Comparison of algorithms for solving the sign problem of the finite μ O(3) model in 1+1 dimensions

Csaba Török

Eötvös University, Budapest

In collaboration with S. Katz, F. Niedermayer and D. Nógrádi

LATTICE 2016, Southampton

Motivation

sign problem:
S is complex $\longrightarrow \mathrm{e}^{-S}$ cannot be interpreted as a probability weight
possible solution: Complex Langevin (CL) algorithm

Motivation

sign problem:
S is complex $\longrightarrow \mathrm{e}^{-S}$ cannot be interpreted as a probability weight
possible solution: Complex Langevin (CL) algorithm

Problem with CL: using certain parameters CL results converge to wrong results.

Motivation

sign problem:
S is complex $\longrightarrow \mathrm{e}^{-S}$ cannot be interpreted as a probability weight
possible solution: Complex Langevin (CL) algorithm

Problem with CL: using certain parameters CL results converge to wrong results.

Two examples:

- XY model in 3 dimensions: CL fails at low β, in the broken phase [Aarts et al., 1005.3468]
- HDQCD: CL fails at low temperature, but its failure depends on β [Seiler et al., 1211.3709]

Question: What happens to CL results towards the continuum?

Motivation

sign problem:
S is complex $\longrightarrow \mathrm{e}^{-S}$ cannot be interpreted as a probability weight
possible solution: Complex Langevin (CL) algorithm

Problem with CL: using certain parameters CL results converge to wrong results.

Two examples:
- XY model in 3 dimensions: CL fails at low β, in the broken phase [Aarts et al., 1005.3468]
- HDQCD: CL fails at low temperature, but its failure depends on β [Seiler et al., 1211.3709]

Question: What happens to CL results towards the continuum?

$\longrightarrow$ 1+1 dimensional O(3) model at finite μ

The model

1+1 dimensional O(3) model at finite μ

The action at a finite Λ lattice:

$$S = 2\beta V - \beta \sum_{x\in\Lambda} \left(\phi_{x+\hat{0}} \mathrm{e}^{\mathrm{i}\mu a t_{12}} \phi_x + \phi_{x+\hat{1}} \phi_x \right),$$

constraints:

$$\sum_{i=1}^{3} \phi_{x,i}^2 = 1 \quad \forall x \in \Lambda.$$

$V = N_x \times N_t$, $\beta = 1/g^2$ and t_{12} is the generator of rotations in the xy-plane.

The model has a severe sign problem due to the chemical potential.

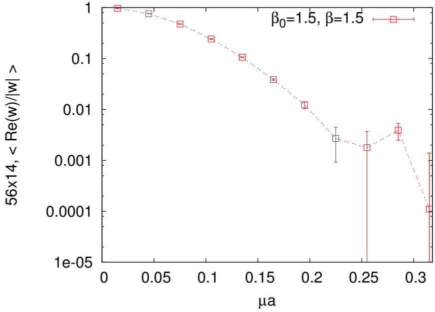
Severity of the sign problem

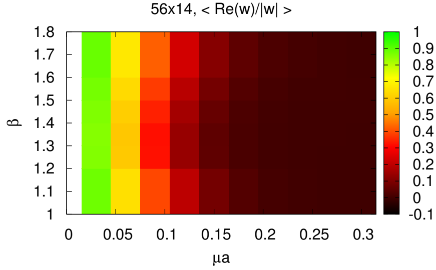
We used multi-parameter reweighting (simulating with the cluster algorithm at $\mu = 0$):

$$\langle O \rangle_{\beta,\mu} = \frac{\langle O w \rangle_{\beta_0,\mu_0=0}}{\langle w \rangle_{\beta_0,\mu_0=0}}, \quad \text{where}$$

$$\langle O \rangle_{\beta,\mu} \equiv \frac{1}{Z} \int \prod_x \mathrm{d}\phi_x \delta(\phi_x^2 - 1) \mathrm{e}^{-S(\beta,\mu)} O$$

$$\text{and} \quad w = \mathrm{e}^{S(\beta_0,\mu_0) - S(\beta,\mu)}.$$





But the sign problem can be eliminated
by using different variables for the model.

Dual variables and worm algorithm

But the sign problem can be eliminated
by using different variables for the model.

[Bruckmann et al., 1507.04253]

Dual variables and worm algorithm

But the sign problem can be eliminated
by using different variables for the model.

