QED Corrections to Hadronic Observables

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In the real world up and down quarks have different masses and electric charges.

Isospin-breaking effects are typically a few percent effects:

\[
\frac{m_u - m_d}{M_p} \simeq 0.3\% \quad \alpha_{\text{EM}} = 0.7\% \quad \frac{M_n - M_p}{M_n} \simeq 0.1\%
\]

From FLAG16 [Aoki et al., arXiv:1607.00299] and [PDG review, Rosner et al., 2016], [Cirigliano et al., Rev. Mod. Phys. 84, 399 (2012)]

\[
f_{\pi^\pm} = 130.2(1.4) \text{ MeV} \quad \text{err} = 1\% \quad \delta_{\text{QED}}^{\text{PT}}(\pi^- \rightarrow \ell^- \bar{\nu}) = 1.8\%
\]

\[
f_{K^\pm} = 155.6(0.4) \text{ MeV} \quad \text{err} = 0.3\% \quad \delta_{\text{QED}}^{\text{PT}}(K^- \rightarrow \ell^- \bar{\nu}) = 1.1\%
\]

\[
f_+(0) = 0.9704(24)(22) \quad \text{err} = 0.5\% \quad \delta_{\text{QED}}^{\text{PT}}(K \rightarrow \pi \ell \bar{\nu}) = [0.5, 3]\%
\]
Semileptonic B decays (measurement of $|V_{cb}|$)

\[ B \rightarrow D^{(*)} \ell \nu \]

relevant for Belle II (data taking starts in 2018). Radiative corrections are expected to be of about 3%.

Bailey et al. (Fermilab Lattice and MILC Collab.), Phys. Rev. D89, 114504 (2014)
Aubert et al. (BaBar), Phys. Rev. Lett. 100, 231803 (2008)
Aubert et al. (BaBar), Phys. Rev. D 79, 012002 (2009)
Amhis et al. (HFAG), arXiv:1207.1158 [hep-ex]

Radiative corrections have three contributions:

- Short-distance contributions (photons coupling to the $W$). These contributions can be systematically accounted for (OPE).

- Long-distance soft-photon contributions, in loops and finale-state radiation, a.k.a. inner-bremsstrahlung. These are analytically calculable.

- Long-distance hard-photon contributions, a.k.a. structure-dependent contributions. These are fully non-perturbative, and they are either neglected or estimated by saturating relevant matrix elements with a few resonances (for light mesons one can use $\chi$PT).

We can and should do better than this.
Two ways for QCD+QED on the lattice

- Expand observables with respect to $\alpha_{em}$ and simulate QCD only.
  

  E.g. Cottingham formula for the mass correction:

  \[
  \Delta m = -\frac{e^2}{4m} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \int d^4x \ e^{-ikx} \langle h|T\{j_\mu(x)j_\mu(0)\}|h\rangle_{c,QCD} + O(e^4)
  \]

  Pros: Only $O(\alpha_{em}^0)$ observables.

  Cons: Complex observables (e.g. a 4-point functions for mass correction), typically involving fermionic disconnected diagrams.
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Only $O(\alpha_{em}^0)$ observables.

Cons:

Complex observables (e.g. a 4-point functions for mass correction), typically involving fermionic disconnected diagrams.

▶ Simulate QCD+QED on the lattice.

Pros:

Simpler observables (e.g. 2-point functions for mass correction).

Cons:

Signal is typically $O(\alpha_{em})$.

I am going to talk about...

- **things I have read about**: Quick review of recent activity and results

- **things I have worked on**: Charged states in a finite box: discussion of proposed methods

- **things I know nothing about, but I find interesting**: Decay rates, IR divergences and potentially large logarithms
Part I – Things I have read about

Quick review of recent activity and results
except RM123-SOTON decay rate calculation
Analytic understanding of power-law finite-volume corrections to masses of stable states, in the fully-relativistic theory.

Davoudi, Savage, Phys. Rev. D 90, no. 5, 054503 (2014)

Large volume simulations: physical size up to $M_\pi L = 8.1$ with a $64 \times 80^3$ lattice.
Taylor expand masses around the SU(3) symmetric point and $\alpha_{\text{EM}} = 0$, neglecting $O(e^4)$ and $O(\delta m^2)$. Move away from the symmetric point by keeping $\delta m_u + \delta m_d + \delta m_s = 0$. 
RBC/UKQCD: HVP contribution to $g_\mu - 2$

Talk by Harrison, Tue 15.00
Talk by Gülpers, Tue 15.20

- Exploratory study to calculate isospin corrections to the HVP contribution to $g_\mu - 2$.

