LATTICE 2016
Southampton

QED Corrections to Hadronic Observables

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Motivations

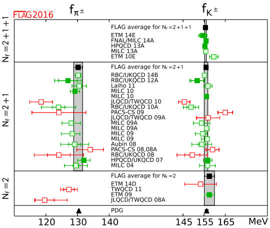
- In the real world up and down quarks have different masses and electric charges.

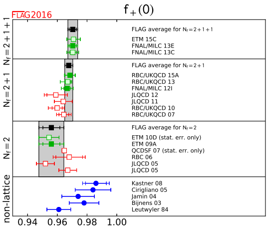
- Isospin-breaking effects are typically a few percent effects:

$$\frac{m_u - m_d}{M_p} \simeq 0.3\% \qquad \alpha_{\mathrm{EM}} = 0.7\% \qquad \frac{M_n - M_p}{M_n} \simeq 0.1\%$$

- From FLAG16 [Aoki *et al.*, arXiv:1607.00299] and [PDG review, Rosner *et al.*, 2016], [Cirigliano *et al.*, Rev. Mod. Phys. **84**, 399 (2012)]

$$\begin{aligned}
f_{\pi^\pm} &= 130.2(1.4)\ \mathrm{MeV} && \mathrm{err} = 1\% && \delta^{\chi\mathrm{PT}}_{\mathrm{QED}}(\pi^- \to \ell^- \bar{\nu}) = 1.8\% \\
f_{K^\pm} &= 155.6(0.4)\ \mathrm{MeV} && \mathrm{err} = 0.3\% && \delta^{\chi\mathrm{PT}}_{\mathrm{QED}}(K^- \to \ell^- \bar{\nu}) = 1.1\% \\
f_+(0) &= 0.9704(24)(22) && \mathrm{err} = 0.5\% && \delta^{\chi\mathrm{PT}}_{\mathrm{QED}}(K \to \pi \ell \bar{\nu}) = [0.5, 3]\%
\end{aligned}$$





Motivations

- Semileptonic B decays (measurement of $|V_{cb}|$)

  $$B \to D^{(*)}\ell\nu$$

  relevant for Belle II (data taking starts in 2018). Radiative corrections are expected to be of about 3%.

  PDG review 2015 update, Olive et al. (PDG), Chin. Phys. C, 38, 090001 (2014)
  Bailey et al. (Fermilab Lattice and MILC Collab.), Phys. Rev. D89, 114504 (2014)
  Aubert et al. (BaBar), Phys. Rev. Lett. **100**, 231803 (2008)
  Adam et al. (CLEO), Phys. Rev. D **67**, 032001 (2003)
  Aubert et al. (BaBar), Phys. Rev. D **79**, 012002 (2009)
  Amhis et al. (HFAG), arXiv:1207.1158 [hep-ex]

- Radiative corrections have three contributions:
  - Short-distance contributions (photons coupling to the W). These contributions can be systematically accounted for (OPE).
  - Long-distance soft-photon contributions, in loops and finale-state radiation, a.k.a. *inner-bremsstrahlung*. These are analytically calculable.
  - Long-distance hard-photon contributions, a.k.a. *structure-dependent* contributions. These are fully non-perturbative, and they are either neglected or estimated by saturating relevant matrix elements with a few resonances (for light mesons one can use χPT).

- We can and should do better than this.

Two ways for QCD+QED on the lattice

- Expand observables with respect to α_{em} and simulate QCD only.

  de Divitiis *et al.* (RM123), Phys.Rev. D87 (2013) 11, 114505.

  E.g. Cottingham formula for the mass correction:

$$\Delta m = -\frac{e^2}{4m} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \int d^4x \; e^{-ikx} \langle h | \mathrm{T}\{ j_\mu(x) j_\mu(0) \} | h \rangle_{c,QCD} + O(e^4)$$

Pros:
Only $O(\alpha_{em}^0)$ observables.

Cons:
Complex observables (e.g. a 4-point functions for mass correction), typically involving fermionic disconnected diagrams.

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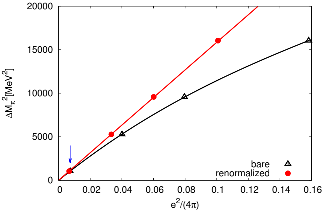
  Only $O(\alpha_{em}^0)$ observables.