[Bruckmann et al., 1507.04253]

$\longrightarrow$ Compare the CL results to worm results.

Dual variables and worm algorithm

But the sign problem can be eliminated
by using different variables for the model.

[Bruckmann et al., 1507.04253]

$\longrightarrow$ Compare the CL results to worm results.

The complex Langevin equation

The CL equation for 3-component scalar fields using discrete timesteps:

$$\phi_{x,i}^{(n+1)} = \phi_{x,i}^{(n)} - \varepsilon_n \frac{\delta S}{\delta \phi_{x,i}}^{(n)} + \sqrt{\varepsilon_n}\eta_{x,i}^{(n)}, \quad i = 1, 2, 3.$$

The noise satisfies

$$\langle \eta_{x,i}(\tau)\eta_{x',j}(\tau') \rangle = 2\delta_{ij}\delta_{xx'}\delta(\tau - \tau'), \quad \langle \eta_{x,i}(\tau) \rangle = 0.$$

Dual variables and worm algorithm

But the sign problem can be eliminated
by using different variables for the model.

[Bruckmann et al., 1507.04253]

$\longrightarrow$ Compare the CL results to worm results.

The complex Langevin equation

The CL equation for 3-component scalar fields using discrete timesteps:

$$\phi_{x,i}^{(n+1)} = \phi_{x,i}^{(n)} - \varepsilon_n \frac{\delta S}{\delta \phi_{x,i}}^{(n)} + \sqrt{\varepsilon_n}\eta_{x,i}^{(n)}, \quad i = 1, 2, 3.$$

The noise satisfies

$$\langle \eta_{x,i}(\tau)\eta_{x',j}(\tau')\rangle = 2\delta_{ij}\delta_{xx'}\delta(\tau - \tau'), \quad \langle \eta_{x,i}(\tau)\rangle = 0.$$

But this equation does not preserve the length of the ϕ vectors.

A) Spherical coordinates

In terms of spherical coordinates $\phi_x = (\sin\vartheta_x \cos\varphi_x, \sin\vartheta_x \sin\varphi_x, \cos\vartheta_x)$, and Z becomes

$$Z = \int \prod_{x_1} \mathrm{d}\varphi_{x_1} \prod_{x_2} \mathrm{d}\vartheta_{x_2} \mathrm{e}^{-\left(S[\varphi,\vartheta] - \sum_x \ln\sin\vartheta_x\right)}$$

$$= \int \prod_{x_1} \mathrm{d}\varphi_{x_1} \prod_{x_2} \mathrm{d}\vartheta_{x_2} \mathrm{e}^{-S_{eff}[\varphi,\vartheta]},$$

Then the discretized complex Langevin steps are

$$\varphi_x(n+1) = \varphi_x(n) + \varepsilon_n K_x^{(\varphi)}(n) + \sqrt{\varepsilon_n}\eta_x^{(\varphi)}(n),$$

$$\vartheta_x(n+1) = \vartheta_x(n) + \varepsilon_n K_x^{(\vartheta)}(n) + \sqrt{\varepsilon_n}\eta_x^{(\vartheta)}(n),$$

where $K_x^{(\varphi)} = -\delta S_{eff}/\delta\varphi_x$ and $K_x^{(\vartheta)} = -\delta S_{eff}/\delta\vartheta_x$.

B) Group space integration

exponentialized
Euler-Maruyama discretization

Using Descartes-coordinates

$$\phi_x = O_x \phi_0, \qquad O_x(n+1) = R_x(\varepsilon_n) O_x(n),$$

where ϕ_0 is fixed, $O_x \in O(3)$ and

$$R_x(\varepsilon_n) = \prod_{a \in (1,2,3)} \exp\left(t_a(\varepsilon_n K_{ax} + \sqrt{\varepsilon_n}\eta_{ax})\right),$$

where the force is

$$K_{ax} = -D_{ax} S[O] = -\partial_\alpha S[\mathrm{e}^{\alpha t_a} O_x]|_{\alpha=0}.$$

Using ϕ_x instead of O_x:

$$\phi_x^{(n+1)} = \prod_{a \in (1,2,3)} \exp\left[(\varepsilon_n K_{ax} + \sqrt{\varepsilon_n}\eta_{ax}) t_a\right] \phi_x^{(n)}.$$

