Heavy-heavy current improvement for calculation of $\bar{B} \to D^{(*)}\ell\bar{\nu}$ semi-leptonic form factors using the Oktay-Kronfeld action

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Abstract

The Oktay-Kronfeld action is a highly improved version of the Fermilab action and systematically reduces heavy quark discretization effects through $\mathcal{O}(\lambda^3)$ in HQET power counting, for the heavy-light meson spectrum. To calculate $\bar{B} \to D^{(*)}\ell\bar{\nu}$ semi-leptonic form factors using Oktay-Kronfeld heavy quarks, we need to improve the heavy quark currents to the same level. We report our progress in calculating the improvement coefficients for currents composed of bottom and charm quarks.

Physical motivation

- Decay rate for $\bar{B} \to D^{(*)}\ell\bar{\nu}$ are proportional to $|V_{cb}|^2$.

$$\frac{d\Gamma}{dw}(\bar{B} \to D^{(*)}\ell\bar{\nu}) = |V_{cb}|^2|F_{D^{(*)}}(w)|^2 \times \text{Const.}$$

$$\times \text{ electroweak.} \times \text{ kinematic term.}$$

- By calculating semi-leptonic form factors $h_\pm(w), h_{A_i}(w), h_V(w)$ from lattice, we can determine $|V_{cb}|$ using experimental measurements of decay rate. $|F_{D^{(*)}}(w)|^2$ is expressed in terms of $h_\pm(w)$, $h_{A_i}(w)$, $h_V(w)$.

$$\frac{\langle D(p_D)|V^\mu|B(p_B)\rangle}{\sqrt{M_D M_B}} = (v_B + v_D)^\mu h_+(w) + (v_B - v_D)^\mu h_-(w)$$

$$\frac{\langle D^*(p_{D^*},\epsilon)|A^\mu|B(p_B)\rangle}{\sqrt{M_{D^*} M_B}} = i[\epsilon^{*\mu} h_{A_1}(w)(1+w) - (\epsilon^* \cdot v_B)(v_B^\mu h_{A_2}(w) + v_{D^*}^\mu h_{A_3}(w))]$$

$$\frac{\langle D^*(p_{D^*},\epsilon)|V^\mu|B(p_B)\rangle}{\sqrt{M_{D^*} M_B}} = \varepsilon^{\mu\nu\rho\sigma}\epsilon_\nu^*(v_B)_\rho(v_{D^*})_\sigma h_V(w),$$

where v_B and v_D is hadron velocity and $w = v_B \cdot v_D$.

Improvement for heavy quark on lattice

- In the case of heavy quark ($m_q a \sim 1$), there needs systematic way to control the discretization errors.
- The discretization errors from heavy quarks can be controlled systematically by lifting axis-interchange symmetry, retaining Wilson's time derivative, and introducing irrelevant operators. [1]
- For relativistic quarks ($m_q a \ll 1$), it reduces to ordinary Symanzik improvement.

Action

- Fermilab action.[1]

$$S_0 = a^4 \sum_x \bar{\psi}(x)[m_0 + \gamma_4 D_4 + \zeta \vec{\gamma} \cdot \vec{D}]\psi(x) - \frac{1}{2} a^5 \sum_x \bar{\psi}(x)[\Delta_4 + r_s \zeta \Delta^{(3)}]\psi(x)$$

$D_\mu = (T_\mu - T_{-\mu})/2a$, $\Delta_\mu = (T_\mu + T_{-\mu} - 2)/a^2$, $\Delta^{(3)} = \sum_{i=1}^{3} \Delta_i$
It is generalized Wilson action which allows time-space axis-interchange asymmetry.

- Dimension-five interactions.

$$S_B = -\frac{1}{2} c_B \zeta a^5 \sum_x \bar{\psi}(x) i \vec{\Sigma} \cdot \vec{B} \psi(x)$$

$$S_E = -\frac{1}{2} c_E \zeta a^5 \sum_x \bar{\psi}(x) \vec{\alpha} \cdot \vec{E} \psi(x)$$

where, $B_i = \frac{1}{2}\epsilon_{ijk} F_{jk}$, $E_i = F_{4i}$, $F_{\mu\nu}$ is field strength with four-leaf clover terms.

