Non-perturbative determination of improvement coefficients using coordinate space methods in $N_f = 2 + 1$ lattice QCD

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work done in collaboration with Gunnar Bali for the RQCD collaboration based on arXiv: 1607.07090

34rd International Symposium on Lattice Field Theory, Southampton, UK
Improvement of LQCD with Wilson fermions

Wilson fermions introduce cut-off effects linear in the lattice spacing \( a \).

One can account for them following Symanzik improvement programme

\[
S_{\text{QCD}}(a(\beta)) = S_{\text{continuum}} + aS_1 + a^2S_2 + \ldots
\]

Improvement of the action requires knowledge of the \( c_{\text{SW}} \) coefficient and of \( b_g \) and \( b_m \) which are proportional to the quark mass.

Apart of the action itself we want to use improved operators. We usually define

\[
A_{\mu}^{jk, I}(x) = \bar{\psi}_j(x)\gamma_\mu\gamma_5\psi_k(x) + ac_A\partial_\mu^{\text{sym}}P^{jk}(x)
\]

and

\[
A_{\mu}^{jk, R}(x) = Z_A(1 + a b_A m_{jk} + a3\bar{b}_A \bar{m})A_{\mu}^{jk, I}(x)
\]

⇒ full, non-perturbative improvement of Wilson fermions needs \( c_{\text{SW}} \), \( b_g \), \( c_J \), \( b_J \).
Introduction

**CLS ensembles**

The CLS initiative is currently generating ensembles with $N_f = 2 + 1$ flavours of non-perturbatively improved Wilson Fermions and the tree-level Lüscher-Weisz gauge action at $\beta = 3.4, 3.46, 3.55$ and $3.7$. This corresponds to lattice spacings of $a \in [0.05, 0.09]$ fm.

**Improvement coefficients**

- $c_{SW} \rightarrow$ Bulava, Schaefer, ’13
- $c_A \rightarrow$ Bulava, Della Morte, Heitger, Wittemeier ’15
- $b_{\Gamma}, \tilde{b}_{\Gamma} \rightarrow$ this talk

**Mass dependent improvement coefficients**

We use coordinate space method proposed by Martinelli et al. (Phys. Lett. B 411, 141 (1997)) to determine $b_J, \tilde{b}_J$ for flavour non-singlet scalar, pseudoscalar, vector and axialvector currents.
We denote quark mass averages as

\[ m_{jk} = \frac{1}{2} (m_j + m_k), \]

where

\[ m_j = \frac{1}{2a} \left( \frac{1}{\kappa_j} - \frac{1}{\kappa_{\text{crit}}} \right). \]

The mass dependence of physical observables can be parameterized in terms of the average quark mass

\[ \bar{m} = \frac{1}{3} (m_s + 2m_{\ell}). \]

We define connected Euclidean current-current correlation functions in a continuum renormalization scheme \( R \), e.g., \( R = \overline{\text{MS}} \), at a scale \( \mu \):

\[ G^R_{j(k)}(x, m_{\ell}, m_s; \mu) = \left\langle \Omega \left| T J^{(jk)}(x) J^{(jk)}(0) \right| \Omega \right\rangle^R. \]
Two observations

1) The continuum correlation function differs from that of the massless case by mass dependent terms

\[ G^R_{J(jk)}(x, m_\ell, m_s; \mu) = G^R_{J(jk)}(x, 0, 0; \mu) \]
\[ \times \left[ 1 + \mathcal{O} \left( m^2 x^2, m^2 \langle FF \rangle x^6, m \langle \bar{\psi} \psi \rangle x^4, m \langle \bar{\psi} \sigma F \psi \rangle x^6 \right) \right] , \]

2) The continuum Green function \( G^R \) above can be related to the corresponding Green function \( G \) obtained in the lattice scheme at a lattice spacing \( a = a(g^2) \) as follows:

\[ G^R_{J(jk)}(x, m_\ell, m_s; \mu) = (Z^R_J)^2 (\tilde{g}^2, a \mu) \]
\[ \times \left( 1 + 2b_J am_{jk} + 6\tilde{b}_J a\bar{m} \right) G_{J(jk),l}(n, am_{jk}, a\bar{m}; g^2) , \]
\[
\frac{G_{J(jk)}(n, \rho, \bar{m}; g^2)}{G_{J(rs)}(n, \rho, \bar{m}; g^2)} = 1 + 2b_Ja \begin{pmatrix} m_{rs}^{(\sigma)} - m_{jk}^{(\rho)} \end{pmatrix} + 6\tilde{b}_Ja \begin{pmatrix} \bar{m}^{(\sigma)} - \bar{m}^{(\rho)} \end{pmatrix} + O(a^2, x^2),
\]

We can construct two useful observables:

\[
R_J(x, \delta m) \equiv \frac{G_{J(12)}(n, \rho, \bar{m}; g^2)}{G_{J(13)}(n, \rho, \bar{m}; g^2)} = 1 + 2b_Ja\delta m
\]

\[
\tilde{R}_J(x, \delta \bar{m}) \equiv \frac{G_{J(12)}(n, \bar{m}; g^2)}{G_{J(12)}(n, \bar{m}; g^2)} = 1 + (2b_J + 6\tilde{b}_J)a\delta \bar{m}
\]

⇒ Let’s see how this works in practice!
Overview of CLS ensembles at $\beta = 3.4$.

