Stochastic reconstruction of charmonium spectral functions at finite temperature

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in collaboration with
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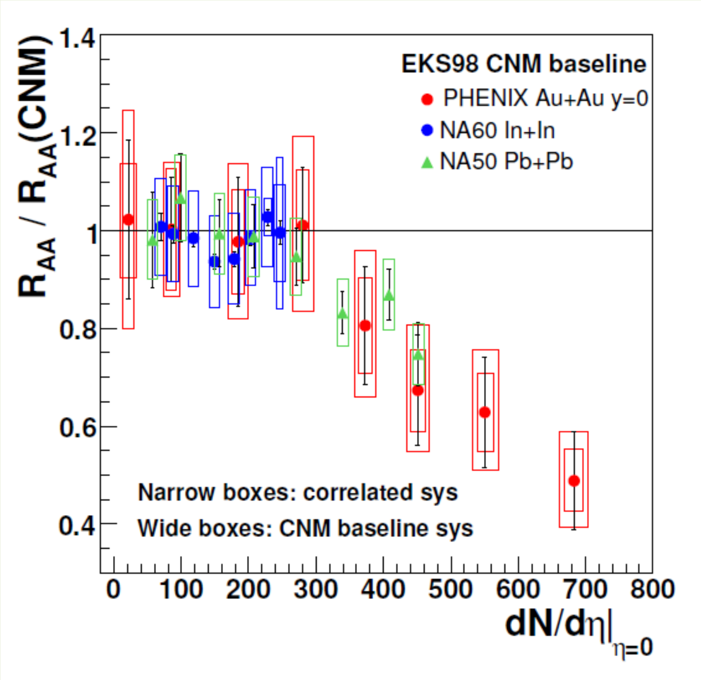
Motivation

- Charmonium spectral function (SPF)
  - has all information about in-medium properties of charmonia

Charmonium dissociation temperature

→ Important to understand charmonium suppression

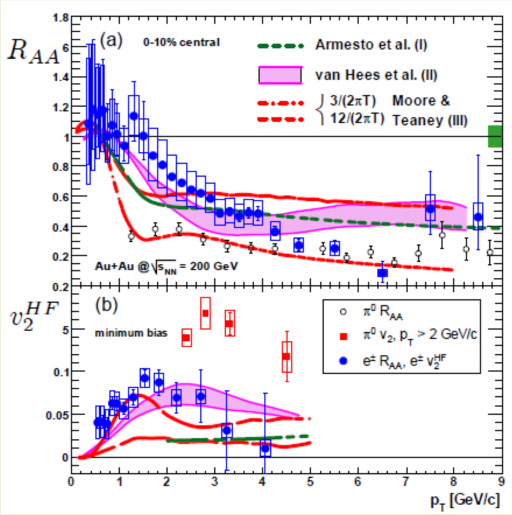


Heavy quark diffusion coefficient

$$D = \frac{1}{6\chi_{00}} \lim_{\omega\to 0} \sum_{i=1}^{3} \frac{\rho_{ii}^V(\omega,\mathbf{0})}{\omega}$$

$\rho_{ii}^V(\omega)$: vector SPF

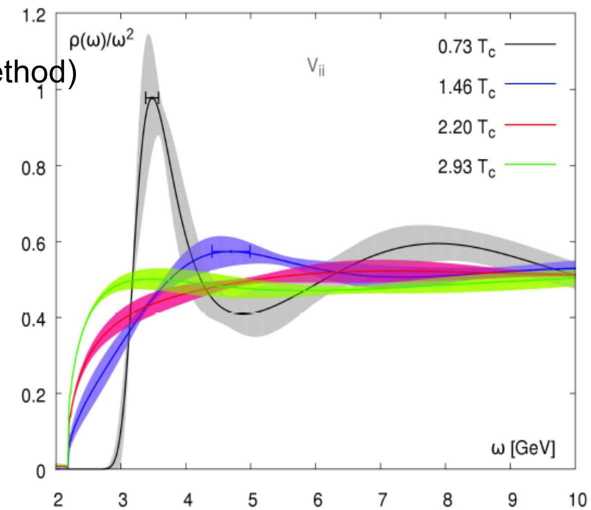
→ Important input for hydro models



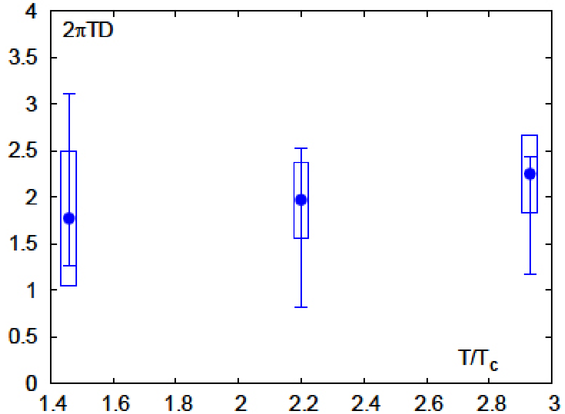
Recent lattice studies on charmonium SPF

- Several studies both in quenched QCD and with dynamical quarks, using MEM (Maximum Entropy Method)
  - Dissociation temperatures are still not conclusive
- The heavy quark diffusion coefficient has been computed

H.-T. Ding *et al.*, PRD 86 (2012) 014509



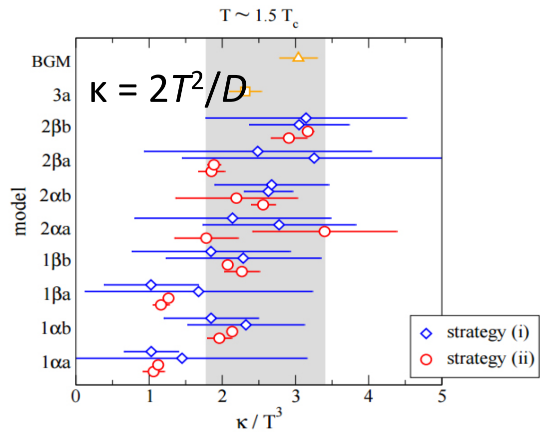
H.-T. Ding *et al.*, PRD 86 (2012) 014509



Quenched QCD:
no continuum limit, with MEM

$2\pi TD \approx 2$

A. Francis, O. Kaczmarek, M. Laine, T. Neuhaus and HO, PRD 92 (2015) no.11, 116003



Heavy quark EFT:
the only continuum result so far, based on theoretically motivated fits

$\kappa/T^3 = 1.8 - 3.4 \rightarrow 2\pi TD = 3.7 - 7.0$

- Perturbative estimate

  $2\pi DT \approx 71.2$ in LO

  Moore and Teaney, PRD 71 (2005) 064904

  $2\pi DT \approx 8.4$ in NLO

  Caron-Hout and Moore, PRL 100 (2008) 052301
- Strong coupling limit

  $2\pi DT \approx 1$

  Kovtun, Son and Starinets, JHEP 0310 (2004) 064

It is important to crosscheck previous results with more precise SPFs given on larger and finer lattice and by different methods.