  Cons:
  Complex observables (e.g. a 4-point functions for mass correction), typically involving fermionic disconnected diagrams.

- Simulate QCD+QED on the lattice.

  Pros:
  Simpler observables (e.g. 2-point functions for mass correction).

  Cons:
  Signal is typically $O(\alpha_{em})$.



Borsanyi *et al.*, Science 347 (2015) 1452-1455.

I am going to talk about...

- **things I have read about**: Quick review of recent activity and results
- **things I have worked on**: Charged states in a finite box: discussion of proposed methods
- **things I know nothing about, but I find interesting**: Decay rates, IR divergences and potentially large logarithms

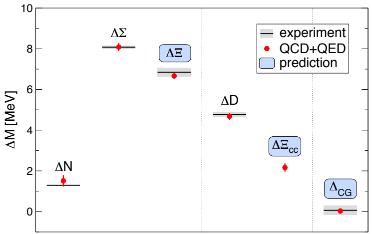
Part I – Things I have read about

Quick review of recent activity and results

except RM123-SOTON decay rate calculation

BMW: Baryon masses

Borsanyi *et al.*, Science 347 (2015) 1452-1455
Talk by Liu, Mon 10.30



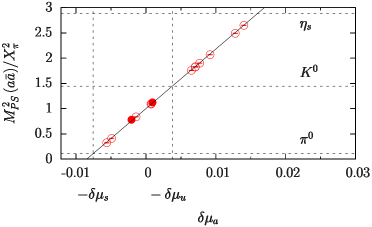
- Analytic understanding of power-law finite-volume corrections to masses of stable states, in the fully-relativistic theory.

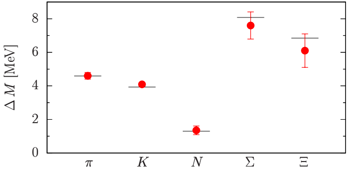
  Fodor *et al.*, Phys. Lett. B **755**, 245 (2016)
  Davoudi, Savage, Phys. Rev. D **90**, no. 5, 054503 (2014)

- Large volume simulations: physical size up to $M_\pi L = 8.1$ with a 64×80^3 lattice.

QCD-SF: Masses

R. Horsley *et al.*, arXiv:1508.06401
R. Horsley *et al.*, JHEP **1604**, 093 (2016)
Talk by Rakow, Wed 9.20
Talk by Young, Wed 9.40
Talk by Liu, Mon 10.30



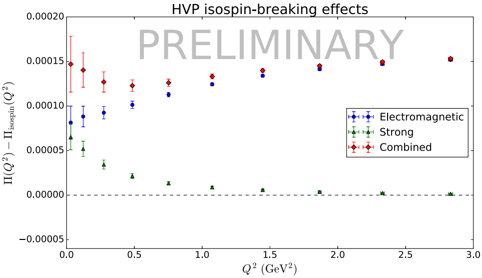


- Taylor expand masses around the SU(3) symmetric point and $\alpha_{\rm EM} = 0$, neglecting $O(e^4)$ and $O(\delta m^2)$. Move away from the symmetric point by keeping $\delta m_u + \delta m_d + \delta m_s = 0$.

RBC/UKQCD: HVP contribution to $g_\mu - 2$

Talk by Harrison, Tue 15.00
Talk by Gülpers, Tue 15.20

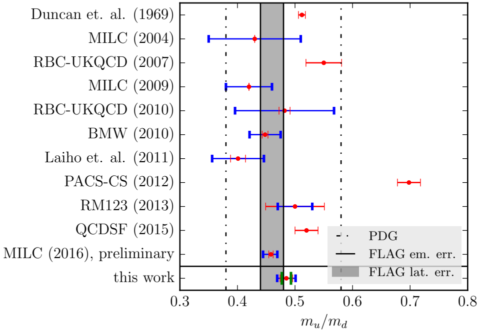
- Exploratory study to calculate isospin corrections to the HVP contribution to $g_\mu - 2$.
- Electroquenched approximation (i.e. gauge configurations are generated with $\alpha_{\mathrm{EM}} = 0$) + $\mathrm{QED_L}$.
- Comparision between stochastic QED (valence Dirac operator = QCD+QED Dirac operator, with free EM field), and RM123 method (observables are expanded in α_{EM} by hand).