- Electroquenched approximation (i.e. gauge configurations are generated with $\alpha_{EM} = 0$) + QED$_L$.

- Comparison between stochastic QED (valence Dirac operator = QCD+QED Dirac operator, with free EM field), and RM123 method (observables are expanded in $\alpha_{EM}$ by hand).

Hadronic Vacuum Polarisation
Preliminary upper bound on $a_{H\Delta ,LO}$

$\mu$ isospin-breaking effects:

$\sim 0.5\%$ at $1\sigma$. 14/17
Violation to Daschen’s theorem.

Ratio of up and down quark masses. Claim: \( m_u / \delta m_u \sim 24 \).

Electroquenched (with some estimate for systematic error) + QED\(_{TL} \) (with correction for masses)

![Graph showing the results of various collaborations on the ratio \( m_u / m_d \)]

- Duncan et al. (1969)
- MILC (2004)
- RBC-UKQCD (2007)
- MILC (2009)
- RBC-UKQCD (2010)
- BMW (2010)
- Laiho et al. (2011)
- PACS-CS (2012)
- RM123 (2013)
- QCDSF (2015)
- MILC (2016), preliminary
- this work

PDG
FLAG em. err.
FLAG lat. err.

\[ M^2 = 2 B^2 m + O(mud \delta \delta, mud \delta m, \delta \delta^2, \delta m, m^2) \]

with \( B^2 \) from [1] to get

\[ m = \frac{M^2 B^2}{\delta m} = 2.41(6)(4)(9) \text{ MeV} \]

Using \( m_{ud} \) from [2] we arrive at

\[ m_u / m_d = 0.485(11)(8)(14) \]
Part II – Things I have worked on

Charged states in a finite box: discussion of proposed methods
In a finite box with periodic boundary conditions, Gauss law forbids states with nonzero charge

\[ Q = \int d^3 x \, j_0(t, x) = \int d^3 x \, \partial_k E_k(t, x) = 0 \]

Some proposed methods

- Remove the global zero-mode of the gauge field (QED_{TL})
- Restrict the global zero-mode of the gauge field
- Remove the spatial zero-mode of the gauge field in each timeslice (QED_L)
- Massive photon.
- C* boundary conditions.

All these approaches are equivalent if the infinite-volume limit is take before any other limit (large-\(t\) limit in 2-point functions, continuum limit, massless photon limit). In general the infinite-volume limit does not commute with the other limits.
Recipe: Remove the global zero-mode of the gauge field

\[ a_\mu = e L_\mu \int d^4 x \ A_\mu(x) = 0 \]
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\[ a_\mu = eL_\mu \int d^4x \, A_\mu(x) = 0 \]

Action

\[ S(\frac{1}{eL} a + B) = S(B) + \frac{i}{L_\mu} a_\mu \int d^4x \, j_\mu(x) \]

Integration over the zero-modes yields a delta function

\[ \int da \, e^{-S(a,B)} = e^{-S(0,B)} \prod_\mu \delta \left( \frac{1}{L_\mu} \int d^4x \, j_\mu(x) \right) \]

Configurations in which a charged state is created between two interpolating operators are excluded by the delta function.
**QED\textsubscript{TL}: Gauss law and zero-modes**


**Recipe:** Remove the global zero-mode of the gauge field

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No transfer matrix (i.e. Hamiltonian) [Borsanyi et al., Science 347 (2015) 1452-1455]. In particular the two-point function does not have a spectral decomposition:

\[ \int d^3x \, \langle \psi(t,x) \bar{\psi}(0) \rangle \neq \sum_{n,m} A_{nm}(L)e^{-t[E_n(L)-E_m(L)]}e^{-TE_m(L)} \]

Infinite-volume limit should be taken before fitting plateaux in effective masses and the continuum limit.
Restriction of zero-modes


Recipe: Restrict the global zero-mode of the gauge field

\[-\pi < a_\mu = e L_\mu \int d^4x \ A_\mu(x) < \pi\]

The restriction can be seen as a nonlocal gauge fixing (for large gauge transformations).