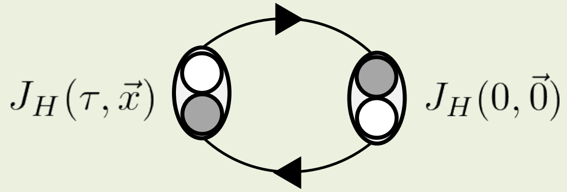
Reconstruction of SPF

Euclidian (imaginary time) meson correlation function

$$G_H(\tau,\vec{p}) \equiv \int d^3x e^{-i\vec{p}\cdot\vec{x}} \langle J_H(\tau,\vec{x}) J_H(0,\vec{0}) \rangle$$

$$= \int_0^\infty \frac{d\omega}{2\pi} \rho_H(\omega,\vec{p}) K(\omega,\tau)$$

Spectral function (SPF)



$$K(\omega,\tau) \equiv \frac{\cosh[\omega(\tau - 1/2T)]}{\sinh(\omega/2T)} \qquad J_H(\tau,\vec{x}) \equiv \bar{\psi}(\tau,\vec{x})\Gamma_H \psi(\tau,\vec{x})$$

- Computing SPF → **Ill-posed problem**
  - # of data points of a correlator is *O*(10) while a SPF needs O(1000) data points.
  - In general, simple χ^2 fitting does not work!
- Several ways to reconstruct SPF
  - MEM M. Asakawa, T. Hatsuda and Y. Nakahara, Prog.Part.Nucl.Phys. 46 (2001) 459-508
  - A new Bayesian method Y. Burnier and A. Rothkopf, PRL 111 (2013) 18, 182003
  - Stochastic methods **→ our approach (See also H.-T. Shu's talk [Thu. 7/28 @ 14:40])**

Stochastic method: comparison with MEM

MEM

- Most likely solution

$$\max P[A|\bar{G}] \leftrightarrow \min F \leftrightarrow \frac{\delta F}{\delta A} = 0$$

where

$$P[A|\bar{G}] \propto e^{-F} \quad F \equiv \chi^2/2 - \alpha S$$

- Prior information

$$S \equiv -\int d\omega\ A(\omega) \ln\left(\frac{A(\omega)}{D(\omega)}\right)$$

- Eliminating α

$$P[\alpha|\bar{G}] \propto P[\alpha] \int \mathcal{D}A\, e^{-F[A]}$$

where

$$P[\alpha] = 1,\ 1/\alpha$$

➡

$$\langle A(\omega)\rangle = \int d\alpha A_\alpha(\omega) P[\alpha|\bar{G}]$$

Stochastic methods

- Most likely solution

$$\mathcal{D}'n \equiv \mathcal{D}n\ \Theta[n]\delta\left(\int dx\ n(x) - \bar{G}(\tau_0)\right)$$

Stochastically evaluate

$$\langle n(x)\rangle_\alpha = \int \mathcal{D}'n\ n(x) e^{-\chi^2/2\alpha} \quad \text{for} \quad n(x) \equiv \frac{A(\phi^{-1}(x))}{D(\phi^{-1}(x))}$$

SAI (Stochastic Analytical Inference)

S. Fuchs *et al.*, PRE81, 056701 (2010)
HO, PoS LATTICE 2015, 175 (2016)

- Prior information

$$x \equiv \phi(\omega) = \frac{1}{N} \int_{-\infty}^{\omega} d\omega' D(\omega')$$

- Basis

δ functions

- Eliminating α

$$P[n|\bar{G}] \propto P[\alpha] \int \mathcal{D}'n\ e^{-\chi^2[n]/2\alpha}$$

➡ $\langle\langle n(x)\rangle\rangle = \int d\alpha \langle n(x)\rangle_\alpha P[n|\bar{G}]$

SOM (Stochastic Optimization Method)

A. S. Mishchenko *et al.*, PRB62, 6317 (2000)
H.-T. Shu *et al*, PoS LATTICE 2015, 180 (2016)

- Prior information

None ($x = \omega$)

dose not rely on DM!

- Basis

Boxes

- Eliminating α

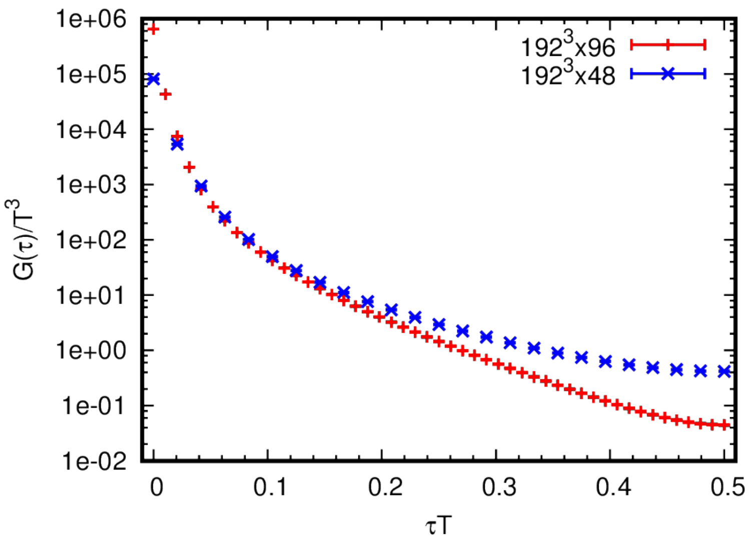
Choosing α
at a critical point of $\langle\chi^2\rangle_\alpha$

Lattice setup

- Standard plaquette gauge action + O(a)-improved Wilson valence quarks
- In the quenched approximation
- On fine and large isotropic lattices
  - $\beta = 7.793 \rightarrow a = 0.009$ fm ($a^{-1} = 22$ GeV)
  - $N_\sigma = 192$

    The scale has been set by r_0=0.49fm and with an interpolation for r_0/a
    in A. Francis, O. Kaczmarec, M. Laine, T. Neuhaus, HO, PRD 91 (2015) 9, 096002
  - $N_\tau = 96, 48 \rightarrow \quad T = 0.75, 1.5T_c$
  - $\kappa = 0.13211 \rightarrow m_V$= 3.234(9) GeV

    Experimental values: $m_{J/\psi}$ = 3.096.916(11) GeV
    J. Beringer *et al*. [PDG], PRD 86 (2012) 010001
- Vector channel



Normalization and relation btw SAI and SOM

- Normalization (to cure divergence of $K(\omega,\tau)$ at $\omega = 0$)

$$A(\omega) = \rho(\omega) K(\omega, \tau_0)$$

$$D'(\omega) = D(\omega) K(\omega, \tau_0)$$

$$\tilde{K}(\omega, \tau) \equiv \frac{K(\omega, \tau)}{K(\omega, \tau_0)} = \frac{\cosh[\omega(\tau - 1/2T)]}{\cosh[\omega(\tau_0 - 1/2T)]}$$

$\tau_0/a = 4$ is chosen to eliminate cutoff effects

SOM is equivalent to SAI with

$$D'(\omega) = 1 \;\Rightarrow\; D(\omega) = K^{-1}(\omega, \tau_0)$$

Default models

- Resonance (Res)

$$\rho(\omega) = \frac{\Gamma M}{(\omega^2 - M^2)^2 + M^2\Gamma^2}\frac{\omega^2}{\pi} \qquad \Gamma = \Theta(\omega - \omega_0)\gamma_0\left(1 - \frac{\omega_0^2}{\omega^2}\right)^5$$

