Perturbative running of the twisted Yang-Mills coupling in the gradient flow scheme

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Introduction

We present our ongoing computation of the running of the twisted Yang-Mills coupling using gradient flow techniques.
In particular, we use the gradient flow method with twisted boundary conditions to perform a perturbative expansion of the expectation value of the Yang-Mills energy density up to fourth order at finite flow time, and regularise the resulting integrals.
Additionally, we will show our ongoing computation of the aforementioned integrals in the particular case of a two-dimensional twist.

Starting point

- The hypothesis of volume independence states that that in a SU(N) theory on a *d*-dimensional twisted torus, the physical size and the size of the gauge group are related in such a way that, up to N^{-2} corrections, they are always combined in an effective length:

$$\tilde{l} = L_g l \qquad L_g = N^{\frac{2}{d_t}}$$
$$n_{\mu\nu} = k L_g \epsilon_{\mu\nu}$$

Where *k* and L_g are two coprime integers, and d_t stands for the number of dimensions with twisted boundary conditions. We consider a torus with length *l* in the twisted directions, and $\tilde{l}$ in the non-twisted ones, which yields a volume:

$$V = l^{d_t} \tilde{l}^{d-d_t}$$

- The Twisted Gradient Flow (TGF) scheme was used in order to obtain the running of the renormalised 't Hooft coupling in terms of the effective size and an arbitrary parameter c :

$$\lambda_{TGF}\left(\tilde{l}\right) = \mathcal{N}^{-1}(c) \left.\frac{t^2 \langle E(t)\rangle}{N}\right|_{t=\frac{1}{8}c^2\tilde{l}^2}$$

The normalisation constant was obtained by matching the coupling to the tree level bare one. It is given after some computations by:

$$\mathcal{N}(c) = \frac{c^4 (d-1)}{128 \tilde{l}^{d-4}} \theta_3^{d-d_t}\left(0, i\pi c^2\right) \times \left\{\theta_3^{d_t}\left(0, i\pi c^2\right) - \theta_3^{d_t}\left(0, i\pi c^2 L_g^2\right)\right\}$$

Where θ_3 denotes the Jacobi theta function:

$$\theta_3(0, it) = \sum_{m\in\mathbb{Z}} e^{-\pi t m^2}$$

- *E(t)* is computed in terms of the gradient flow field $B_\mu(x,t)$, which is defined as a field following the flow equations:

$$\partial_t B_\nu(x,t) = D_\mu G_{\mu\nu}(x,t) + D_\mu \partial_\nu B_\nu(x,t)$$

With the initial condition that B_μ matches the usual A_μ gauge field for *t=0* .

Computing <E>/N in perturbation theory

- The observable was computed in perturbation theory up to order g_0^4 :

$$\frac{1}{N}\langle E(t)\rangle = \frac{1}{2N}\langle \mathrm{Tr}\, G_{\mu\nu}(t) G^{\mu\nu}(t)\rangle$$
$$B_\mu(x,t) = \sum_k g_0^k(x,t) B_\mu^{(k)}(x,t)$$

- The fields were be expanded in a basis *Γ(p)* in momentum space with structure constants *F(p,q)*:

$$B_\mu^{(i)}(x,t) = V^{-1/2} \sum_q e^{iqx} B_\mu^{(i)}(q) \hat{\Gamma}(q) \qquad \theta_{\mu\nu} = \frac{\theta \tilde{l}^2}{4\pi^2}\tilde{\epsilon}_{\mu\nu}$$
$$\left[\hat{\Gamma}(p), \hat{\Gamma}(q)\right] = iF(p,q)\hat{\Gamma}(p+q) \qquad \theta = \frac{2\pi\bar{k}}{L_g} = 2\pi\tilde{\theta}$$
$$F(p,q) = -\sqrt{\frac{2}{N}} \sin\left(\frac{1}{2}\theta_{\mu\nu} p_\mu q_\nu\right) \qquad \tilde{\epsilon}_{\mu\nu}\epsilon_{\nu\lambda} = \delta_{\mu\lambda}$$
$$k\bar{k} = 1 \bmod L_g$$