The graph illustrates the relationship between $t_0(2m^2_K - m^2_\pi)$ and $m_s$, showing a decrease as $t_0 m^2_\pi$ increases.

- $m_s = \text{const.}$
- Physical point
- $m_l = m_s$
- $TrM = \text{const.}$
List of ensembles

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>name</th>
<th>$\kappa_l$</th>
<th>$\kappa_s$</th>
<th># conf.</th>
<th>step</th>
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<td>3.7</td>
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<td>0.1367546608</td>
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<td>10</td>
</tr>
</tbody>
</table>
Example: $R_P - 1$ for $n = (0, 1, 1, 1)$
Example: $\tilde{R}_P - 1$ for $n = (0, 1, 2, 2)$
Tree-level cut-off effects

![Graph showing the relationship between $\delta_J$ and $x/a$ with data points for different values of $J$.]
Applying OPE to the ratio of correlation functions one gets

\[
\frac{G_J^{(12)}(x)}{G_J^{(34)}(x)} = 1 + (A_J^{12} - A_J^{34})x^2 \\
+ \left[ (A_J^{34})^2 - A_J^{12}A_J^{34} + B_J^{12} - B_J^{34} \right] x^4 + \cdots ,
\]

with the mass dependent coefficients

\[
A_{jk}^J = -\frac{1}{4} \left( m_j^2 + m_k^2 + \frac{m_j m_k}{s_J} \right), \\
B_{jk}^J = \frac{\pi^2}{32N} \langle FF \rangle + \frac{m_j^2 m_k^2}{16} + \frac{\pi^2}{8N} \frac{2 + s_J}{s_J} (m_j + m_k) \langle \bar{\psi} \psi \rangle.
\]

We correct our observables \( R_J \) and \( \tilde{R}_J \) by subtracting the leading continuum corrections.
Improved observables

\[ B_J(x, \delta m) \equiv \left[ R_J(x, \delta m) - R^{\text{tree}}_J(x, \delta m) + \frac{2 + s_J}{8N} \frac{\pi^2}{s_J} (M^2 - M^2_K) F^2_0 x^4 \right] \times \left( \frac{1}{\kappa_s} - \frac{1}{\kappa_\ell} \right)^{-1} = b_J + O(x^6) + O(g^2 a^2) + \cdots, \]

\[ \tilde{B}_J(x, \delta \bar{m}) \equiv \left[ \tilde{R}_J(x, \delta \bar{m}) - \tilde{R}^{\text{tree}}_J(x, \delta \bar{m}) + \frac{2 + s_J}{8N} \frac{\pi^2}{s_J} \left( \delta M^2 \right) F^2_0 x^4 \right] \times \left( \frac{1}{\kappa(\sigma)} - \frac{1}{\kappa(\rho)} \right)^{-1} = b_J + 3\tilde{b}_J + O(x^6) + O(g^2 a^2) + \cdots. \]
Example of numerical data
Results in the scalar channel

$\beta = 3.4$

$\beta = 3.46$

$\beta = 3.55$

$\beta = 3.7$
Final results: rational parametrizations

\[ b_S(g^2) = 1 + 0.11444(1) C_F \left( 1 - 0.439(50) g^2 \right) \left( 1 - 0.535(14) g^2 \right)^{-1} \]

\[ b_P(g^2) = 1 + 0.0890(1) C_F \left( 1 - 0.354(54) g^2 \right) \left( 1 - 0.540(11) g^2 \right)^{-1} \]

\[ b_V(g^2) = 1 + 0.0886(1) C_F \left( 1 + 0.596(111) g^2 \right) \]

\[ b_A(g^2) = 1 + 0.0881(1) C_F \left( 1 - 0.523(33) g^2 \right) \left( 1 - 0.554(10) g^2 \right)^{-1} \]
### Conclusions

We implemented a coordinate space method to determine improvement coefficients proportional to quark mass. With a negligible numerical effort, we can achieve a 5% - 10% precision on $b_J$. Improvement coefficients proportional to the trace of the mass matrix are accessible, but need a better statistical precision.

#### $\tilde{b}_J$ improvement coefficients at $\beta = 3.4$

<table>
<thead>
<tr>
<th>Improvement Coefficient</th>
<th>Value</th>
<th>Stat. Error</th>
<th>Sys. Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{b}_S$</td>
<td>2.0 (1.3)Stat</td>
<td>(0.3)Sys</td>
<td></td>
</tr>
<tr>
<td>$\tilde{b}_P$</td>
<td>-3.4 (1.3)Stat</td>
<td>(0.6)Sys</td>
<td></td>
</tr>
<tr>
<td>$\tilde{b}_V$</td>
<td>-0.1 (0.4)Stat</td>
<td>(0.1)Sys</td>
<td></td>
</tr>
<tr>
<td>$\tilde{b}_A$</td>
<td>1.4 (0.4)Stat</td>
<td>(0.9)Sys</td>
<td></td>
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