- Transport peak (Trans)

$$\rho(\omega) = \frac{\omega\eta}{\omega^2 + \eta^2}$$

- Free Wilson (Free)

$$\rho(\omega) = \frac{N_c}{L^3}\sum_{\boldsymbol{k}} \sinh\left(\frac{\omega}{2T}\right)\left[b^{(1)} - b^{(2)}\frac{\sum_{i=1}^{3}\sin^2 k_i}{\sinh^2 E_{\boldsymbol{k}}(m)}\right] \times \frac{\delta(\omega - 2E_{\boldsymbol{k}}(m))}{(1+\mathcal{M}_{\boldsymbol{k}}(m))^2\cosh^2 E_{\boldsymbol{k}}(m)}$$

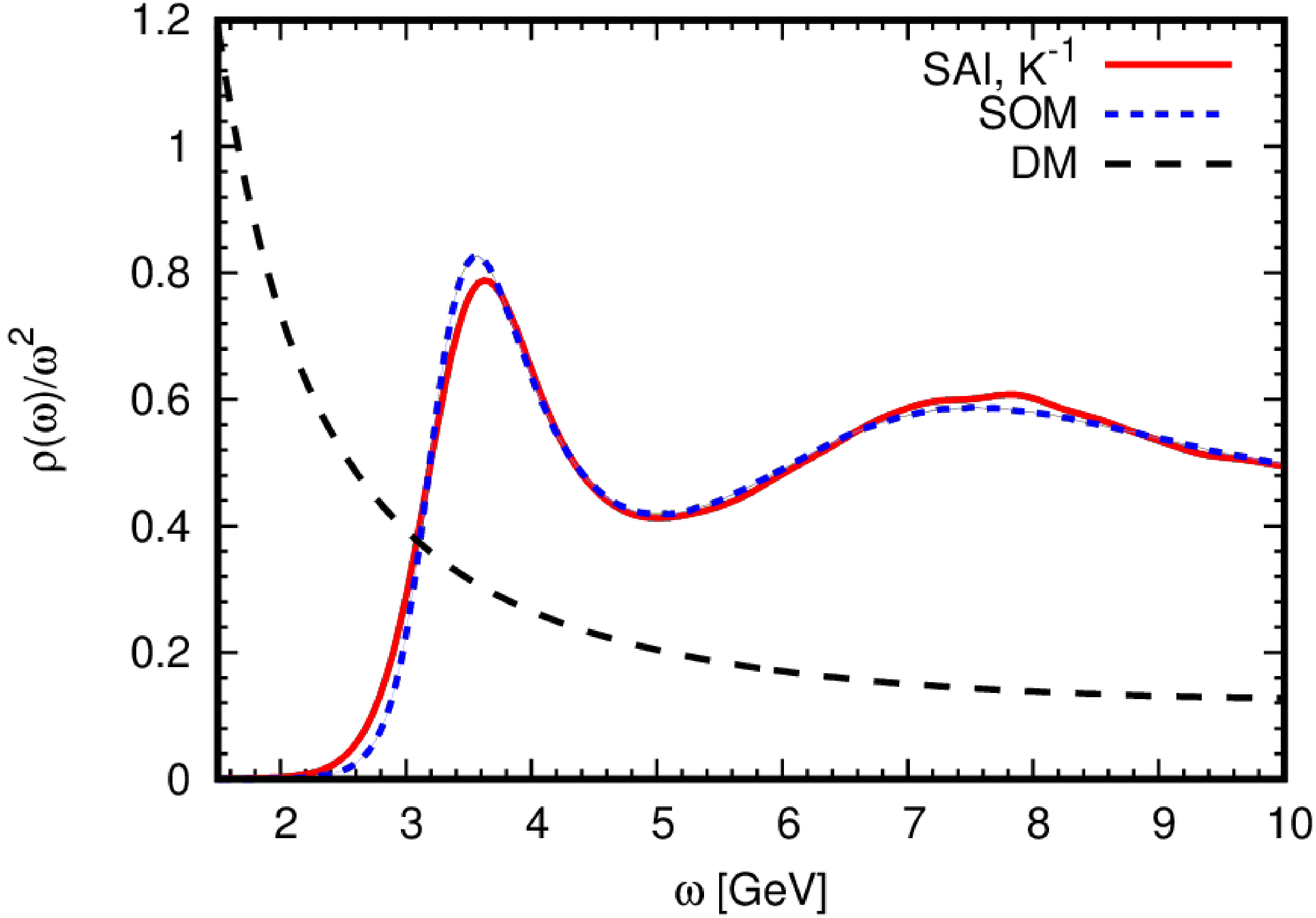
$$\cosh E_{\boldsymbol{k}}(m) = 1 + \frac{\mathcal{K}_{\boldsymbol{k}}^2 + \mathcal{M}_{\boldsymbol{k}}^2(m)}{2(1+\mathcal{M}_{\boldsymbol{k}}(m))}$$

$$\mathcal{K}_{\boldsymbol{k}} = \sum_{i=1}^{3}\gamma_i \sin k_i$$

$$\mathcal{M}_{\boldsymbol{k}}(m) = \sum_{i=1}^{3}(1 - \cos k_i) + m$$

$b^{(1)} = 3$, $b^{(2)} = 1$ for the V channel

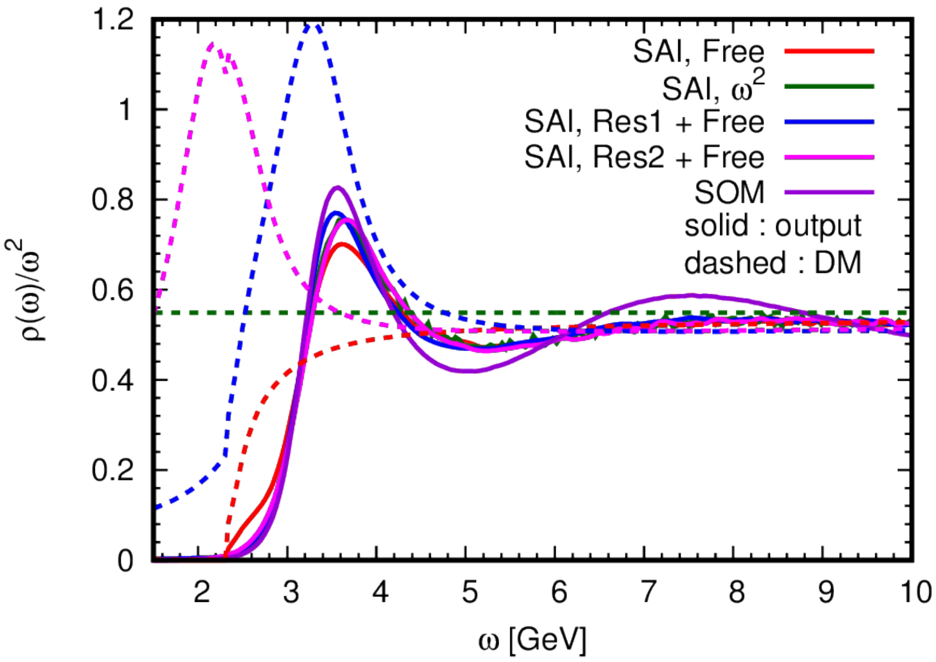
Comparison between SAI and SOM at $0.75T_c$