C) Direct method to include the constraint in Descartes-coordinates

standard Euler-Maruyama discretization with Dirac-delta

$$Z = \int \prod_x \mathrm{d}\phi_x \delta(\phi_x^2 - 1) \mathrm{e}^{-S[\phi]} = \int \prod_x \mathrm{d}\phi_x \mathrm{e}^{-(S[\phi] - \sum_y \ln \delta(\phi_y^2 - 1))}.$$

Dirac-delta is approximated with a sharp Gaussian:

$$\frac{1}{\sqrt{2\pi} b} \exp\left\{ - \frac{(\phi_y^2 - 1)^2}{2b^2} \right\} \to \delta(\phi_y^2 - 1), \quad \text{as} \quad b \to 0.$$

The force is then

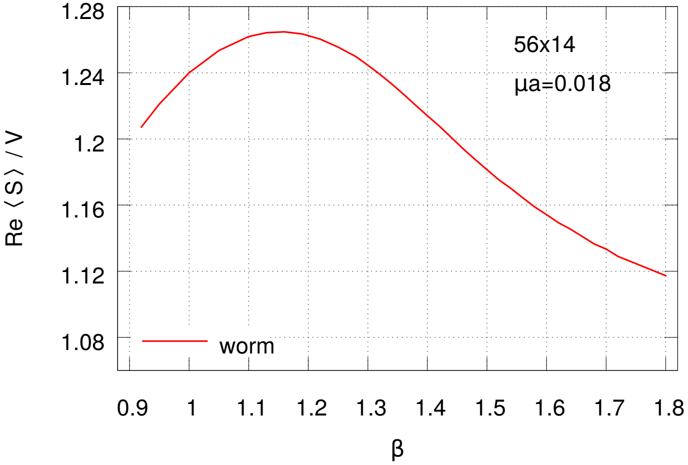
$$K_x = \beta \left(\phi_{x+\hat{0}} \mathrm{e}^{\mathrm{i}\mu a t_3} + \mathrm{e}^{\mathrm{i}\mu a t_3} \phi_{x-\hat{0}} + \phi_{x+\hat{1}} + \phi_{x-\hat{1}} \right) - \frac{2}{b^2} (\phi_x^2 - 1) \phi_x.$$

Then fields evolve according to

$$\phi_{x,i}^{(n+1)} = \phi_{x,i}^{(n)} + \varepsilon_n K_{x,i}^{(n)} + \sqrt{\varepsilon_n} \eta_{x,i}^{(n)}.$$

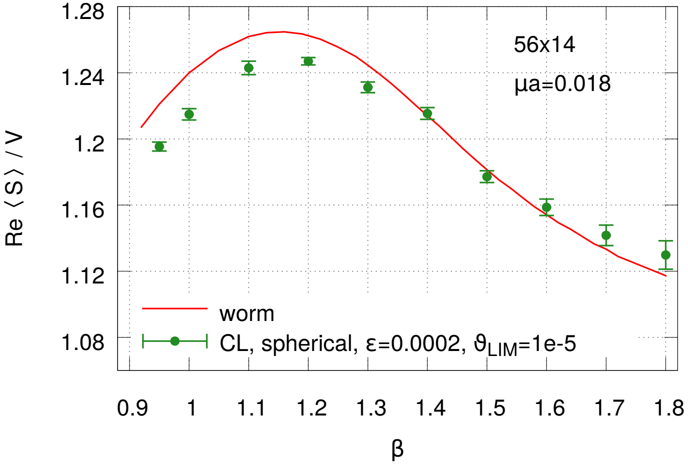
Comparison of the results – the action density

$\mu/T = 0.25$



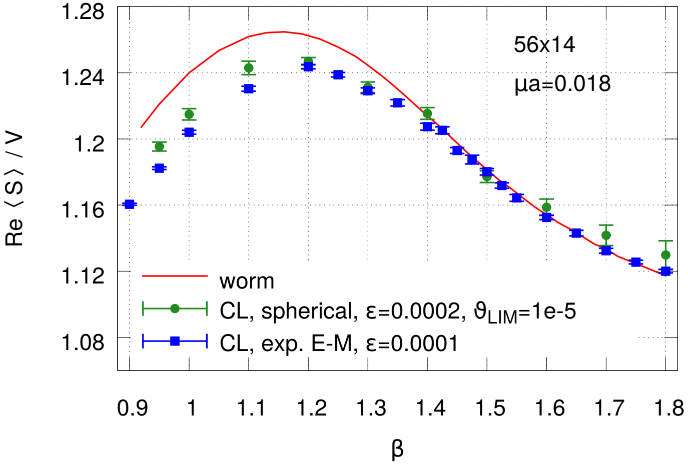
Comparison of the results – the action density