Action

- Oktay-Kronfeld(OK) action [2] contains six more non-zero couplings at dimension 6 and 7 to improve in the level of $\mathcal{O}(\lambda^3)$.
  $(\lambda \sim a\Lambda, \Lambda/m_q)$ at tree-level.

$$
\begin{aligned}
S_6 &= a^6 \sum_x \bar{\psi}(x)\big[c_1 \sum_i \gamma_i D_i \Delta_i + c_2\{\vec{\gamma}\cdot\vec{D}, \Delta^{(3)}\}\big]\psi(x) \\
&\quad + a^6 \sum_x \bar{\psi}(x)\big[c_3\{\vec{\gamma}\cdot\vec{D}, i\vec{\Sigma}\cdot\vec{B}\} + c_{EE}\{\gamma_4 D_4, \vec{\alpha}\cdot\vec{E}\}\big]\psi(x) \\
S_7 &= a^7 \sum_x \bar{\psi}(x) \sum_i \Big[c_4 \Delta_i^2 \psi(x) + c_5 \sum_{j\neq i}\{i\Sigma_i B_i, \Delta_j\}\Big]\psi(x)
\end{aligned}
$$

- Coefficients c_i are fixed by matching dispersion relation, one-gluon vertex, Compton scattering.

Heavy-heavy current

- The improved lattice operator can be constructed by linear combination of lattice operators with the same quantum number as continuum operator.
- In the case of flavor-changing bilinear,[1]

$$\bar{c}\Gamma b = Z_\Gamma \left[\bar{\psi}_c \Gamma \psi_b + \sum_{d\geq 4} \sum_i C^d_{\Gamma i} O^d_{\Gamma i} \right],$$

Z_Γ : matching factor.
$\bar{c}\Gamma b$: continuum bilinear.
$\bar{\psi}_c \Gamma \bar{\psi}_b$: leading lattice bilinear.
$O^d_{\Gamma i}$: higher dimensional lattice operator with dimension d.

- By matching on-shell matrix elements, we obtain coefficients Z_Γ and $C^d_{\Gamma i}$.
- We need improved current in the level of $\mathcal{O}(\lambda^3)$ in HQET power counting.

Heavy-heavy current

- For λ^3 improvement, we define improved field as follows [3], $(a = 1)$

$$\begin{aligned}\Psi_I(x) = & e^{M_1/2}\Big[1 + d_1\vec{\gamma}\cdot\vec{D} + \frac{1}{2}d_2\Delta^{(3)} + \frac{1}{2}id_B\vec{\Sigma}\cdot\vec{B} + \frac{1}{2}d_E\vec{\alpha}\cdot\vec{E} \\ & + d_{r_E}\{\vec{\gamma}\cdot\vec{D},\vec{\alpha}\cdot\vec{E}\} + d_{z_E}\gamma_4(\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D}) \\ & + \frac{1}{6}d_3\gamma_i D_i\Delta_i + \frac{1}{2}d_4\{\vec{\gamma}\cdot\vec{D},\Delta^{(3)}\} + d_5\{\vec{\gamma}\cdot\vec{D}, i\vec{\Sigma}\cdot\vec{B}\} \\ & + d_{EE}\{\gamma_4 D_4,\vec{\alpha}\cdot\vec{E}\} + d_{z_3}\vec{\gamma}\cdot(\vec{D}\times\vec{B} + \vec{B}\times\vec{D})\Big]\psi(x)\end{aligned}$$

- We can define flavor-changing current using improved quark field,[1]

$$\bar{\psi}_c\Gamma\psi_b \to \bar{\Psi}_{cI}\Gamma\Psi_{bI},$$

  where Γ is Dirac structure of the current.
- Then, obtaining improved flavor-changing current is equivalent to obtaining improvement parameters d_i which suffice to match on-shell matrix element to continuum QCD.

Matching of two-quark matrix element

- $\langle c(\eta_c, p_c)|\bar{\Psi}_{cI}\Gamma\Psi_{bI}|b(\eta_b, p_b)\rangle_{\mathsf{lat}} = \langle c(\eta_c, p_c)|\bar{c}\Gamma b|b(\eta_b, p_b)\rangle_{\mathsf{cont}}$

  At tree-level [1],

  $$\begin{aligned} &\to \mathcal{N}_c(p_c)\mathcal{N}_b(p_b)\bar{u}_c^{\mathsf{lat}}(\eta_c, p_c)\bar{R}_c^{(0)}(p_c)\Gamma R_b^{(0)}(p_b)u_b^{\mathsf{lat}}(\eta_b, p_b) \\ &= \sqrt{\frac{m_c}{E_c}}\sqrt{\frac{m_b}{E_b}}\bar{u}_c(\eta_c, p_c)\Gamma u_b(\eta_b, p_b), \end{aligned}$$

  where $R^{(0)}(p)$ is the zero-gluon vertex of improved current.