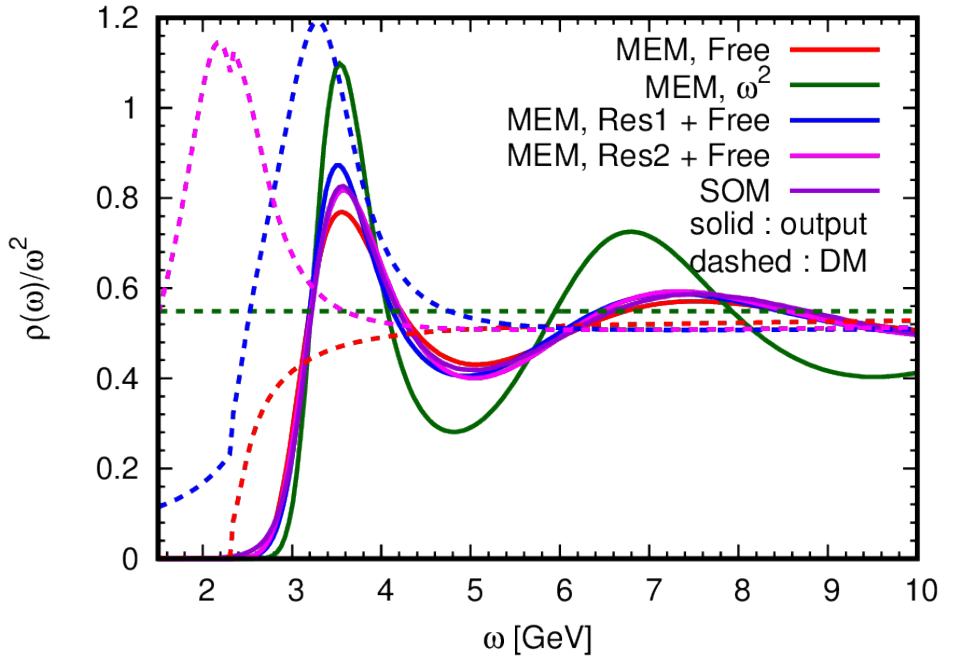
The SOM result is almost consistent with the SAI result with $D(\omega) = K^{-1}(\omega,\tau_0)$.

Default model dependence at $0.75T_c$ (1)

SAI



MEM



DM = Free, ω^2, Res1 + Free, Res2 + Free
Res1 : peak location ~ J/Ψ mass
Res2 : peak location < J/Ψ mass

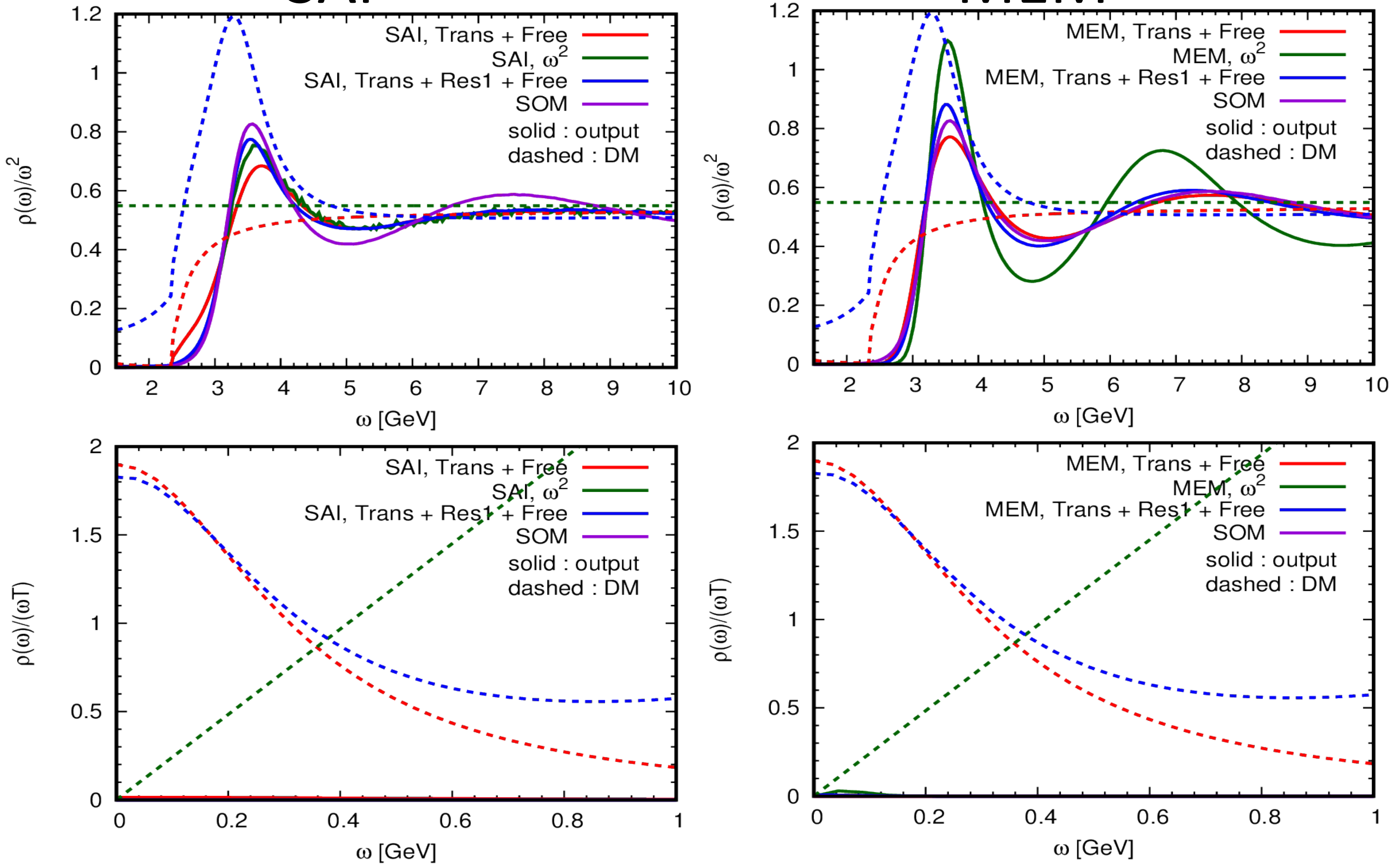
SAI and MEM have weak DM dependence except for $D(\omega) \sim \omega^2$ for MEM.

Default model dependence at $0.75T_c$ (2)