$\mu/T = 0.25$



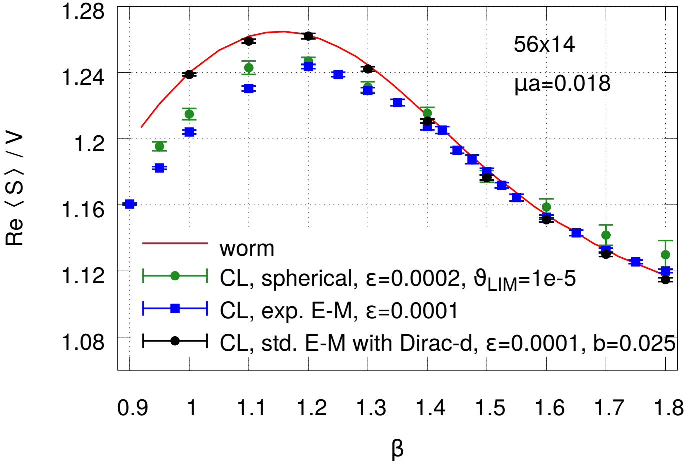
Comparison of the results – the action density

$\mu/T = 0.25$

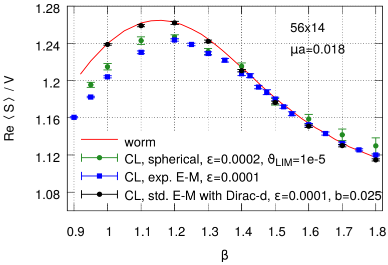


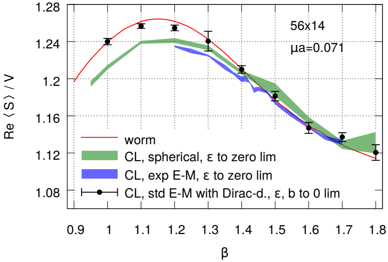
Comparison of the results – the action density

$$\mu/T = 0.25$$



Comparison of the results – the action density





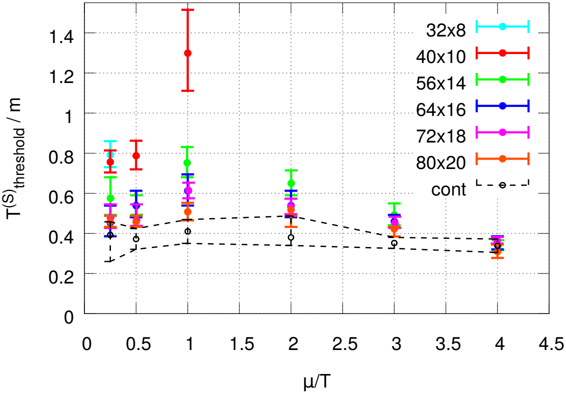
After taking $\varepsilon \to 0$ and $b \to 0$:

- CL, spherical and CL with exp. E-M discr. both fail at low β even when the sign problem is mild
- CL std. E-M discr. with Dirac-delta is correct for all β (but the errors become larger at small betas as the sign problem becomes more severe)

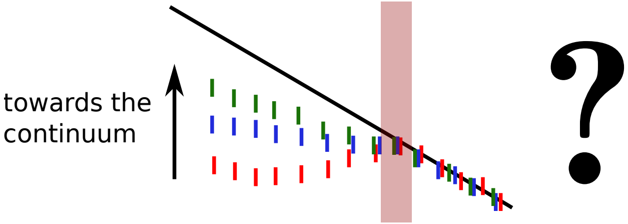
Comparison of the results – the action density

Below $T^{(S)}_{threshold}/m$ the exp. E-M discretization develops wrong results for the action.

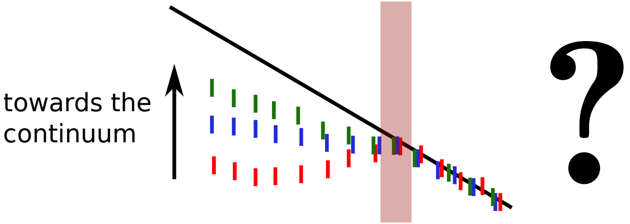
Continuum estimate: $T_{threshold}/m \sim 0.4$.