A transfer matrix interpretation of the two-point function is not possible, and the decomposition in exponentials is not guaranteed. Infinite-volume limit should be taken before fitting plateaux in effective masses and the continuum limit.
Recipe: Remove the spatial zero-mode of the gauge field in each timeslice

\[ \int d^3x \ A_\mu(t, x) = 0 \]
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\[ \int d^3 x \ A_\mu (t, x) = 0 \]

QED_L has a transfer matrix. It is a nonlocal prescription. Locality is a core property of QFT, it is a fundamental assumption behind

- Renormalizability by power counting
- Volume-independence of renormalization constants
- Operator product expansion
- Effective-theory description of long-distance physics
- Symanzik improvement program
- ...

Infinite-volume limit should be taken before the continuum limit.
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Infinite-volume limit should be taken before the continuum limit.

What is the status on these issues?

- Operators with dimension \( \leq 4 \) are renormalized at \( O(\alpha) \) by the infinite-volume counterterms.
- Higher dimensional operators generate nonlocal divergences.
Simpler case: $\lambda \phi^4$ scalar theory, with the constraint

$$\int d^3 x \, \phi(t, x) = 0$$
UV cutoff and subtraction of spatial zero-modes

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$$\int d^3x \phi(t, x) = 0$$

- Explicit calculation in heat-kernel regularization with cutoff $\Lambda$ yields

$$\left(\Box \phi\right)^2(x) = \sum_{dO \leq 6} \Lambda^{6-dO} c_O [O(x)]_R - \frac{\lambda}{4(2\pi)^{1/2}} \frac{\Lambda}{L^6} \int d^3z \left[\phi^2(x_0, z)\right]_R + O(\lambda^2)$$

- It is impossible to define $\left(\Box \phi\right)^2(x)$, i.e. a local and finite operator that coincides with $\left(\Box \phi\right)^2(x)$ at tree level.
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- It it impossible to define $[\Box \phi^2(x)]_R$, i.e. a local and finite operator that coincides with $(\Box \phi^2)(x)$ at tree level.

- In the (would-be) Symanzik expansion of observables there are terms proportional to $aL^{-3}$.

$$\Lambda^{-2} \int d^4 x (\Box \phi)^2(x) = -\frac{\lambda}{4(2\pi)^{1/2}} \frac{1}{\Lambda L^3} \int d^4 x [\phi^2(x)]_R + \text{local contr.s} + O(\lambda^2)$$
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- Once locality is violated, there is a number of unexpected and counterintuitive phenomena happening. A systematic analysis of the effects of the non-locality of $\text{QED}_L$ is desirable, especially in view of calculations of more complex observables than masses.


Recipe: Landau gauge + mass term for photon.

Local prescription. Gauge invariance is broken in a controlled way (softly broken). Continuum limit can be consistently take before infinite-volume limit and \( m_\gamma \to 0 \) limit. Infinite-volume limit must be taken before the \( m_\gamma \to 0 \) limit.
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Integration over the zero-modes

$$\int da \ e^{-S(a,B)} \propto e^{-S(0,B)} \exp \left\{ -\frac{e^2}{2m_\gamma^2 TL^3} \sum_{\mu} \left( \int d^4 x \ j_\mu (x) \right)^2 \right\}$$

Competing effect

- The $m_\gamma \to 0$ limit suppresses charged states.
- The $T, L \to \infty$ limit allows charged states.
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\]

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- The \( T, L \to \infty \) limit allows charged states.

In particular

\[
\lim_{m_\gamma \to 0} \langle \psi(x) \bar{\psi}(0) \rangle = 0 + \text{contact terms}
\]
Recipe: Landau gauge + mass term for photon.

Local prescription. Gauge invariance is broken in a controlled way (softly broken). Continuum limit can be consistently take before infinite-volume limit and $m_\gamma \to 0$ limit. Infinite-volume limit must be taken before the $m_\gamma \to 0$ limit.