SAI MEM

High ω

Low ω



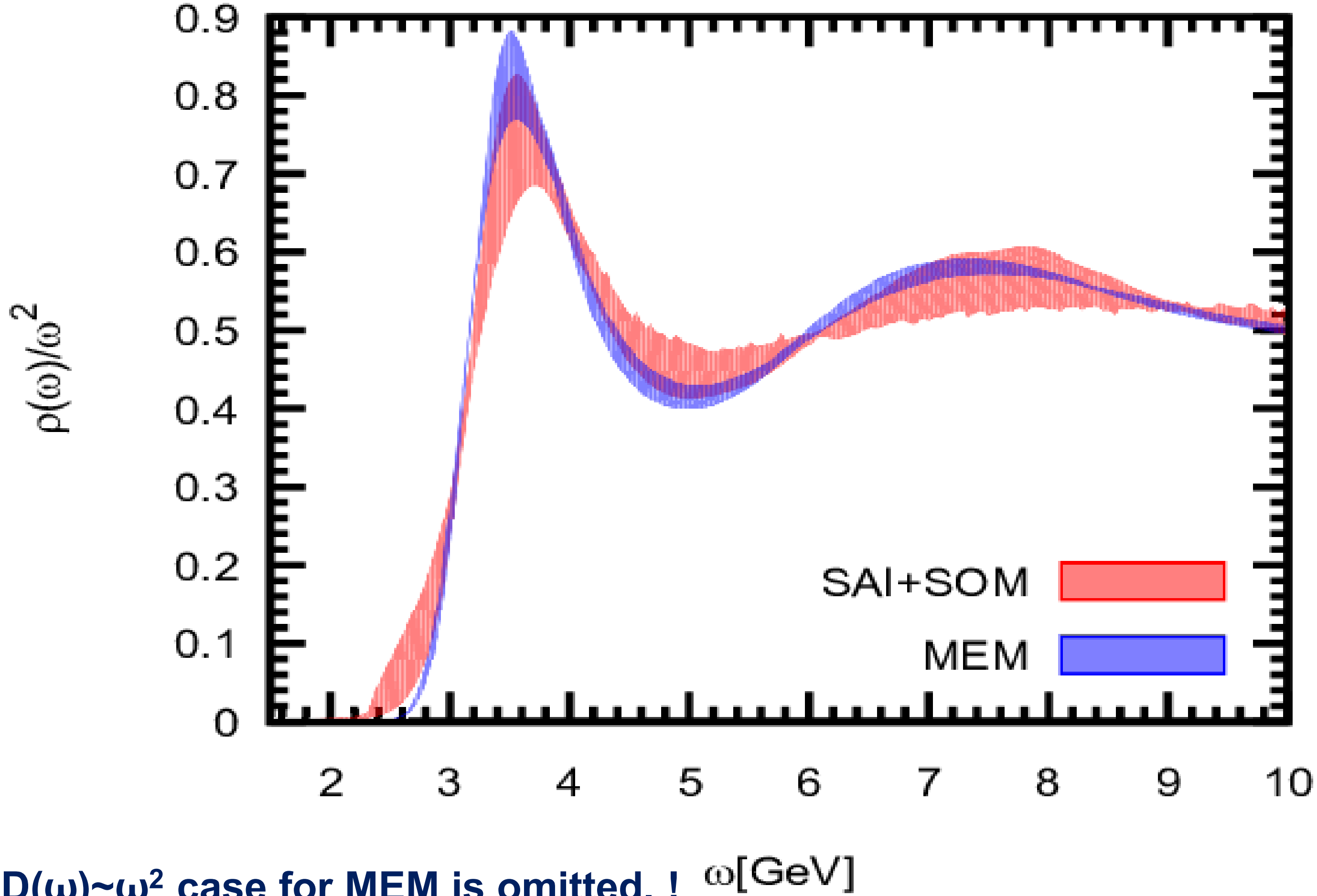
DM = Trans + Free, ω^2, Trans + Res1 + Free

The high ω part is insensitive to the transport peak of DM.

The intercept is quite small.

Systematic uncertainties at 0.75T_c

Band = Lowest—Highest values among all DM

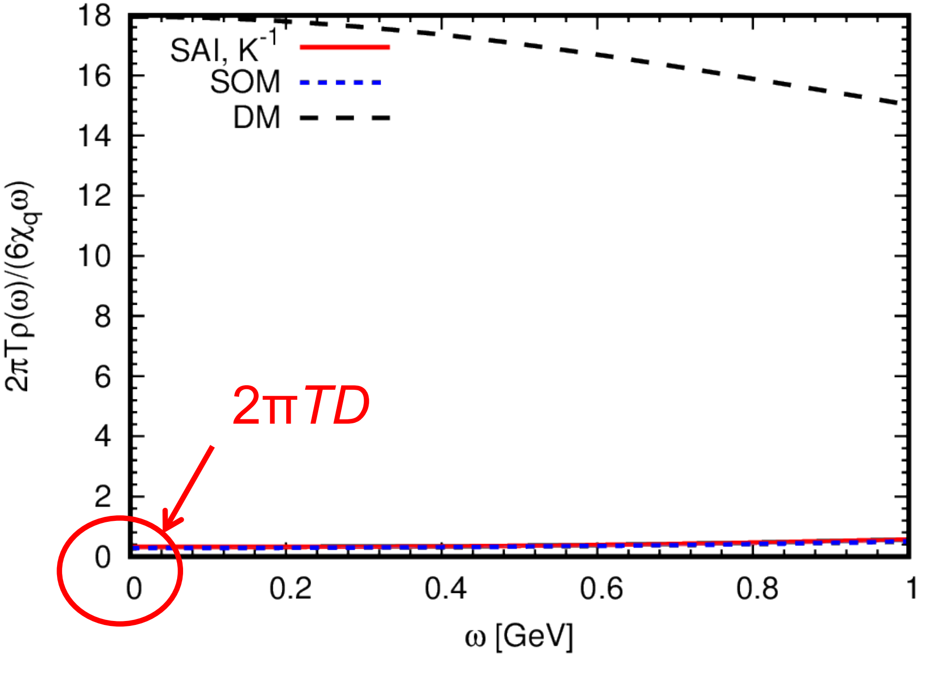


! The D(ω)~ω² case for MEM is omitted. !

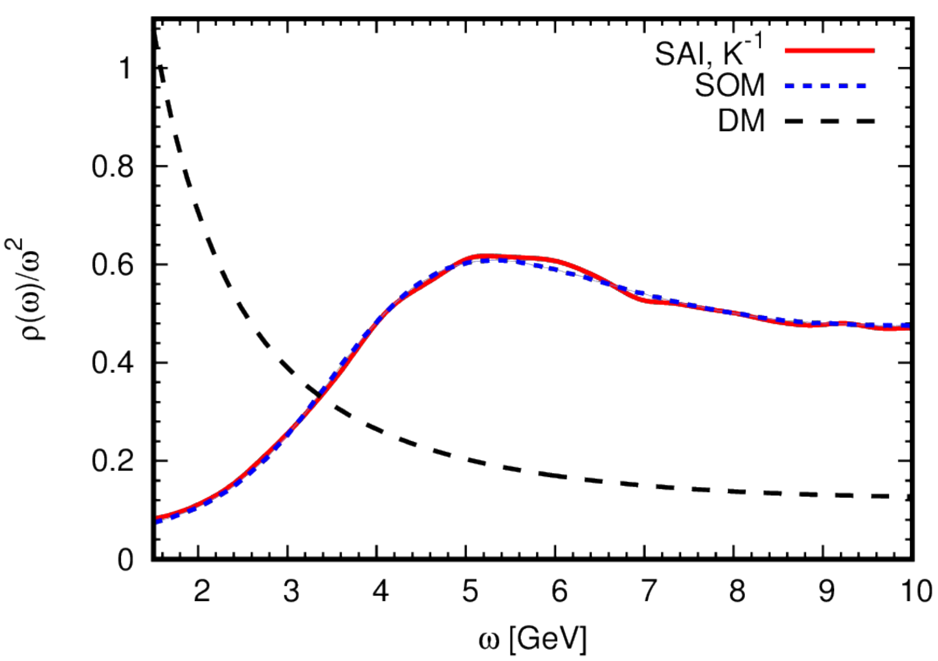
The first peak is stable. → J/Ψ exists.