Further speculation on taking the continuum limit

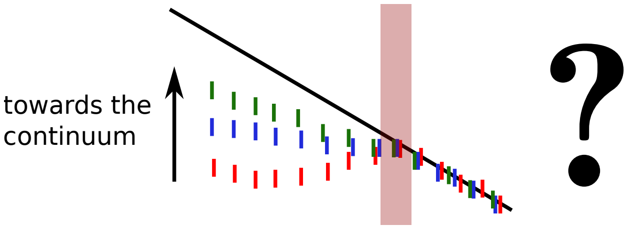


Further speculation on taking the continuum limit



Can wrong results become less and less wrong towards the continuum?

Further speculation on taking the continuum limit

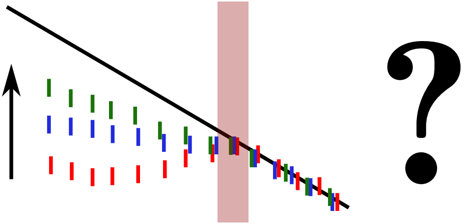


Can wrong results become less and less wrong towards the continuum?

- trace anomaly (θ/T^2)
- particle density (n/m)

Further speculation on taking the continuum limit

towards the continuum



Can wrong results become less and less wrong towards the continuum?

- trace anomaly (θ/T^2)
- particle density (n/m)

Not in this model.

(At least not with these discretizations.)

Comparison of the results – the trace anomaly

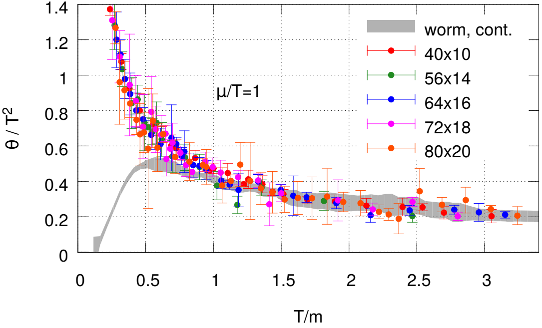
The trace anomaly is:

$$\frac{\theta}{T^2} = \frac{\epsilon - p}{T^2} = -\frac{N_t}{N_x} a \frac{\mathrm{d}\log Z}{\mathrm{d}a} = \frac{N_t}{N_x} \frac{(am)}{\frac{\mathrm{d}(am)}{\mathrm{d}\beta}} \left\langle \frac{S_{ren}}{\beta} \right\rangle .$$

where S_{ren} is the renormalized action

$$\langle S_{ren}(\beta, T, \mu) \rangle = \langle S(\beta, T, \mu) \rangle - \langle S(\beta, T = 0, \mu = 0) \rangle .$$

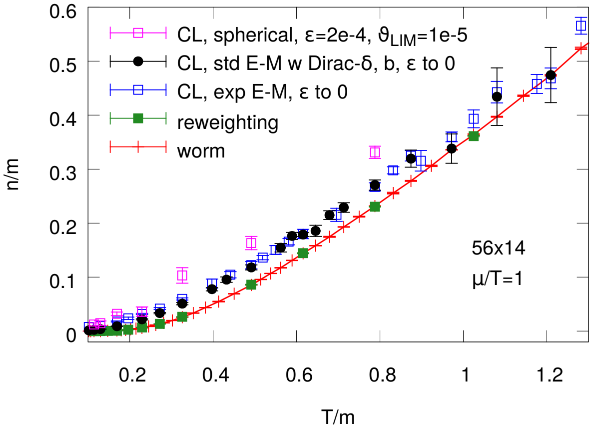
For the exp. E-M discretization:



Comparison of the results – the density

The density is

$$n = \frac{T}{V_{sp}} \frac{\partial \log Z}{\partial \mu} = \frac{T}{V_{sp}} \frac{1}{Z} \frac{\partial Z}{\partial \mu} = m \frac{1}{N_t N_x} \frac{1}{am} \left\langle -\frac{\partial S}{\partial (\mu a)} \right\rangle .$$