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A bit more in detail

$$\int d^3 z \ e^{-ipx} \langle \psi(x) \bar{\psi}(0) \rangle \simeq \frac{m_\gamma^3 (2\pi TL^3)^{3/2}}{e^3 \langle \delta Q \rangle_0} \ e^{-\frac{e^2}{2m_\gamma^2 T} x_0^2} \langle \delta Q(T),0 \psi(x_0, 0) \bar{\psi}(0) \rangle_0$$

where $\langle \cdot \rangle_0$ is the expectation value in QED$_{TL}$. Notice that at LO the expectation value does not depend on $p$!
Recipe: Use C* boundary conditions along spatial directions for all fields

\[ A_\mu (x + Lk) = -A^*_\mu (x) \]
\[ \psi(x + Lk) = C^{-1} \bar{\psi}^T (x) \]

The flux of electric fields across the boundaries in not forced to vanish

\[ Q(t) = \int d^3x \, j_0(t, x) = \int d^3x \, \partial_k E_k (t, x) \neq 0 \]
Recipe: Use $C^*$ boundary conditions along spatial directions for all fields

$$A_\mu(x + Lk) = -A_\mu^*(x)$$

$$\psi(x + Lk) = C^{-1}\bar{\psi}^T(x)$$

The flux of electric fields across the boundaries is not forced to vanish

$$Q(t) = \int d^3x \, j_0(t, x) = \int d^3x \, \partial_k E_k(t, x) \neq 0$$

Local prescription. Gauge invariance is preserved. Continuum limit can be consistently taken before infinite-volume limit. Flavour and charge conservation are partially violated.

- This generates unphysical decay of a few hadrons, but most of them are protected. In $n$-point functions involving the non-protected hadrons, infinite-volume limit must be taken before the large-$t$ limit.
- Non-physical decay is exponentially suppressed with the volume.
- Flavour symmetry is broken only by boundary effects. Composite operators renormalize as if flavour symmetry were intact.
Gauss law forbids non-zero charge states in a finite box with periodic boundary conditions.

Several proposal to work around this problem involve tampering with (global or spatial) zero modes of the gauge fields.

The implications of non locality are not systematically understood. This might be an issue, especially for simulations at unphysically large values of $\alpha_{EM}$, and for complex observables.

In view of a target precision of 1%, it would be safer to use setups that are theoretically under control.
Part III – Things I know nothing about, but I find interesting

Decay rates, IR divergences and potentially large logarithms

Bonus: quick review of RM123-SOTON decay rate calculation
The decay amplitude $\pi \rightarrow \ell \bar{\nu}$ is infinite at order $\alpha_{\text{EM}}$ because of IR divergences.

[ Bloch and Nordsieck, Phys. Rev. 52, 54 (1937) ] The physical quantity is the decay rate of $\pi \rightarrow \ell \bar{\nu}$ plus an arbitrary number of undetected soft photons (i.e. photons with energy lower than the detector resolution $\Delta E$) in the final states. At order $\alpha_{\text{EM}}$ only one photon matters.

The proposed method uses in a smart way the decomposition of decay amplitudes in a perturbative universal part and a non-universal structure-dependent part.
Calculation of $\Gamma(\Delta E)$ with $\Delta E \sim 30\text{MeV}$. Crucial ingredients:

- Finite volume regulates the IR divergences.
- Full calculation of the finite structure-dependent part of $\pi \rightarrow \ell\bar{\nu}$.
- The structure-dependent part of $\pi \rightarrow \ell\bar{\nu}\gamma$ is shown to be negligible.
- The universal part of $\pi \rightarrow \ell\bar{\nu}$ is calculated analytically in a $1/L$ expansion plus $\ln L$. Finite volume corrections to the structure-dependent part vanish like $1/L^2$.
- Exploratory electroquenced + QED$_L$.

\[ \frac{\Gamma_{\text{NLO}}(\Delta E \sim 30\text{MeV})}{\Gamma_{\text{LO}}(\Delta E \sim 30\text{MeV})} = 1.0210(15)(\ldots)_{\text{QED}} \]
IR divergences cancel in physical observables. Large logarithms may appear in physical observables as remnants of IR divergences. As a consequence a reliable estimate of the radiative corrections is problematic. In the last part of my talk I want to argue that these logs may be unexpectedly large.

- **Cancellation of IR divergences** in inclusive decay rates and cross sections.
  
  Bloch and Nordsieck, Phys. Rev. **52**, 54 (1937)

- **Factorisation**, exponentiation and cancellation of IR divergences. Universality.
  
  Grammer and Yennie, Phys. Rev. D **8**, 4332 (1973)

- IR divergences as a failure of perturbation theory for transition amplitudes.
  
  Lee and Nauenberg, Phys. Rev. **133**, B1549 (1964)

- More on universality.
  