Comparison between SAI and SOM at $1.5T_c$

Low ω



High ω



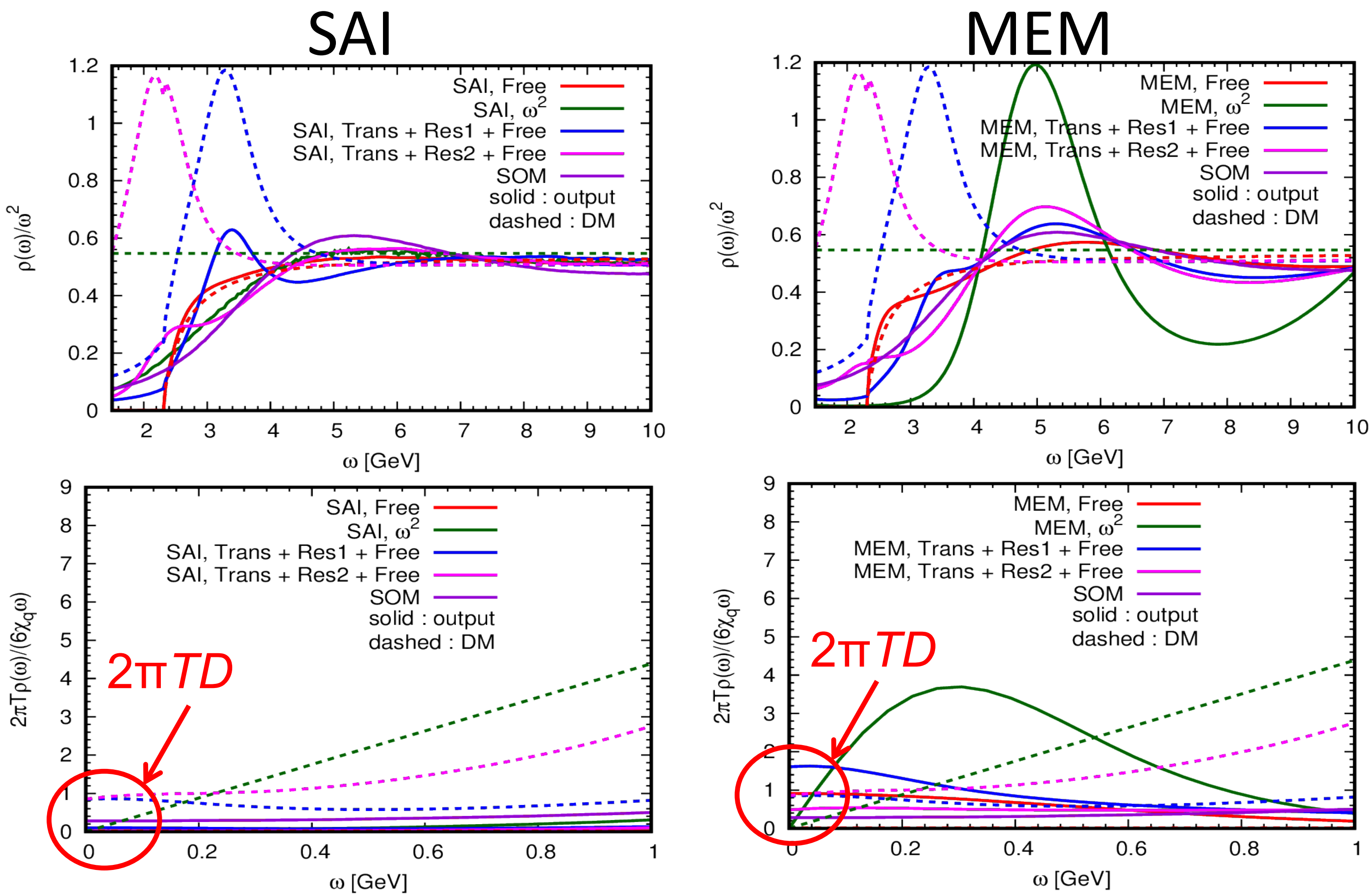
The SOM result is almost consistent with the SAI result with $D(\omega) = K^{-1}(\omega,\tau_0)$

The intercept = $2\pi TD$ is very small

Default model dependence at 1.5T_c (1)

High ω

Low ω

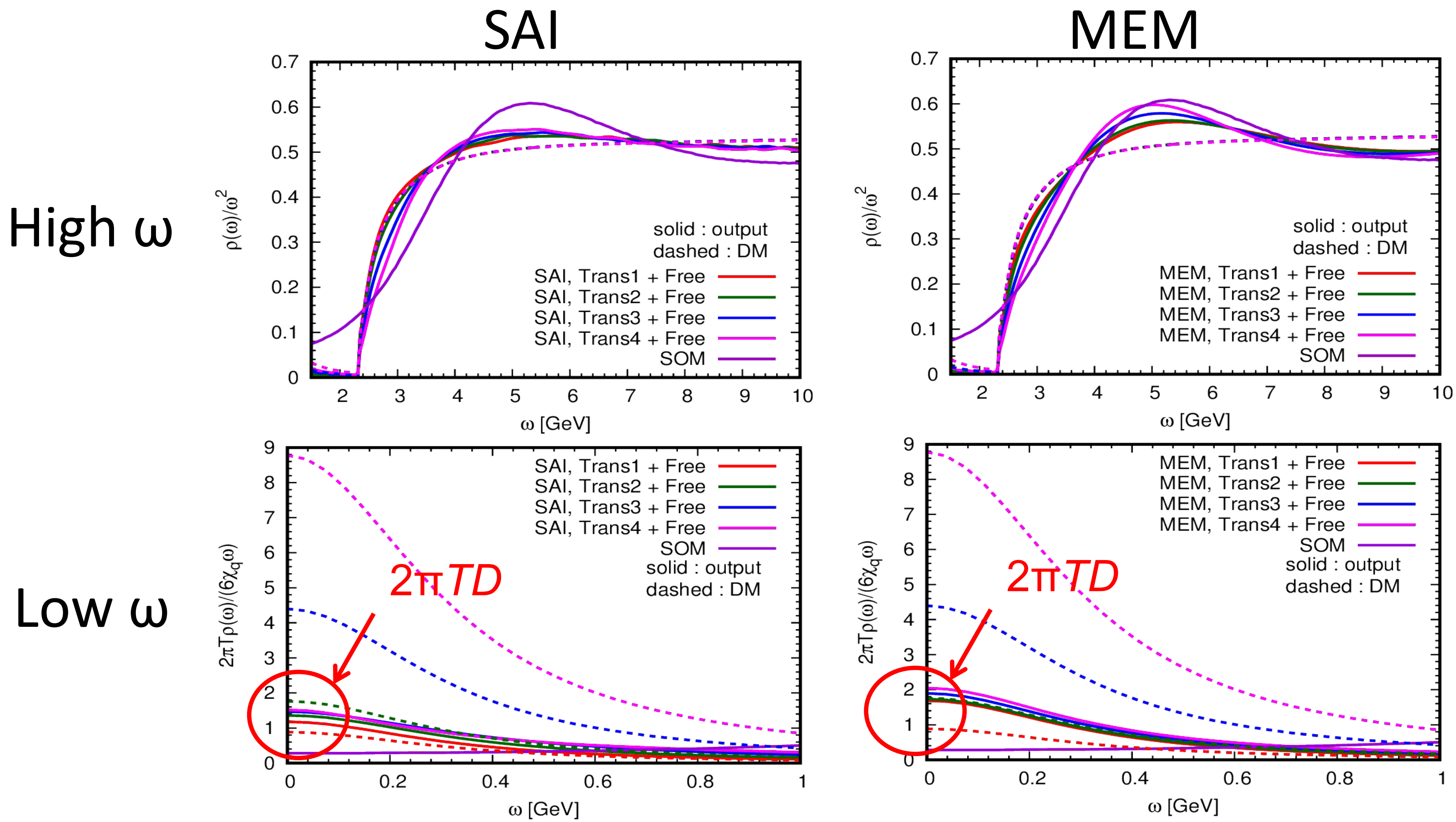


DM = Free, ω^2, Trans1 + Res1 + Free, Trans1 + Res2 + Free   Trans1 → 2πTD=1

There is no clear peak around J/Ψ, except for Trans+Res1+Free DM result for SAI.