- The observable can be expressed as a combination of several integrals:

$$\frac{1}{N}\langle E(t)\rangle = \frac{\lambda_0 c_0}{V_{eff}}\left(1 + \lambda_0 \sum_{i=1}^{15} c_i I_i\right)$$
$$\lambda_0 = g_0^2 N \qquad V_{eff} = \tilde{l}^d$$

A few representative examples of these integrals include:

$$I_1 = \mathcal{A}\int_0^\infty dx_1 \int_0^\infty dx_2 e^{-(2t+x_1)p^2 - (2t+x_2)r^2 - t(2p\cdot r)}$$
$$I_4 = \mathcal{A}\int_0^\infty dx_1 \int_0^t ds e^{-2tp^2 - (2t+x_1)r^2 - (2t-s)(2p\cdot r)}$$
$$I_7 = \mathcal{A}\int_0^t s ds e^{-2tp^2 - 2tr^2 - (2t-s)(2p\cdot r)}$$
$$I_{13} = \mathcal{A}\int_0^1 dx \int_0^t s ds e^{-(2t+2sx)p^2 - 2tr^2 - (2t+s(x-1))(2p\cdot r)}$$
$$\mathcal{A} = \frac{1}{V_{eff}} \sum_{pr} N F^2(r,p)$$

The presence of the *NF²(r,p)* terms will be crucial to what follows, and specific to finite volume.

- After replacing *t* by its value in terms of *c*, and rescaling the integration variables, the integrals can be rewritten in terms of Siegel theta functions:

$$\Theta\left(0|iA\left(s,u,v,\tilde{\theta}\right)\right) = \sum_{M\in\mathbb{Z}^{2d}} \exp\left\{-\pi M^t A M\right\}$$
$$A\left(s,u,v,\tilde{\theta}\right) = \frac{\pi c^2}{2}\begin{pmatrix} s\mathbb{I}_d & v\mathbb{I}_d + i\tilde{\theta}\tilde{\epsilon} \\ v\mathbb{I}_d - i\tilde{\theta}\tilde{\epsilon} & u\mathbb{I}_d \end{pmatrix}$$

The theta functions can then be reabsorbed into an auxiliary function:

$$F_c\left(s,u,v,\tilde{\theta}\right) = \frac{c^4}{64\tilde{l}^{d-4}} \mathrm{Re}\left(\Theta\left(0|iA(s,u,v,0)\right) - \Theta\left(0|iA\left(s,u,v,\tilde{\theta}\right)\right)\right)$$

- This structure of a difference of two theta functions comes directly from expanding the sin function from the *F²(r,p)* form factors in complex exponentials.

Regularisation

- In terms of F_c functions and after some algebra, some of the previous examples of the integrals to compute read:

$$I_1 = \int_0^1 dx \int_0^\infty y dy F_c(2+xy, 2+(1-x)y, 1)$$
$$I_4 = \int_0^1 dx \int_0^\infty dy F_c(2, 2x+y, x)$$
$$I_7 = \int_0^1 x dx F_c(2, 2x, x)$$
$$I_{13} = \int_0^1 dx \int_0^1 y dy F_c(2, 2y, (x-1)y)$$

- There is an alternative formula for the theta functions using the inverse:

$$\Theta\left(0|iA\left(s,u,v,\tilde{\theta}\right)\right) = (\det A)^{-1/2} \sum_M e^{-\pi M^t A^{-1} M}$$

This diverges for *det A=0,* which occurs for *u=v=0* (or in points that can be taken there through a momentum shift).

- In order to make the divergence more manifest, we partially inverted the theta function:

$$\Theta\left(0|iA\left(s,u,v,\tilde{\theta}\right)\right) = (\hat{c}u)^{-\frac{d}{2}} \sum_m \exp\left\{-\pi\hat{c}sm^2\right\} \times \sum_n \left\{-\frac{\pi}{\hat{c}u}\left(n - icvm - \tilde{\theta}\tilde{\epsilon}m\right)^2\right\}$$
$$M = \begin{pmatrix} m \\ n \end{pmatrix} \qquad \hat{c} = \frac{1}{2}\pi c^2$$

