Polyakov-line phases with complex-spin phases $\phi^a_{\vec{x}}$, $\sum_a \phi^a_{\vec{x}} = 0$

$$[d(\vec{x})]_{ak,bl} = \eta_4(\vec{x}) \sin \tfrac{\pi + \phi^a_{\vec{x}} + 2\pi k}{N_T} \delta_{ab}\delta_{kl}$$

Spin dynamics mimicking that of Polyakov lines

$$\beta H_{\text{noise}} = -\frac{\beta}{N_c} \sum_{\vec{x},j,a} \cos(\phi^a_{\vec{x}+\hat{\jmath}} - \phi^a_{\vec{x}}) - \frac{2h}{N_c(N_c-1)} \sum_{\vec{x},a<b} \cos(\phi^a_{\vec{x}} - \phi^b_{\vec{x}})$$

Ordered phase $\rightarrow$ N_c vacua $\phi^a_{\vec{x}} = \frac{2\pi}{N_c}$ $\forall a, \vec{x}$ (analogues of the center sectors)

Toy Model II

Hopping terms from "toy" gauge links $u_j(t,\vec{x})$

$$[v_{\pm j}(\vec{x})]_{ak,bl} = \frac{1}{N_T} \sum_{t=0}^{N_T-1} e^{i\frac{2\pi t}{N_T}(l-k)} e^{i\frac{t}{N_T}[\phi^b_{\vec{x}\pm\hat{\jmath}} - \phi^a_{\vec{x}}]} [u_{\pm j}(t,\vec{x})]_{ab}$$

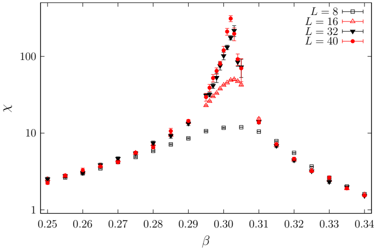
"Toy" gauge links obey a simpler dynamics: Wilson action without spatial plaquettes, no fermion determinant, Polyakov line as external field

$$S_u = -\hat{\beta}\,\mathrm{Re}\,\mathrm{Tr} \sum_{\vec{x}} \sum_{j=1}^{3} \Big\{ \Big[\sum_{t=0}^{N_T-2} u_j(t,\vec{x}) u_j^\dagger(t+1,\vec{x}) \Big] + u_j(N_T-1,\vec{x}) p(\vec{x}+\hat{\jmath}) u_j^\dagger(0,\vec{x}) p^\dagger(\vec{x}) \Big\}$$

$$p(\vec{x}) = \mathrm{diag}(e^{i\phi^a_{\vec{x}}})$$

Expectation values: $$\langle \mathcal{O} \rangle = \frac{\int D\phi\, e^{-\beta H_{\mathrm{noise}}[\phi]} \left[\frac{\int Du\, e^{-S_u[\phi,u]} \mathcal{O}[\phi,u]}{\int Du\, e^{-S_u[\phi,u]}} \right]}{\int D\phi\, e^{-\beta H_{\mathrm{noise}}[\phi]}}$$

Minimal Toy Model: $N_c = N_T = 2$



Single phase $\phi_{\vec{x}} = \phi^1_{\vec{x}} = -\phi^2_{\vec{x}}$

$$\begin{aligned}\beta H^{N_c=2}_{\text{noise}} = & -\beta \textstyle\sum_{\vec{x},j} \cos(\phi_{\vec{x}+\hat{\jmath}} - \phi_{\vec{x}}) \\ & - h \textstyle\sum_{\vec{x}} \cos(2\phi_{\vec{x}})\end{aligned}$$

U(1) symmetry broken to $\mathbb{Z}_2$ at $h \neq 0$

$\beta < \beta_c / \beta > \beta_c$ disordered/ordered phase ($\sim$ confined/deconfined phase)

In general

$$\sin \omega_{ak'}(\vec{x}) = -\sin \omega_{ak}(\vec{x}) \quad \text{if} \quad k' = \frac{N_T}{2} + k \mod N_T$$

$$\sin \frac{-\phi + \pi + 2\pi k}{N_T} = \sin \frac{\phi + \pi + 2\pi\left(\frac{N_T}{2} - 1 - k\right)}{N_T}$$

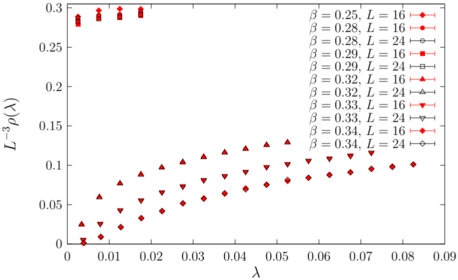
For $N_c = N_T = 2$ only one relevant Matsubara frequency $\omega(\vec{x}) = \frac{\phi_{\vec{x}} + \pi}{2}$

"Unperturbed" eigenvalues: $\pm \eta_4(\vec{x}) \cos \frac{\phi_{\vec{x}}}{2}$

Numerical Results: Chiral Symmetry

Fix "gauge coupling" $\hat{\beta} = 5.0$ and symmetry-breaking term $h = 1.0$, study dependence on β (i.e., ordering of the spin system)