- Expand continuum and lattice spinor in $\mathcal{O}(\vec{p}^{\,3})$ [3], $(a = 1)$

$$\sqrt{\frac{m}{E}}u(\eta, p) = \Big[1 - \frac{i\vec{\gamma}\cdot\vec{p}}{2m} - \frac{\vec{p}^{\,2}}{8m^2} + \frac{3i(\vec{\gamma}\cdot\vec{p})\vec{p}^{\,2}}{16m^3}\Big]u(\eta, 0) + \mathcal{O}(\vec{p}^{\,4})$$

$$\mathcal{N}(\vec{p})u^{\mathsf{lat}}(\eta, p) = \Big[1 - \frac{i\zeta\vec{\gamma}\cdot\vec{p}}{2\sinh M_1} - \frac{\vec{p}^{\,2}}{8M_X^2} + \frac{1}{6}iw_3\sum_{k=1}^{3}\gamma_k p_k^3 + \frac{3i(\vec{\gamma}\cdot\vec{p})\vec{p}^{\,2}}{16M_Y^3}\Big]u(\eta, 0) + \mathcal{O}(\vec{p}^{\,4})$$

Matching of two-quark matrix element

- M_X, M_Y, and rotation breaking parameter w_3 are expressed by couplings of action [1, 3],

$$\frac{1}{M_X^2} = \frac{\zeta^2}{\sinh^2 M_1} + \frac{2r_s\zeta}{e^{M_1}}, \qquad w_3 = \frac{3c_1 + \zeta/2}{\sinh M_1},$$
$$\frac{1}{M_Y^3} = \frac{8}{3\sinh M_1}\Big\{2c_2 + \frac{1}{4}e^{-M_1}\Big[\zeta^2 r_s(2\coth M_1 + 1) + \frac{\zeta^3}{\sinh M_1}\Big(\frac{e^{-M_1}}{2\sinh M_1} - 1\Big)\Big] + \frac{\zeta^3}{4\sinh^2 M_1}\Big\}.$$

- The improvement vertex remedies mismatch of continuum and lattice spinor in order $\mathcal{O}(\vec{p}^{\,3})$.
  $\rightarrow$ obtain d_1, d_2, d_3, d_4.

Matching of four-quark matrix element

- To obtain improvement parameters other than d_1, d_2, d_3, d_4, we are considering matrix elements.

$$\langle \ell(\eta_2, p_2) u(\eta', p') | \bar{\psi}_u \Gamma \Psi_{bI} | b(\eta, p) \ell(\eta_1, p_1) \rangle_{\text{lat}}$$

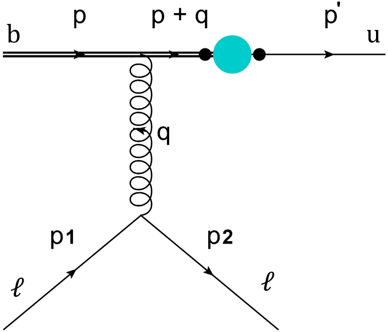

Figure: A diagram for gluon exchange from the OK action one-gluon vertex of b-quark for improved flavor-changing current at tree-level. The large circles represent electroweak current insertions, and the small dots represent field improvement.

Matching of four-quark matrix element

- The matching equation for the one-gluon exchange from b-quark line gives constraint on the improvement parameter d_i,

$$
\begin{aligned}
n_\mu(q)\Big[R_b^{(0)}(p+q)S^{\mathsf{lat}}(p+q)(-gt^a)\Lambda_\mu(p+q,p) \\
+ R_{b\mu}^{(1)}(-gt^a)(p+q,p)\Big]\mathcal{N}_b(p)u_b^{\mathsf{lat}}(\eta,p) \\
= \frac{m_b - i\gamma\cdot(p+q)}{m_b^2+(p+q)^2}\gamma_\mu\sqrt{\frac{m_b}{E_b}}u_b(\eta,p).
\end{aligned}
$$

p, q : momentum of b-quark and gluon, respectively.
$\mathcal{N}_b$: lattice spinor normalization factor for OK action.
$n_\mu(q)$: wave-function factor for gluon.
$R_b^{(0)}$, $R_{b\mu}^{(1)}$: zero and one-gluon vertex for improved field which contains d_i.
Λ_μ : one-gluon vertex of OK-action.
$S^{\mathsf{lat}}(p+q)$: b-quark propagator for OK action.