The intercept is quite small for SAI. ω^2 DM result for MEM has strange behavior.

Default model dependence at 1.5T_c (2)



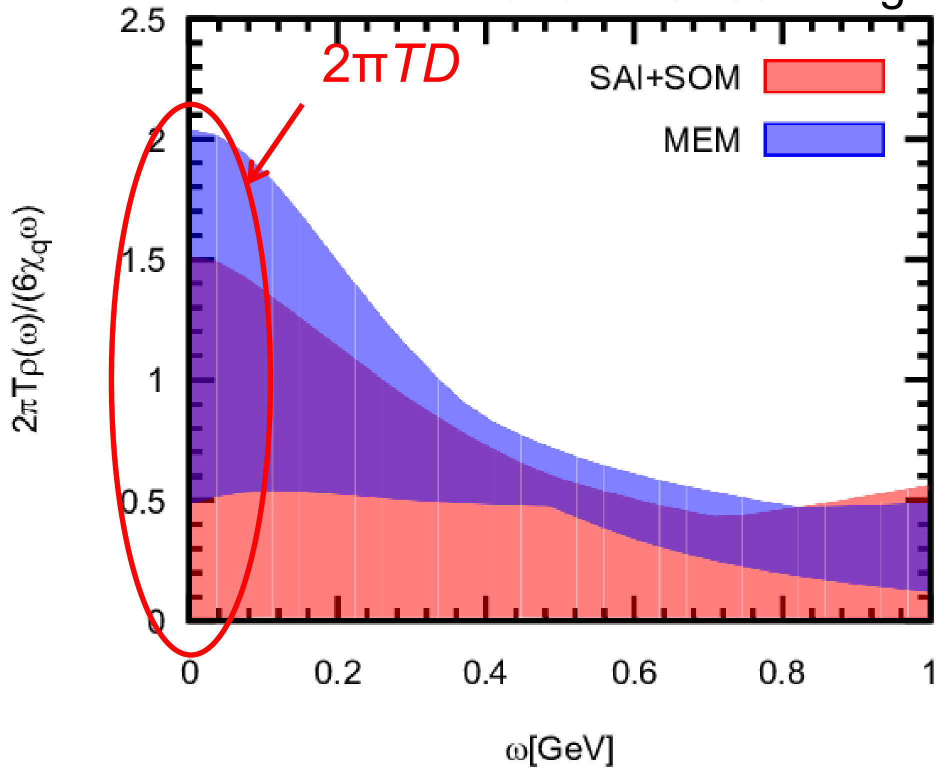
DM = Trans(1-4) + Free Trans(1-4) → 2π*TD*~1, 2, 4, 9 (fixed width)

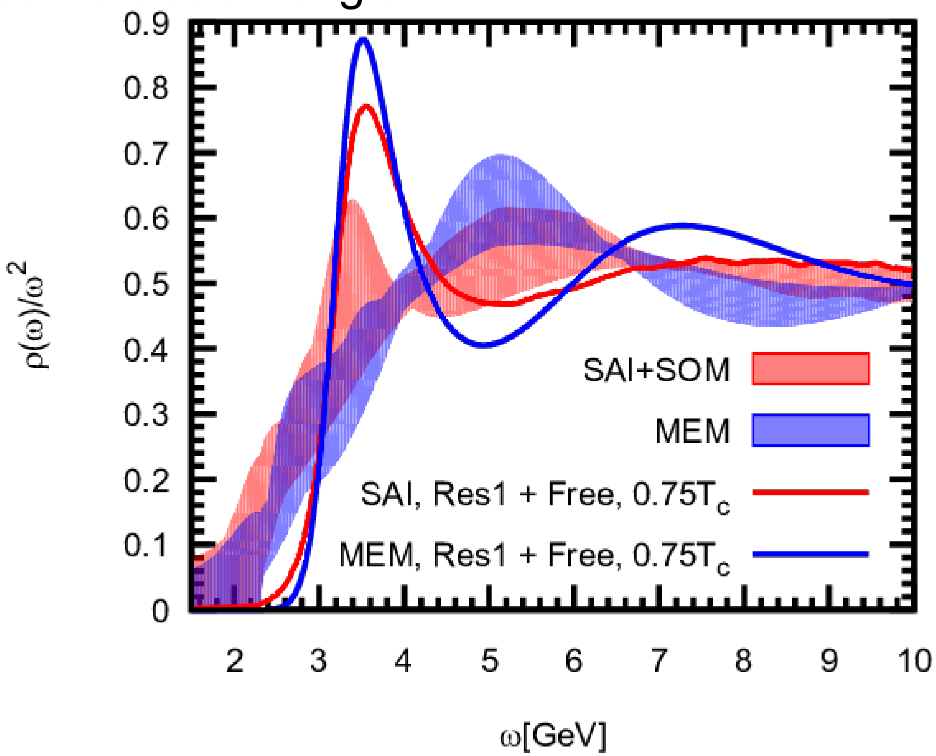
There is no clear peak around J/Ψ. The intercept seems to have an upper bound.

The intercept for SAI → 2π*TD* ~ 1.5, for MEM → 2π*TD* ~ 2

Systematic uncertainties at 1.5T_c

Band = Lowest—Highest values among all DM





! The D(ω)~ω² case for MEM is omitted. !

Melting of J/Ψ is not conclusive so far, although most of the cases in our analysis suggests no clear J/Ψ peak.

There seems to be an upper bound of 2π*TD*, which is 1.5—2 in this study, while a lower bound is not clear.