BMW: Up and down quark masses

Fodor *et al.*, arXiv:1604.07112 [hep-lat]
Talk by Varnhost, Tue 15.20

- Violation to Daschen's theorem.
- Ratio of up and down quark masses. Claim: $m_u/\delta m_u \sim 24$.
- Electroquenched (with some estimate for systematic error) + $\mathrm{QED_{TL}}$ (with correction for masses)



Part II – Things I have worked on

Charged states in a finite box: discussion of proposed methods

Charge states in a finite box

In a finite box with periodic boundary conditions, Gauss law forbids states with nonzero charge

$$Q = \int \mathrm{d}^3x \; j_0(t, \mathbf{x}) = \int \mathrm{d}^3x \; \partial_k E_k(t, \mathbf{x}) = 0$$

Some proposed methods

- Remove the global zero-mode of the gauge field ($\mathrm{QED}_{\mathrm{TL}}$)
- Restrict the global zero-mode of the gauge field
- Remove the spatial zero-mode of the gauge field in each timeslice ($\mathrm{QED}_{\mathrm{L}}$)
- Massive photon.
- $\mathrm{C}^\star$ boundary conditions.

All these approaches are equivalent if the infinite-volume limit is take before any other limit (large-t limit in 2-point functions, continuum limit, massless photon limit). In general the infinite-volume limit does not commute with the other limits.

QED_{TL}: Gauss law and zero-modes

Duncan *et al.*, Phys. Rev. Lett. **76**, 3894 (1996)

Recipe: Remove the global zero-mode of the gauge field

$$a_\mu = eL_\mu \int \mathrm{d}^4x \; A_\mu(x) = 0$$

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$$a_\mu = eL_\mu \int \mathrm{d}^4x\ A_\mu(x) = 0$$

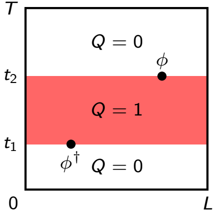
Action

$$S(\tfrac{1}{eL}a + B) = S(B) + \frac{i}{L_\mu} a_\mu \int d^4x\ j_\mu(x)$$

Integration over the zero-modes yields a delta function

$$\int \mathrm{d}a\ e^{-S(a,B)} = e^{-S(0,B)} \prod_\mu \delta\left(\frac{1}{L_\mu}\int d^4x\ j_\mu(x)\right)$$

Configurations in which a charged state is created in between two interpolating operators are excluded by the delta function.



$$\frac{1}{T}\int d^4x\ j_0(x) = \frac{t_2 - t_1}{T}$$

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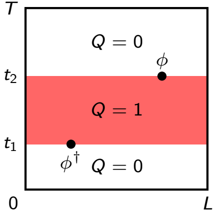

$$\frac{1}{T} \int d^4 x \; j_0(x) = \frac{t_2 - t_1}{T}$$

No transfer matrix (i.e. Hamiltonian) [Borsanyi *et al.*, Science 347 (2015) 1452-1455]. In particular the two-point function does not have a spectral decomposition:

$$\int \mathrm{d}^3 x \; \langle \psi(t, \mathbf{x}) \bar{\psi}(0) \rangle \neq \sum_{n,m} A_{nm}(L) e^{-t[E_n(L) - E_m(L)]} e^{-TE_m(L)}$$

Infinite-volume limit should be taken before fitting plateaux in effective masses and the continuum limit.

Restriction of zero-modes

Gockeler *et al.*, Nucl. Phys. B **334**, 527 (1990)

Recipe: Restrict the global zero-mode of the gauge field

$$-\pi < a_\mu = eL_\mu \int \mathrm{d}^4 x \; A_\mu(x) < \pi$$

The restriction can be seen as a nonlocal gauge fixing (for large gauge transformations).

A transfer matrix interpretation of the two-point function is not possible, and the decomposition in exponentials is not guaranteed. Infinite-volume limit should be taken before fitting plateaux in effective masses and the continuum limit.