- Estimate radiative corrections by separating universal and **structure-dependent** parts.
  
  Sirlin, Rev. Mod. Phys. **50**, 573 (1978)

- IR divergences as failure of standard (Haag and Ruelle) scattering theory.
  
  Immense work of Buchholz, Jackiw, Zwanziger...
Soft divergences at LO

- Photon propagator $k^{-2}$ is not enough to generate IR divergences.
- Euclidean $n$-point functions are IR finite. Soft divergences when matter propagators go on shell

\[ \bar{p}^2 = -M^2 \implies \frac{1}{(\bar{p} + k)^2 + M^2} = \frac{1}{2\bar{p}k + k^2} \approx \frac{1}{2\bar{p}k} \]
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Self energy

$$\Sigma(\bar{p}) \sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{2\bar{p}k} = \text{IR finite}$$

Scattering amplitude

$$A(\bar{p}, \bar{p}') \sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{1}{2\bar{p}k} = \text{IR divergent}$$

Wave function normalization:

$$\frac{1}{2\bar{p}k}$$
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$$\bar{p}^2 = -M^2 \Rightarrow \frac{1}{(\bar{p} + k)^2 + M^2} = \frac{1}{2\bar{p}k + k^2} \simeq \frac{1}{2\bar{p}k}$$

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Wave function normalization:

\[ \frac{\partial \Sigma}{\partial \bar{p}}(\bar{p}) \sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \left( \frac{1}{2\bar{p}k} \right)^2 = \text{IR divergent} \]

- At higher order the analysis is complicated by nested divergences.
Universality of soft divergences at LO

Soft logarithms involving hadrons are the same one would calculate in an effective theory in which hadrons are treated as point-like particles.

Effective theory:

\[
\pi^b L \ell \bar{\nu} \quad A \propto \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \left(-2p_\pi-k\right) \frac{1}{2p_\pi k + k^2} \gamma^\mu \frac{-i(p_\ell - k) + m_\ell}{-2p_\ell k + k^2} \gamma^\rho
\]
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Effective theory:

\[
[A]_{\text{IR div.}} \propto F_\pi \bar{p}_\pi^\mu \int_k \frac{1}{k^2} (-2 \bar{p}_\pi^\rho) \frac{1}{2 \bar{p}_\pi k} \gamma_\mu \frac{-i \bar{p}_\ell + m_\ell}{-2 \bar{p}_\ell k} \gamma_\rho
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Full theory:

- Effective (1PI) vertices, dressed propagators.
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**Effective theory:**

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\]

**Full theory:**

- Effective (1PI) vertices, dressed propagators.
- Effective (1PI) vertices are analytic around \( k = 0 \).
- Residues are analytic around the mass-shell.

IR finite
Universality of soft divergences at LO

Soft logarithms involving hadrons are the same one would calculate in an effective theory in which hadrons are treated as point-like particles.

Effective theory:

\[ [A]_{\text{IR div.}} \propto F_{\pi} \bar{p}_\pi^\mu \int_k \frac{1}{k^2} \left( -2\bar{p}_\pi^\rho \right) \frac{1}{2\bar{p}_\pi k} \gamma_\mu \frac{-i\bar{p}_\ell + m_\ell}{-2\bar{p}_\ell k} \gamma_\rho \]

Full theory:

- Effective (1PI) vertices, dressed propagators.
- Effective (1PI) vertices are analytic around \( k = 0 \).
- Residues are analytic around the mass-shell.

\[ [A]_{\text{IR div.}} \propto \Gamma_{\pi \gamma \bar{\pi}}^L(\bar{p}_\pi) \int_k \frac{Z_\gamma(0)}{k^2} \Gamma_{\rho \gamma \bar{\pi}}^\pi(\bar{p}_\pi, 0) \frac{Z_\pi(\bar{p}_\pi)}{2\bar{p}_\pi k} \gamma_\mu \frac{-i\bar{p}_\ell + m_\ell}{-2\bar{p}_\ell k} \gamma_\rho \]
Universality of soft divergences at LO

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\]

Full theory:

- Effective (1PI) vertices, dressed propagators.
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[A]_{\text{IR div.}} \propto \Gamma_{\mu}^{\pi\gamma\bar{\pi}}(\bar{p}_{\pi}) \int_k \frac{Z_{\gamma}(0)}{k^2} \Gamma_{\rho}^{\pi\gamma\bar{\pi}}(\bar{p}_{\pi}, 0) \frac{Z_{\pi}(\bar{p}_{\pi})}{2\bar{p}_{\pi} k} \gamma_{\mu} \frac{-i\bar{p}_{\ell} + m_{\ell}}{-2\bar{p}_{\ell} k} \gamma_{\rho}
\]

By definition \( \Gamma_{\mu}^{\pi L}(\bar{p}_{\pi}) \equiv F_{\pi} \bar{p}_{\pi}^{\mu} \)

Canonical normalization \( Z_{\gamma}(0) = Z_{\pi}(\bar{p}_{\pi}) = 1 \)

Ward identity \( k_\rho \Gamma_{\rho}^{\pi\gamma\bar{\pi}}(\bar{p}_{\pi}, k) = \Delta^{-1}(\bar{p}) - \Delta^{-1}(\bar{p} + k) = -2\bar{p}k + O(k^2) \)

\[ \Rightarrow \Gamma_{\rho}^{\pi\gamma\bar{\pi}}(\bar{p}_{\pi}, k) = -2\bar{p}_{\rho} + O(k) \]
Factorisation of soft divergences at LO

Introduce an IR regulator \( \mu \) that preserves unitarity, e.g. photon mass \( (\mu = m_\gamma) \).

\[
\pi \ell \bar{\nu} = \pi \ell \bar{\nu}
\]

\[
= \alpha_{EM} B(\bar{p}_\pi, \bar{p}_\ell) \ln \frac{m_\gamma}{\Lambda} \times \left[ \pi \ell \bar{\nu} \right] + \pi \ell \bar{\nu}
\]

\[
A_{NLO}(\alpha \to \beta; m_\gamma) = \left[ 1 + \frac{\alpha_{EM}}{2} R(\alpha \to \beta) \ln \frac{m_\gamma}{\Lambda} \right] \times A_{LO}(\alpha \to \beta) + A_{NLO, k^2 > \Lambda^2}(\alpha \to \beta)
\]

1-loop amplitude with \( m_\gamma > 0 \)

universal (known) function

tree-level amplitude with \( m_\gamma = 0 \)

1-loop amplitude with \( m_\gamma = 0 \) and \( k^2 > \Lambda^2 \) restriction
Physics interpretation: from the experimental point of view it is impossible to differentiate between
\[ h \rightarrow \ell + \bar{\nu}, \]
\[ h \rightarrow \ell + \bar{\nu} + N\gamma, \]
- if each photon is emitted with a lower energy than the detector resolution \( \Delta E \);
- and the total energy carried away by the undetected photons is (roughly) less than the resolution \( \Delta E \) with which we can reconstruct the lepton energy.
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The physical quantity is the decay rate integrated over soft photons, which is finite.\(^1\)

\[
\Gamma(\Delta E) = \lim_{m_{\gamma} \to 0} \frac{1}{2m_{\pi}} \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\sum_{k_{\alpha} < \Delta E}^{k_{\alpha} < \Delta E}} \mathrm{d} \Phi_{N\gamma} \left| \langle \pi | H_W | \ell, \bar{\nu}, N\gamma \rangle \right|^2 =
\]

\[
= \frac{1}{2m_{\pi}} \left| \begin{array}{c} \pi \\ \ell \end{array} \right| + \frac{1}{2m_{\pi}} \left| \begin{array}{c} \pi \\ \bar{\nu} \end{array} \right| + \frac{1}{2m_{\pi}} \left| \begin{array}{c} \pi \\ \bar{\nu} \end{array} \right|
\]

\(^1\)The diagrammatic expansion is wrong. I am deliberately neglecting the wave-function renormalization for sake of presentation.
**Bloch-Nordsieck prescription**

Physics interpretation: from the experimental point of view it is impossible to differentiate between

\[
    h \rightarrow \ell + \bar{\nu}, \\
    h \rightarrow \ell + \bar{\nu} + N\gamma,
\]

- if each photon is emitted with a lower energy than the detector resolution \(\Delta E\);
- and the total energy carried away by the undetected photons is (roughly) less than the resolution \(\Delta E\) with which we can reconstruct the lepton energy.