- And then the divergence occurs for *n=0* and for two different situations:
  - $\tilde{\theta} = 0$
  - $m = L_g \hat{m} \qquad \hat{m}\in\mathbb{Z}$

- Defining:

$$H\left(s,u,v,\tilde{\theta}\right) = \sum_{m\neq\hat{m}L_g} e^{-\pi\hat{c}\left(sm^2 + un^2 + v(2m\cdot n) + 2i\tilde{\theta}m\tilde{\epsilon}n\right)}$$
$$H_{div}(s,u,v) = (\hat{c}u)^{-\frac{d}{2}} \sum_{m\neq\hat{m}L_g} e^{-\pi\hat{c}\frac{su-v^2}{u}m^2}$$

- The finite part is then given by:

$$F_c^{fin}\left(s,u,v,\tilde{\theta}\right) = \frac{c^4}{64\tilde{l}^{d-4}}\left(H(s,u,v,0) - H_{div}(s,u,v) - H\left(s,u,v,\tilde{\theta}\right)\right)$$

- Dimensional regularisation is feasible by expressing the divergent part in terms of Jacobi theta functions. For instance, for a four-dimensional twist:

$$F_c^{div}(s,u,v) = (\hat{c}u)^{-\frac{d}{2}}\left\{\theta_3^d(0|a) - \theta_3^d\left(0|aL_g^2\right)\right\}$$
$$a = \frac{\hat{c}}{u}\left(su - v^2\right) \qquad d = 4 - 2\epsilon$$

- In dimensional regularisation, the term proportional to 1/ε reproduces the universal one-loop divergence of the bare coupling as expected.

- A determination of the *Λ* parameter in this scheme is ongoing.

References

- **A. González-Arroyo, M. Okawa**, *Twisted-Eguchi-Kawai model: A reduced model for large-N lattice gauge theory*, Phys. Rev. D 27, 2397 (1983)
- **R. Narayanan and H. Neuberger**, *Infinite N phase transitions in continuum Wilson loop operators*, JHEP 0603 (2006) 064, arXiv:hep-th/0601210
- **A. González-Arroyo, M. Okawa**, *Large N reduction with the Twisted Eguchi-Kawai model*, HEP 1007 (2010) 043, arXiv:1005.1981
- **M. Lüscher**, *Trivializing maps, the Wilson flow and the HMC algorithm*, 293 (2010) 899–919, arXiv:0907.5491
- **M. Lüscher**, *Properties and uses of the Wilson flow in lattice QCD*, JHEP 1008 (2010) 071, arXiv:1006.4518
- **R. Lohmayer and H. Neuberger**, *Continuous smearing of Wilson Loops*, PoS LATTICE2011 (2011) 249, arXiv:1110.3522
- **A. Ramos**, *The gradient flow running coupling with twisted boundary conditions*, JHEP 1411 (2014) 101, arXiv:1409.1445
- **M. García Pérez, A. González-Arroyo and M. Okawa**, *Volume independence for Yang–Mills fields on the twisted torus*, Int.J.Mod.Phys. A29 (2014) no.25, 1445001 (2014), arXiv:1406.5655
- **M. García Pérez, A. González-Arroyo and M. Okawa**, *The SU(∞) twisted gradient flow running coupling*, JHEP 1501 (2015) 038, arXiv:1412.0941
- **L. Keegan and A. Ramos**, *(Dimensional) twisted reduction in large N gauge theories*, PoS LATTICE2015 (2016) 290, arXiv:1510.08360

Numerical Simulations

- In order to compute the integrals, a numeric code using trapezoid integration was prepared to compute and integrate the F_c functions and their flow time derivatives.
- The simulations were run for a two-dimensional twist, and for $c = 0.7,\ \bar{k} = 1,\ L_g = 3$
- Some temporary results include:

$$I_7 = \int_0^1 x dx F_c(2, 2x, x) = 0.008538263(9)$$
$$I_7 = \int_0^1 x dx F_c(2, 2, x) = 0.003242711(3)$$
$$I_{13} = \int_0^1 dx \int_0^1 y dy F_c(2, 2y, (x-1)y) = 0.00541510(13)$$

- The computation of the rest of the integrals is ongoing.