$\mathrm{QED_L}$: spatial zero-modes

Hayakawa and Uno, Prog. Theor. Phys. **120**, 413 (2008)
Borsanyi *et al.*, Science 347 (2015) 1452-1455

Recipe: Remove the spatial zero-mode of the gauge field in each timeslice

$$\int \mathrm{d}^3x \; A_\mu(t, \mathbf{x}) = 0$$

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QED_L has a transfer matrix. It is a nonlocal prescription. Locality is a core property of QFT, it is a fundamental assumption behind

- Renormalizability by power counting
- Volume-independence of renormalization constants
- Operator product expansion
- Effective-theory description of long-distance physics
- Symanzik improvement program
- ...

Infinite-volume limit should be taken before the continuum limit.

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What is the status on these issues?

- Operators with dimension ≤ 4 are renormalized at $O(\alpha)$ by the infinite-volume counterterms.
- Non-relativistic EFT description breaks down at $O(\alpha)$, as antiparticles do not decouple in the NR limit. [Fodor *et al.*, Phys. Lett. B **755**, 245 (2016)]
- Higher dimensional operators generate nonlocal divergences.

UV cutoff and subtraction of spatial zero-modes

Simpler case: $\lambda\phi^4$ scalar theory, with the constraint

$$\int \mathrm{d}^3x \; \phi(t, \mathbf{x}) = 0$$

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- Explicit calculation in heat-kernel regularization with cutoff Λ yields

$$(\Box\phi)^2(x) = \sum_{d_O \leq 6} \Lambda^{6-d_O} c_O [O(x)]_\mathrm{R} - \frac{\lambda}{4(2\pi)^{1/2}} \frac{\Lambda}{L^6} \int \mathrm{d}^3z\ [\phi^2(x_0, \mathbf{z})]_\mathrm{R} + O(\lambda^2)$$

- It it impossible to define $[(\Box\phi)^2(x)]_\mathrm{R}$, i.e. a local and finite operator that coincides with $(\Box\phi)^2(x)$ at tree level.

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- In the (would-be) Symanzik expansion of observables there are terms proportional to aL^{-3}.

$$\Lambda^{-2} \int \mathrm{d}^4x \; (\Box\phi)^2(x) = -\frac{\lambda}{4(2\pi)^{1/2}} \frac{1}{\Lambda L^3} \int \mathrm{d}^4x \; [\phi^2(x)]_\mathrm{R} + \text{local contr.s} + O(\lambda^2)$$

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- Once locality is violated, there is a number of unexpected and counterintuitive phenomena happening. A systematic analysis of the effects of the non-locality of $\mathrm{QED_L}$ is desirable, expecially in view of calculations of more complex observables than masses.

Massive photon

Endres *et al.*, arXiv:1507.08916

Recipe: Landau gauge + mass term for photon.

Local prescription. Gauge invariance is broken in a controlled way (soflty broken). Continuum limit can be consistently take *before* infinite-volume limit and $m_\gamma \to 0$ limit. Infinite-volume limit must be taken before the $m_\gamma \to 0$ limit.

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Integration over the zero-modes

$$\int \mathrm{d}a \; e^{-S(a,B)} \propto e^{-S(0,B)} \exp\left\{ -\frac{e^2}{2m_\gamma^2 \, TL^3} \sum_\mu \left(\int d^4x \; j_\mu(x) \right)^2 \right\}$$

Competing effect

- The $m_\gamma \to 0$ limit suppresses charged states.
- The $T, L \to \infty$ limit allows charged states.

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In particular

$$\lim_{m_\gamma \to 0} \langle \psi(x) \bar{\psi}(0) \rangle = 0 + \text{contact terms}$$

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Competing effect

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- The $T, L \to \infty$ limit allows charged states.

A bit more in detail

$$\int \mathrm{d}^3z \; e^{-i\mathbf{px}} \langle \psi(x) \bar{\psi}(0) \rangle \simeq \frac{m_\gamma^3 (2\pi TL^3)^{3/2}}{e^3 \langle \delta_Q \rangle_0} e^{-\frac{e^2}{2m_\gamma^2 V} x_0^2} \langle \delta_{Q(T),0} \psi(x_0, \mathbf{0}) \bar{\psi}(0) \rangle_0$$

where $\langle \cdot \rangle_0$ is the expectation value in $\mathrm{QED_{TL}}$. Notice that at LO the expectation value does not depend on $\mathbf{p}$!