"Chiral symmetry breaking" = nonzero spectral density at the origin



below β_c: nonzero $\rho(0)$, weakly dependent on β

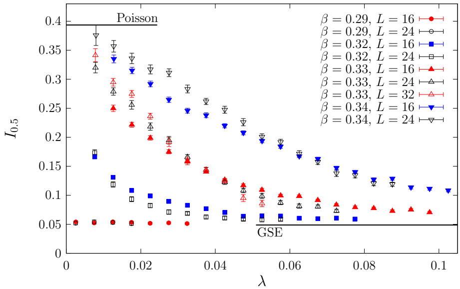
above β_c: zero $\rho(0)$, $\rho(\lambda)$ suppressed as β is increased

Disordered phase ($\beta < \beta_c$): chiral symmetry broken (χSB)

Ordered phase ($\beta > \beta_c$): chiral symmetry restored (χSR)

Numerical Results: Localisation

Detect localisation from the statistical properties of the spectrum: Poisson for localised, Gaussian Symplectic Ensemble of RMT for delocalised modes



Integrated unfolded level spacing distribution

$$I_{0.5} = \int_0^{0.5} ds\, P_\lambda(s)$$

$$s_i = \frac{\lambda_{i+1} - \lambda_i}{\langle \lambda_{i+1} - \lambda_i \rangle_\lambda}$$

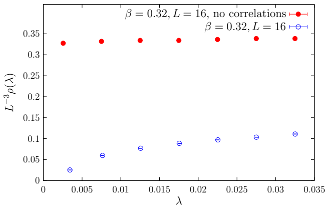
Disordered phase ($\beta < \beta_c$): all modes delocalised

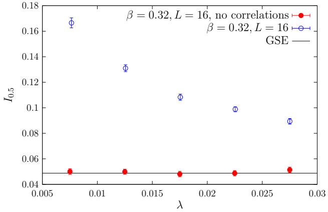
Ordered phase ($\beta > \beta_c$): localisation of lowest modes

Toy model reproduces qualitatively the properties of the QCD spectrum and eigenmodes

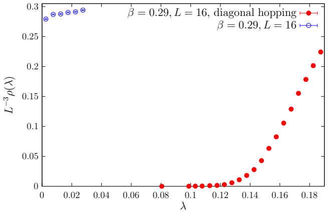
Variations on the Toy Model: Tweaking the Hopping Term

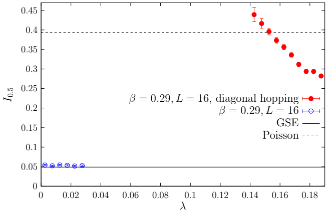
No correlations among gauge links across time slices (ordered phase)





No hopping between different temporal momenta (disordered phase)

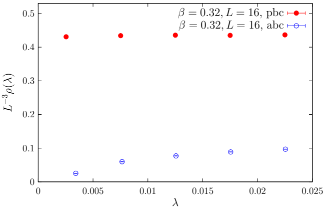


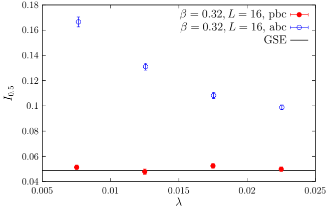


Mixing of t-mom components crucial for χSB, reduced mixing for χSR

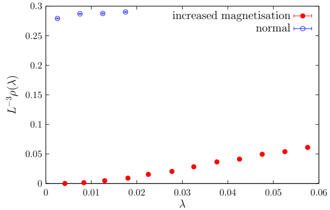
Variations on the Toy Model: Tweaking the Diagonal Term

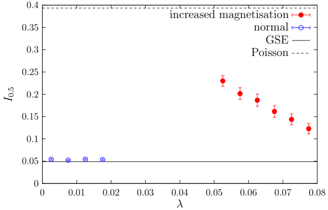
Periodic boundary condition in the temporal direction (ordered phase)





Increased magnetisation (disordered phase) $z_{\vec{x}} = \cos\frac{\phi_{\vec{x}}}{2} \to \frac{2z_{\vec{x}}^{\tau}}{1+z_{\vec{x}}^{2\tau}}$, $\tau = 0.2$





Large enough density of small unperturbed eigenvalues needed for χSB

Summary and Outlook

- Staggered Dirac operator equivalent to an Anderson-type Hamiltonian with internal degrees of freedom
- Chiral symmetry restoration and localisation of lowest modes depend on the ordering of the Polyakov lines:
  - reduced density of small unperturbed eigenvalues
  - reduced mixing of temporal-momentum components
- Tested in a toy model: properties of QCD eigenmodes reproduced, ordering and correlation of spatial links only relevant features

Open issues:

- Does localisation appear exactly at the transition in the toy model?
- Study in this formalism: imaginary chemical potential, magnetic field, adjoint fermions. . .

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