The physical quantity is the decay rate integrated over soft photons, which is finite.\(^1\)

\[
    \Gamma(\Delta E) = \lim_{m_\gamma \to 0} \frac{1}{2m_\pi} \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\sum_\alpha k_\alpha < \Delta E}^{k_\alpha < \Delta E} d\Phi_{N\gamma} |\langle \pi | H_W | \ell, \bar{\nu}, N\gamma \rangle|^2 = \\
    = \frac{1}{2m_\pi} \left| \pi \rightarrow \ell + \bar{\nu} + \pi \rightarrow \ell + \bar{\nu} \right|^2 + \frac{1}{2m_\pi} \int_{1\gamma} \left| \pi \rightarrow \ell + \bar{\nu} + \pi \rightarrow \ell + \bar{\nu} \right|^2
\]

The logarithm in the photon mass is traded for a logarithm in the energy resolution:

\[
    \langle \pi | H_W | \ell, \bar{\nu} \rangle = \left[ 1 + \frac{\alpha_{EM}}{2} R \ln \frac{m_\gamma}{\Lambda} \right] A_{LO} + A_{NLO, k^2 > \Lambda^2} + O(\alpha_{EM}^2)
\]

\[
    \Rightarrow \quad \Gamma(\Delta E) = \left[ 1 + \alpha_{EM} \text{Re} R \ln \frac{\Delta E}{\Lambda} \right] \Gamma_{LO} + \Gamma_{NLO, k^2 > \Lambda^2} + O(\alpha_{EM}^2)
\]

\(^1\) The diagrammatic expansion is wrong. I am deliberately neglecting the wave-function renormalization for sake of presentation.
Large collinear logarithms

We consider the (phenomenologically irrelevant) decay process

\[ B^- \rightarrow e^- + \bar{\nu}_e \]

\[ \Gamma(\Delta E) = \left[ 1 + \alpha_{\text{EM}} \text{Re} R \ln \left( \frac{\Delta E}{\Lambda} \right) \right] \times \Gamma_{\text{LO}} + \Gamma_{\text{NLO}, k^2 > \Lambda^2} + O(\alpha_{\text{EM}}) \]

- Back of the envelope calculation
  - Re \( R \approx -2 + \ln(m_B^2/m_e^2) \approx 16.5 \)
  - \( \Lambda \approx m_B, \Delta E/m_B \approx 10\% \)
  - \( \alpha_{\text{EM}} \text{Re} R \ln \Delta E/m_B = 28\% \)
  - \( \frac{1}{2} (\alpha_{\text{EM}} \text{Re} R \ln \Delta E/m_B)^2 = 4\% \)

1-loop decay rate with \( k^2 > \Lambda^2 \) restriction on photon loop momenta
We consider the (phenomenologically irrelevant) decay process

\[ B^- \rightarrow e^- + \bar{\nu}_e \]

\[
\Gamma(\Delta E) = \left[ 1 + \alpha_{\text{EM}} \text{Re} R \ln \frac{\Delta E}{\Lambda} \right] \times \Gamma_{\text{LO}} + \Gamma_{\text{NLO}, k^2 > \Lambda^2} + \mathcal{O}(\alpha_{\text{EM}})
\]

Back of the envelope calculation

\[
\text{Re} R \simeq -2 + \ln \left( \frac{m_B^2}{m_e^2} \right) \simeq 16.5
\]

\[
\Lambda \simeq m_B, \quad \Delta E/m_B \simeq 10\%
\]

- \( \alpha_{\text{EM}} \text{ Re} R \ln \Delta E/m_B = 28\% \)
- \( \frac{1}{2} (\alpha_{\text{EM}} \text{ Re} R \ln \Delta E/m_B)^2 = 4\% \)

This hard collinear logarithm is not universal, it reads the structure of the B meson and has to be calculated nonperturbatively!
Conclusions

▶ When aiming at the percent precision, isospin breaking corrections must be included.

▶ Activity in this direction has been growing significantly in the past few years.

▶ QED and QCD are very different theories. Inclusion of QED effects implies a shift in the standard paradigm of lattice simulations.

▶ Description of charged states in a finite box is somewhat challenging. Effects of nonlocality are not systematically understood. I advocate the use of setups that are theoretically under control.

▶ Numerical calculations of masses are already at an advanced stage from the technical point of view. The challenge ahead is the full calculation of radiative corrections to decay rates.

▶ (Almost) IR divergences may generate large logarithms in heavy meson decay rates. Potentially lattice QCD can have a big impact there.

Thank you!