C* boundary conditions

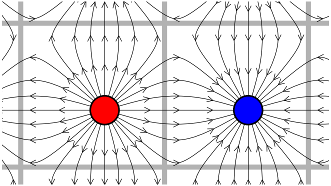
Wiese, Nucl. Phys. B **375**, 45 (1992)
Polley, Z. Phys. C **59**, 105 (1993)
Kronfeld and Wiese, Nucl. Phys. B **357**, 521 (1991)
Lucini *et al.*, JHEP **1602**, 076 (2016)

Recipe: Use C* boundary conditions along spatial directions for all fields

$$A_\mu(x + L\mathbf{k}) = -A^*_\mu(x)$$
$$\psi(x + L\mathbf{k}) = C^{-1}\bar{\psi}^T(x)$$

The flux of electric fiels across the boundaries in not forced to vanish

$$Q(t) = \int \mathrm{d}^3x \; j_0(t, \mathbf{x}) = \int \mathrm{d}^3x \; \partial_k E_k(t, \mathbf{x}) \neq 0$$



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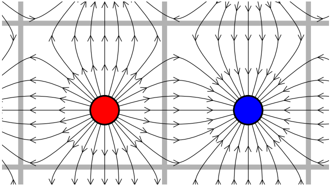

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Local prescription. Gauge invariance is preserved. Continuum limit can be consistently take *before* infinite-volume limit. Flavour and charge conservation are partially violated.

- This generates unphysical decay of a few hadrons, but most of them are protected. In n-point functions involving the non-protected hadrons, infinite-volume limit must be taken before the large-t limit.
- Non-physical decay is exponentially suppressed with the volume.
- Flavour symmetry is broken only by boundary effects. Composite operators renormalize as if flavour symmetry were intact.

Message

- Gauss law forbids non-zero charge states in a finite box with periodic boundary conditions.
- Several proposal to work around this problem involve tampering with (global or spatial) zero modes of the gauge fields.
- The implications of non locality are not systematically understood. This might be an issue, expecially for simulations at unphysically large values of α_{EM}, and for complex observables.
- In view of a target precision of 1%, it would be safer to use setups that are theoretically under control.

Part III – Things I know nothing about, but I find interesting

Decay rates, IR divergences and potentially large logarithms

Bonus: quick review of RM123-SOTON decay rate calculation

RM123-SOTON: Pion and kaon leptonic decay rate

Carrasco *et al.*, Phys.Rev. D91 (2015) 7, 074506
Talk by Tantalo, Wed 10.50
Talk by Simula, Tue 11.10

- The decay amplitude $\pi \to \ell\bar{\nu}$ is infinite at order α_{EM} because of IR divergences.

- [Bloch and Nordsieck, Phys. Rev. **52**, 54 (1937)] The physical quantity is the decay rate of $\pi \to \ell\bar{\nu}$ plus an arbitrary number of undetected soft photons (i.e. photons with energy lower than the detector resolution ΔE) in the final states. At order α_{EM} only one photon matters.

- The proposed method uses in a smart way the decomposition of decay amplitudes in a perturbative universal part and a non-universal structure-dependent part.

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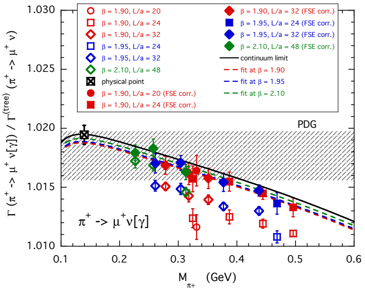
Calculation of $\Gamma(\Delta E)$ with $\Delta E \sim 30$MeV. Crucial ingredients:

- Finite volume regulates the IR divergences.
- Full calculation of the finite structure-dependent part of $\pi \to \ell\bar{\nu}$.
- The structure-dependent part of $\pi \to \ell\bar{\nu}\gamma$ is shown to be negligible.
- The universal part of $\pi \to \ell\bar{\nu}$ is calculated analytically in a $1/L$ expansion plus $\ln L$. Finite volume corrections to the structure-dependent part vanish like $1/L^2$.
- Exploratory electroquenced + $\mathrm{QED_L}$.



$$\frac{\Gamma_{\mathrm{NLO}}(\Delta E \simeq 30\mathrm{MeV})}{\Gamma_{\mathrm{LO}}(\Delta E \simeq 30\mathrm{MeV})} = 1.0210(15)(\ldots)_{\mathrm{qQED}}$$

A textbook story...