- We assume $m_b, 1/a \gg \vec{p}, q_\mu$, $m_b a \simeq 1$ and take expansion over $\frac{\vec{p}}{m_b}$, $\frac{q_\mu}{m_b}$ and $\vec{p}a$, $q_\mu a$.

Matching of four-quark matrix element

- The matching condition gives constraints on improvement parameters d_i in terms of coefficients in the OK action : r_s, ζ, c_i, m_0.
- In the zeroth order of $\vec{p}$, we obtain constraint on d_i as follows,
    - $\mathcal{O}(\vec{q}/q_4)$ term from spatial component gives constraint on d_1.
    - $\mathcal{O}(\vec{q})$ term from spatial and temporal component gives constraint on d_2 and d_E, respectively.
    - $\mathcal{O}(\vec{q}^2)$ and $\mathcal{O}(q_4\vec{q})$ term from temporal component gives constraint on $d_{r_E} - d_{z_E}$ and d_{EE}, respectively.
    - $\mathcal{O}(\vec{q}^2)$ term from spatial component gives constraint on $d_5 - d_{z_3}$.

Results from four-quark matrix element

- We obtain preliminary results for d_E, $d_{r_E} - d_{z_E}$, d_{EE}.

$$
\begin{aligned}
d_E &= \frac{1}{2m_b^2} - \frac{\zeta(1+m_0)(m_0^2+2m_0+2)}{[m_0(2+m_0)]^2} + \frac{\zeta(1+m_0)(1-c_E)}{m_0(2+m_0)}, \\
d_{r_E} - d_{z_E} &= -\frac{1}{8m_b^3} + \frac{r_s\zeta}{24(1+m_0)} + \frac{\zeta c_{EE}(2+2m_0+m_0^2)}{2m_0(1+m_0)(2+m_0)} + \frac{\zeta^2 c_E(2+2m_0+m_0^2)}{[2m_0(2+m_0)]^2} \\
&\quad + \frac{\zeta^2(12+24m_0+16m_0^2+4m_0^3+m_0^4)}{12m_0^3(2+m_0)^3} - d_1 \frac{\zeta(1+m_0)[2+m_0(2+m_0)c_E]}{2m_0^2(2+m_0)^2}, \\
d_{EE} &= \frac{1+m_0}{(m_0^2+2m_0+2)}\Big[-\frac{1}{4m_b^3} + \frac{\zeta(1+m_0)(m_0^2+2m_0+2)}{[m_0(2+m_0)]^3} \\
&\quad + \frac{\zeta c_E(1+m_0)}{[m_0(2+m_0)]^2} + \frac{(2+2m_0+m_0^2)c_{EE}}{m_0(2+m_0)}\Big]
\end{aligned}
$$

Consistency checks

- We cross-check by performing two independent calculations by JAB and JL.
- We are checking whether the improved current satisfies matching condition from four quark matrix element up to third order of $\vec{p}$ and q_μ.
- In the calculation of zeroth order in $\vec{p}$, we find that identical equations appear in temporal and spatial one-gluon component in many cases. (d_E, $d_{r_E} - d_{z_E}$, and d_{EE})
- The constraints are consistent with the results for d_{1-4} from matching the two-quark matrix elements [3] and the results for c_{1-5}, c_B, and c_E from matching the Oktay-Kronfeld action [2].

Summary

- We obtain preliminary results for d_E, d_{EE}, $d_{r_E} - d_{z_E}$ by matching four-quark flavor-changing matrix elements.
- In addition to them, we obtain d_B and $d_5 - d_{z_3}$, but they are not confirmed yet.
- We expect that there is redundant operator in improved current and are working to figure it out.
- Tuning these improvement parameters d_i is sufficient to satisfy four-quark matrix elements matching up to λ^3-order.
- By using improved current, we expect to reduce the discretization error of hadronic matrix elements for $\bar{B} \to D^{(*)}\ell\bar{\nu}$ decay in λ^3-order in HQET.

Bibliography

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