Failure of Complex Langevin and the phase diagram

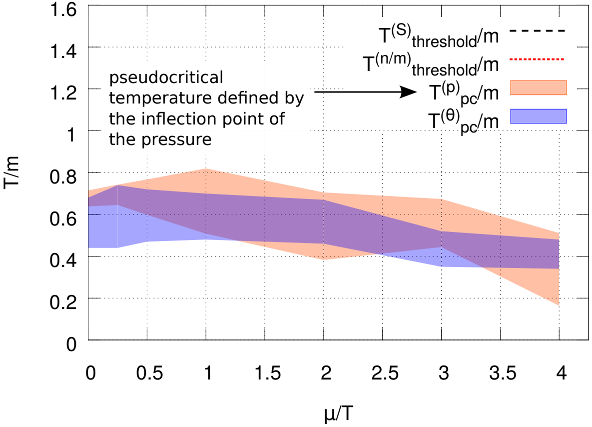
No spontaneous symmetry breaking in 1+1 dimensions at finite temperature

$\longrightarrow$ no phase transition (Coleman–Mermin–Wagner theorem).

Failure of Complex Langevin and the phase diagram

No spontaneous symmetry breaking in 1+1 dimensions at finite temperature

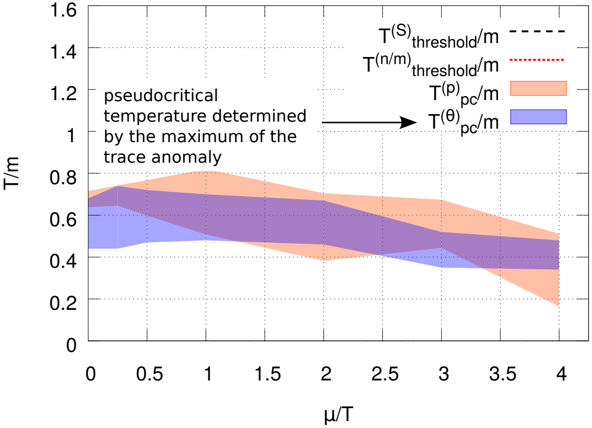
$\longrightarrow$ no phase transition (Coleman–Mermin–Wagner theorem).



Failure of Complex Langevin and the phase diagram

No spontaneous symmetry breaking in 1+1 dimensions at finite temperature

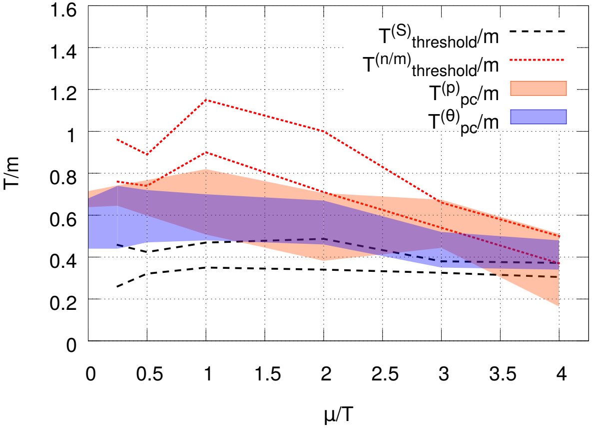
$\longrightarrow$ no phase transition (Coleman–Mermin–Wagner theorem).



Failure of Complex Langevin and the phase diagram

No spontaneous symmetry breaking in 1+1 dimensions at finite temperature

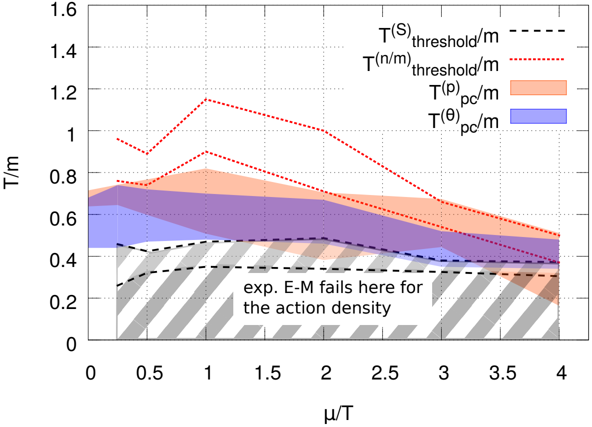
$\longrightarrow$ no phase transition (Coleman–Mermin–Wagner theorem).