IR divergences cancel in physical observables. Large logarithms may appear in physical observables as remnants of IR divergences. As a consequence a reliable estimate of the radiative corrections is problematic. In the last part of my talk I want to argue that these logs may be unespectedly large.

- Cancellation of IR divergences in inclusive decay rates and cross sections.
  Bloch and Nordsieck, Phys. Rev. **52**, 54 (1937)
- Factorisation, exponentiation and cancellation of IR divergences. Universality.
  Yennie, Frautschi and Suura, Annals Phys. **13**, 379 (1961)
  Weinberg, Phys. Rev. **140**, B516 (1965)
  Grammer and Yennie, Phys. Rev. D **8**, 4332 (1973)
- IR divergences as a failure of perturbation theory for transition amplitudes.
  Lee and Nauenberg, Phys. Rev. **133**, B1549 (1964)
- More on universality.
  Low, Phys. Rev. **96**, 1428 (1954)
  Gell-Mann and Goldbarger, Phys. Rev. **96**, 1433 (1954)
- Estimate radiative corrections by separating universal and structure-dependent parts.
  Sirlin, Rev. Mod. Phys. **50**, 573 (1978)
- IR divergences as failure of standard (Haag and Ruelle) scattering theory.
  Kulish and Faddeev, Theor. Math. Phys. **4**, 745 (1970)
  Immense work of Buchholz, Jackiw, Zwanziger...

Soft divergences at LO

- Photon propagator k^{-2} is not enough to generate IR divergences.
- Euclidean n-point functions are IR finite. Soft divergences when matter propagators go on shell

$$\bar{p}^2 = -M^2 \quad \Rightarrow \quad \frac{1}{(\bar{p}+k)^2 + M^2} = \frac{1}{2\bar{p}k + k^2} \simeq \frac{1}{2\bar{p}k}$$

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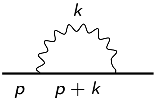
Self energy

$$\Sigma(\bar{p}) \sim \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{2\bar{p}k} = \text{IR finite}$$



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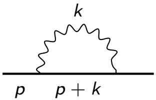
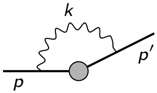


Scattering amplitude

$$A(\bar{p}, \bar{p}') \sim \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{2\bar{p}k} \frac{1}{2\bar{p}' k} = \text{IR divergent}$$



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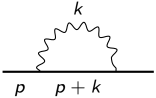

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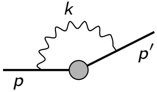
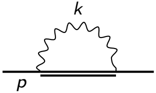


Wave function normalization:

$$\frac{\partial \Sigma}{\partial p}(\bar{p}) \sim \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2} \left(\frac{1}{2\bar{p}k} \right)^2 = \text{IR divergent}$$

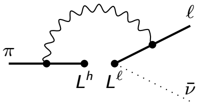


- At higher order the analysis is complicated by nested divergences.

Universality of soft divergences at LO

Soft logarithms involving hadrons are the same one would calculate in an effective theory in which hadrons are treated as point-like particles.

Effective theory:



$$A \propto \int_k F_\pi(\bar{p}_\pi + k)_\mu \frac{1}{k^2}(-2\bar{p}_\pi - k)_\rho \frac{1}{2\bar{p}_\pi k + k^2} \gamma_\mu \frac{-i(\bar{\not{p}}_\ell - \not{k}) + m_\ell}{-2\bar{p}_\ell k + k^2} \gamma_\rho$$

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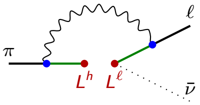

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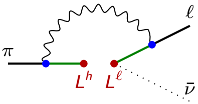

$$[A]_{\text{IR div.}} \propto F_\pi \, \bar{p}_\pi^\mu \int_k \frac{1}{k^2} (-2\bar{p}_\pi^\rho) \frac{1}{2\bar{p}_\pi k} \gamma_\mu \frac{-i\bar{\not{p}}_\ell + m_\ell}{-2\bar{p}_\ell k} \gamma_\rho$$

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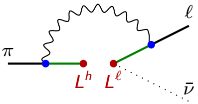

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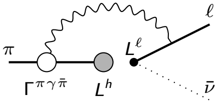
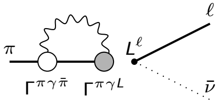
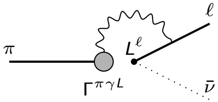
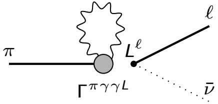


- Effective (1PI) vertices, dressed propagators.







