On the loop formula for the fermionic determinant
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Introduction and a simple example
We discuss here a formula which represents the lattice fermionic determinant (a large order polynomial) as an infinite product of determinants of a smaller, fixed dimension. The formula is based on the loop expansion [3] and has been used for HD-QCD [2]. It provides a systematic approximation to QCD, which can however lead to minor simplifications. We rediscuss here this formula explicitly and discuss its features in detail.

2. A simple example
We consider
\[ \det (1 - i(x + y)) = \exp(-\ln(1-x-y)) \]
(1)
The traces distinguish between the strings XXXY and XYXY, say, but identify cycle permutations, such as XXXY and XYXY.

Expanding Eq. (1) we obtain
\[ \det(1-i(x+y)) = (x+y)(x+2y+1) - \cdots \]
\[ = -i4x^4 + i5x^3y + 2i4xy^3 + i3y^4 \]
\[ = -i(4x^4 + 5x^3y + 4xy^3 + 3y^4) \]
(2)
with further regrouping of the terms observing in (1) which are the monomials in the products \((X + Y)(X + Y)\) and \((X + Y)(Y + X)\).

We immediately see that assuming the terms which are powers of a lowest order monomial (what we call “renormalization”) we get the series
\[ \text{ln}(1 - x - y) = -x - y + \frac{1}{2}(x + y)^2 - \frac{1}{3}(x + y)^3 + \cdots \]
\[ = -x - y - \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{3}xy - \frac{1}{3}yx + \cdots \]
(3)

The loop formula
For QCD at chemical potential \(\mu\) > 0 the coefficients of primary loops of length \(l = 1\) with positive net winding number \(r > 0\) in the time direction are
\[ q_{r, s} = \epsilon^r \epsilon^s C_{r, s}, \quad l_{r, s} = 1 - r + s, \quad s = 0, \ldots, r. \]
(17)
\[ \epsilon = \mu > 0, \quad \nu, \quad \zeta = \kappa = \sigma = 0. \]
(18)

The CL process associated to a partition function \(Z\) with complex measure proceeds in the manifold of a complex variable \(z\) and involves a drift force \(k\) as the logarithmic derivative of the measure
\[ d\mu(z) = e^{-k(z)} \mu(\epsilon z) \]
(25)
with \(k\) an appropriately normalized random noise. A CL process to simulate QCD at nonzero \(\mu\) takes place in the complexified space of the link variables \(U \in SU(N_c)\) [4, 5]. The drift incorporates the logarithmic derivative of the determinant and needs the evaluation of the inverse of \(W\) (Eq. 6) which is a large matrix of size \(N_c^2 \times N_c^2\).

Using the loop formula Eq. (10) we have
\[ U(z) = \sum_{l} \frac{1}{d\mu_l(z)} \frac{d\mu_l(z)}{d\mu(z)} \left( \frac{1 - \epsilon l z}{(1 - \epsilon z)} \right) \]
(20)
\[ = \sum_{l} \frac{1}{d\mu_l(z)} \frac{d\mu_l(z)}{d\mu(z)} \left( \frac{1 - \epsilon l z}{(1 - \epsilon z)} \right) \]
(21)
\[ \alpha = C_1(C_2)^{-1}, \quad \beta = a_C; \]
(22)
\[ \eta = \alpha C_1 + C_2^{-1}, \quad \nu = \alpha C_2 \]
(23)

3. Calculating the logarithm series

By identifying the 2d(1-\(\mu l(N_c-1)\))-sharpest truncated polynomial loops. The second factor in Eq. (21), however, is an infinite product. For a small enough to converge we can cut the product, e.g. \(\kappa = r\), which was done in [2] for a rewriting simulation to produce the phase diagram of QCD with 3 flavors of heavy quarks.

The chain has only primary loop \(P = U_1 U_2\) and the loop formula reproduces the exact det W
\[ \text{det} W = 1 + 4(\epsilon^2 + \epsilon) + 4 \epsilon^2 (2 + \epsilon) + 16 \epsilon^3 \]
(27)

In the second example there are 12 primary loops of length \(l \leq 6\), listed here with their weights: \(l_{1, 2}, l_{1, 4}, l_{2, 4}, l_{1, 3}, l_{2, 3}, l_{1, 5}, l_{2, 5}, l_{3, 5}, l_{1, 6}, l_{2, 6}, l_{3, 6}, l_{4, 6}\). Using the loop formula Eq. (10) we have
\[ U(z) = \sum_{l} \frac{1}{d\mu_l(z)} \frac{d\mu_l(z)}{d\mu(z)} \left( \frac{1 - \epsilon l z}{(1 - \epsilon z)} \right) \]
(20)
\[ = \sum_{l} \frac{1}{d\mu_l(z)} \frac{d\mu_l(z)}{d\mu(z)} \left( \frac{1 - \epsilon l z}{(1 - \epsilon z)} \right) \]
(21)
\[ \alpha = C_1(C_2)^{-1}, \quad \beta = a_C; \]
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\[ \eta = \alpha C_1 + C_2^{-1}, \quad \nu = \alpha C_2 \]
(23)

4. Examples and discussion

As appealing as the loop formula appears its use is involved. The formula does not allow an interpretation as “linear factors” decomposition, but provides a systematic approximation in approaching the true determinant in the convergence domain. Fixing the parameters we may enquire which is the true zero at \(z = 1\) (see the figure in [2]). The true zero at \(z = 1\) is the only one in the complex plane. The loop formula robustly works on its correct interpretation.

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References