Failure of Complex Langevin and the phase diagram

No spontaneous symmetry breaking in 1+1 dimensions at finite temperature

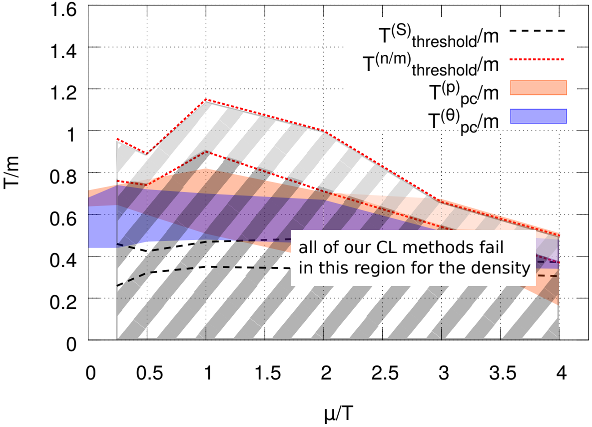
$\longrightarrow$ no phase transition (Coleman–Mermin–Wagner theorem).



Failure of Complex Langevin and the phase diagram

No spontaneous symmetry breaking in 1+1 dimensions at finite temperature

$\longrightarrow$ no phase transition (Coleman–Mermin–Wagner theorem).



Summary

Complex Langevin algorithm was compared to the **worm algorithm** and **reweighting**.

Three different implementations for CL were tested:

A) spherical coordinates,

B) group space integration (exp. E-M),

C) direct method (std. E-M discretization with Dirac-delta).

Results at **low temperature**:

	S/V	θ/T^2	n/m
A)	✗	✗	✗
B)	✗	✗	✗
C)	✓	✓	✗

Continuum limit:
incorrect results at low temperature **do not improve**.

Summary

Complex Langevin algorithm was compared to the **worm algorithm** and **reweighting**.

Three different implementations for CL were tested:

A) spherical coordinates,

B) group space integration (exp. E-M),

C) direct method (std. E-M discretization with Dirac-delta).

Results at **low temperature**:

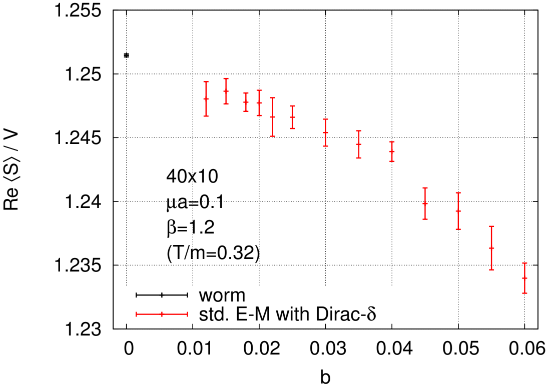
	S/V	θ/T^2	n/m
A)	✗	✗	✗
B)	✗	✗	✗
C)	✓	✓	✗

Continuum limit:
incorrect results at low temperature **do not improve**.

Thank you for your attention!

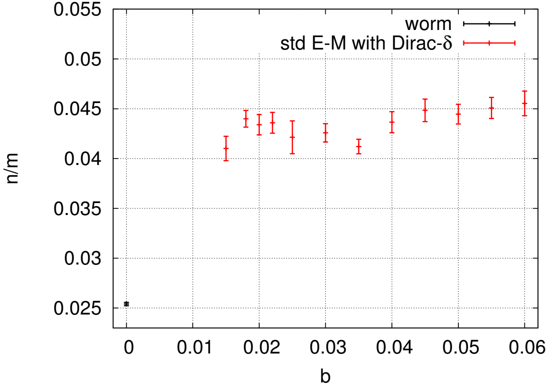
Backup slides

Standard discretization with Dirac-delta – some more figures



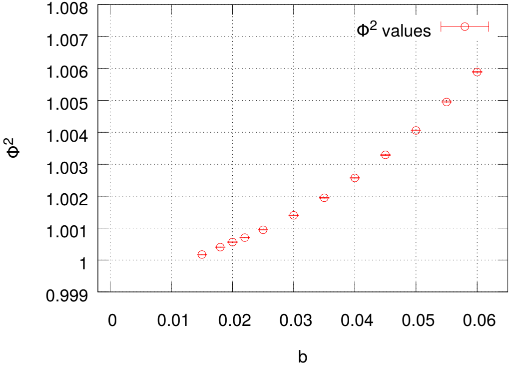
using adaptive stepsize, starting with $\varepsilon = 5 \cdot 10^{-5}$

Standard discretization with Dirac-delta – some more figures



using adaptive stepsize, starting with $\varepsilon = 5 \cdot 10^{-5}$

Standard discretization with Dirac-delta – some more figures



using adaptive stepsize, starting $\varepsilon = 5 \cdot 10^{-5}$