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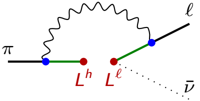

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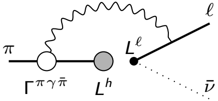
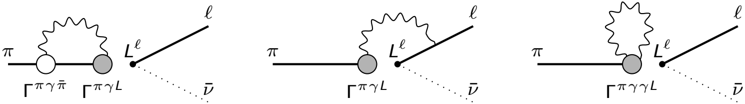


- Effective (1PI) vertices, dressed propagators.
- Effective (1PI) vertices are analytic around $k = 0$.
- Residues are analytic around the mass-shell.



IR finite

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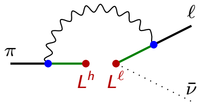

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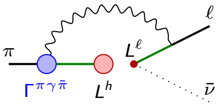

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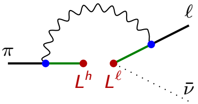

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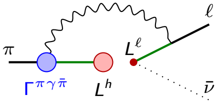

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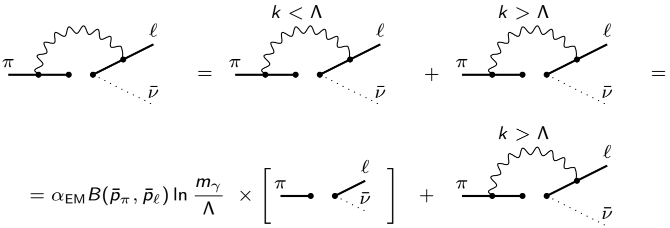
By definition $\Gamma_\mu^{\pi L}(\bar{p}_\pi) \equiv F_\pi \bar{p}_\pi^\mu$
Canonical normalization $Z_\gamma(0) = Z_\pi(\bar{p}_\pi) = 1$
Ward identity $k_\rho \Gamma_\rho^{\pi\gamma\bar\pi}(\bar{p}_\pi, k) = \Delta^{-1}(\bar{p}) - \Delta^{-1}(\bar{p}+k) = -2\bar{p}k + O(k^2)$

$$\Rightarrow \Gamma_\rho^{\pi\gamma\bar\pi}(\bar{p}_\pi, k) = -2\bar{p}_\rho + O(k)$$

Factorisation of soft divergences at LO

Introduce an IR regulator μ that preserves unitarity, e.g. photon mass ($\mu = m_\gamma$).



$$\underbrace{A_{\mathrm{NLO}}(\alpha \to \beta; m_\gamma)}_{\text{1-loop amplitude with } m_\gamma > 0} = \left[1 + \frac{\alpha_{\mathrm{EM}}}{2} \underbrace{R(\alpha \to \beta)}_{\text{universal (known) function}} \ln \frac{m_\gamma}{\Lambda}\right] \times \underbrace{A_{\mathrm{LO}}(\alpha \to \beta)}_{\text{tree-level amplitude with } m_\gamma = 0} + \underbrace{A_{\mathrm{NLO},k^2>\Lambda^2}(\alpha \to \beta)}_{\text{1-loop amplitude with } m_\gamma = 0 \text{ and } k^2 > \Lambda^2 \text{ restriction}}$$

Bloch-Nordsieck prescription

Physics interpretation: from the experimental point of view it is impossible to differentiate between

$$h \to \ell + \bar{\nu} \ ,$$
$$h \to \ell + \bar{\nu} + N\gamma \ ,$$

- if each photon is emitted with a lower energy than the detector resolution ΔE;
- and the total energy carryed away by the undetected photons is (roughly) less than the resolution ΔE with which we can reconstruct the lepton energy.

[1] The diagrammatic expansion is wrong. I am deliberately neglecting the wave-function renormalization for sake of presentation.

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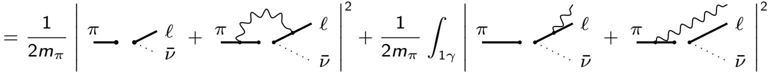
The physical quantity is the decay rate integrated over soft photons, which is finite.[1]

$$\Gamma(\Delta E) = \lim_{m_\gamma \to 0} \frac{1}{2m_\pi} \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\substack{k_\alpha < \Delta E \\ \sum_\alpha k_\alpha < \Delta E}} \mathrm{d}\Phi_{N\gamma} \ \left| \langle \pi | \mathcal{H}_{\mathrm{W}} | \ell, \bar{\nu}, N\gamma \rangle \right|^2 =$$



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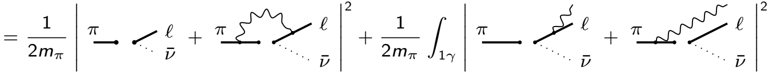

The logarithm in the photon mass is traded for a logarithm in the energy resolution:

$$\langle \pi | \mathcal{H}_\mathrm{W} | \ell, \bar{\nu} \rangle = \left[1 + \frac{\alpha_\mathrm{EM}}{2} R \ln \frac{m_\gamma}{\Lambda} \right] A_\mathrm{LO} + A_{\mathrm{NLO}, k^2 > \Lambda^2} + O(\alpha_\mathrm{EM}^2)$$

$$\Rightarrow \qquad \Gamma(\Delta E) = \left[1 + \alpha_\mathrm{EM} \,\mathrm{Re}\, R \ln \frac{\Delta E}{\Lambda} \right] \Gamma_\mathrm{LO} + \Gamma_{\mathrm{NLO}, k^2 > \Lambda^2} + O(\alpha_\mathrm{EM}^2)$$

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We consider the (phenomenologically irrelevant) decay process

$$B^- \to e^- + \bar{\nu}_e$$

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tree-level decay rate with $m_\gamma = 0$

Back of the envelope calculation

$\mathrm{Re}\, R \simeq -2 + \ln(m_B^2/m_e^2) \simeq 16.5$

$\Lambda \simeq m_B, \;\; \Delta E/m_B \simeq 10\%$

- $\alpha_{\rm EM}\, \mathrm{Re}\, R \ln \Delta E/m_B = 28\%$
- $\frac{1}{2}\,(\alpha_{\rm EM}\, \mathrm{Re}\, R \ln \Delta E/m_B)^2 = 4\%$

1-loop decay rate with $k^2 > \Lambda^2$ restriction on photon loop momenta

Large collinear logarithms

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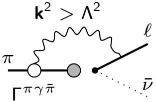

$$\subset \int_{\mathbf{k}^2 > \Lambda^2} \frac{\mathrm{d}^3 k}{(2\pi)^3 2k} \left[\frac{\gamma_\mu \Gamma_\mu^{B\gamma\bar{B}}(\bar{p}_B, k)}{(\bar{p}_B k)\left\{|\mathbf{k}|\sqrt{m_e^2 + \mathbf{p}_e^2} - \mathbf{k}\mathbf{p}_e\right\}}\right]_{k_0 = i|\mathbf{k}|} \propto \ln \frac{m_B}{m_e}, \;\; \ln \frac{\Lambda_{\rm QCD}}{m_e}$$

This hard collinear logarithm is not universal, it reads the structure of the B meson and has to be calculated nonperturbatively!

Conclusions

- When aiming at the percent precision, isospin breaking corrections must be included.
- Activity in this direction has been growing significantly in the past few years.
- QED and QCD are very different theories. Inclusion of QED effects implies a shift in the standard paradigm of lattice simulations.
- Description of charged states in a finite box is somewhat challenging. Effects of nonlocality are not systematically understood. I advocate the use of setups that are theoretically under control.
- Numerical calculations of masses are already at an advanced stage from the technical point of view. The challenge ahead is the full calculation of radiative corrections to decay rates.
- (Almost) IR divergences may generate large logarithms in heavy meson decay rates. Potentially lattice QCD can have a big impact there.

Thank you!