Summary & outlook

- We investigated vector charmonium SPFs
  - on very large and fine quenched lattices
  - with both MEM and stochastic methods (SAI and SOM)
  - at $0.75T_c$ and $1.5T_c$
- Both MEM and the stochastic methods gave almost DM-independent stable SPFs having a clear bound state peak at $0.75T_c$.
- Most of the results suggest that J/Ψ may be melted around $1.5T_c$ but more detailed study is needed to conclude.
- So far we observed an upper bound of $2\pi TD$, which is 1.5－2 at $1.5T_c$ in this preliminary study.
- Work in progress:
  - further checks of the DM-dependence and other systematic uncertainties
  - analysis of the temperature and quark mass dependence as well as other channels
  - continuum extrapolation

End

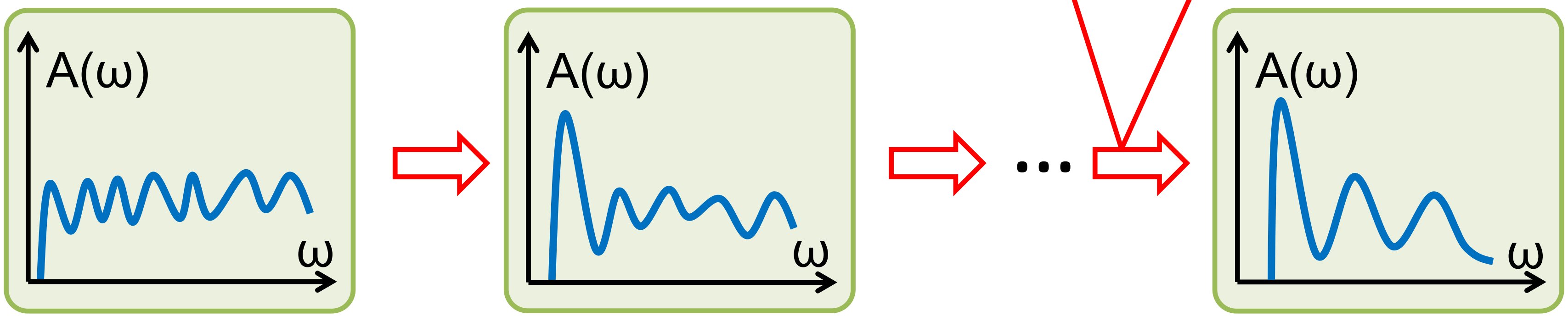
Backup slides

Stochastic method: basic idea

For given α (fictitious temperature, regularization parameter),

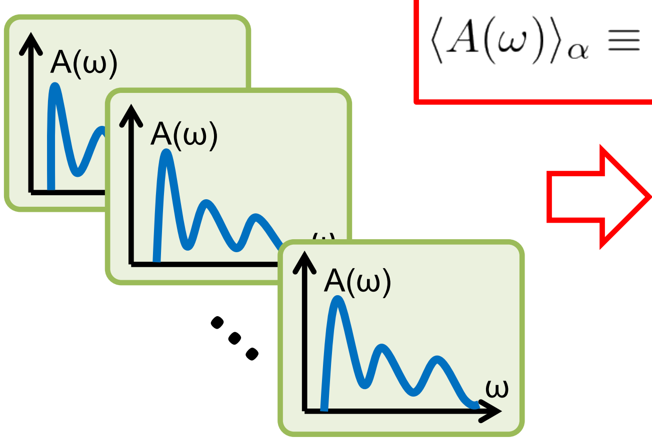
1. generate SPFs stochastically

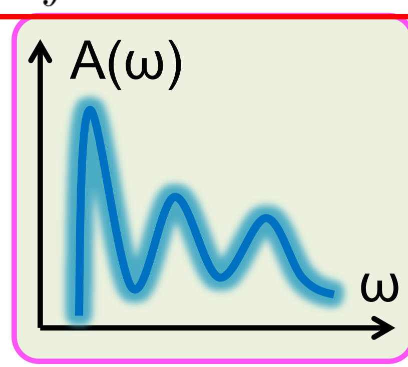
$$P = \min\{1, e^{-\chi^2/2\alpha}\}$$



2. average over all possible spectra

$$\langle A(\omega)\rangle_\alpha \equiv \frac{1}{Z}\int \mathcal{D}A\; A(\omega) e^{-\chi^2/2\alpha} \qquad Z \equiv \int \mathcal{D}A\; e^{-\chi^2/2\alpha}$$





$$\chi^2 \equiv \sum_{\tau,\tau'} \Delta G(\tau) C^{-1}_{\tau,\tau'} \Delta G(\tau')$$

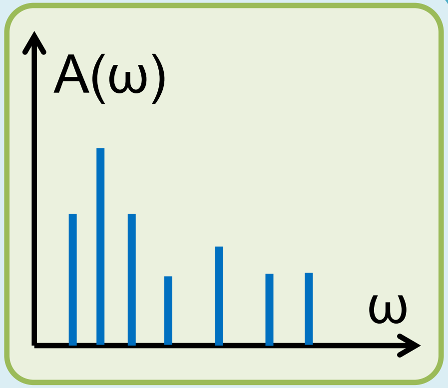
$C^{-1}_{\tau,\tau'}$: covariance matrix

$$\Delta G(\tau) \equiv \bar{G}(\tau) - \int d\omega\; A(\omega)\tilde{K}(\omega,\tau)$$

Stochastic method: basis

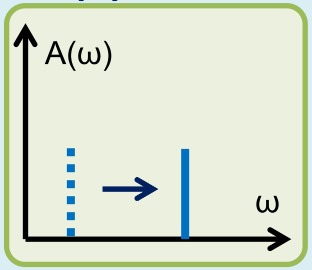
δ functions

$$A(\omega) = \sum_i r_i \delta(\omega - a_i)$$

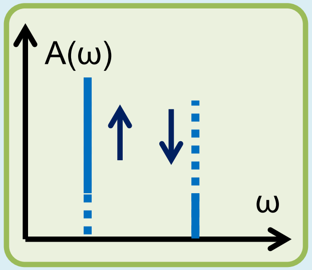


- Update schemes

(a) Shift



(b) Change residues



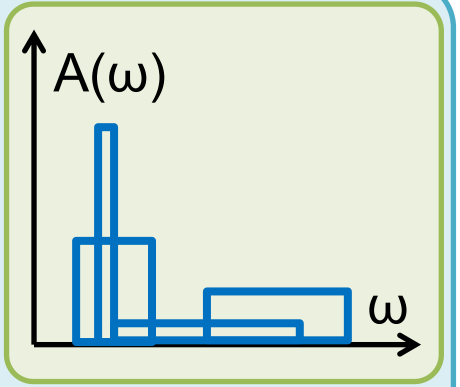
- Constraint

$$\sum_i r_i = \bar{G}(\tau_0)$$

Boxes

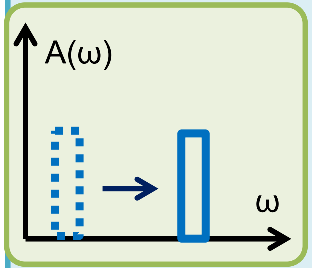
$$A(\omega) = \sum_i h_i R(\omega, a_i, w_i)$$

$$R(\omega, a_i, w_i) \equiv \begin{cases} 1 & (a_i - \frac{w_i}{2} \le \omega \le a_i + \frac{w_i}{2}) \\ 0 & (\text{otherwise}) \end{cases}$$

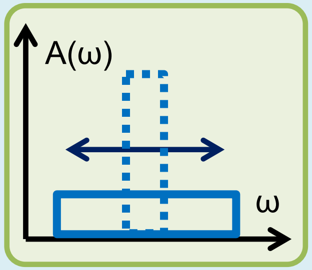


- Update schemes

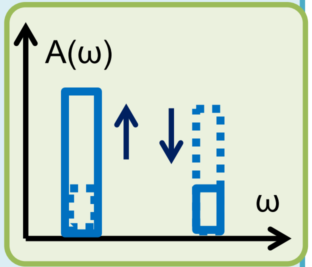
(a) Shift



(b) Change width



(c) Change heights



- Constraint

$$\sum_i h_i w_i = \bar{G}(\tau_0)$$

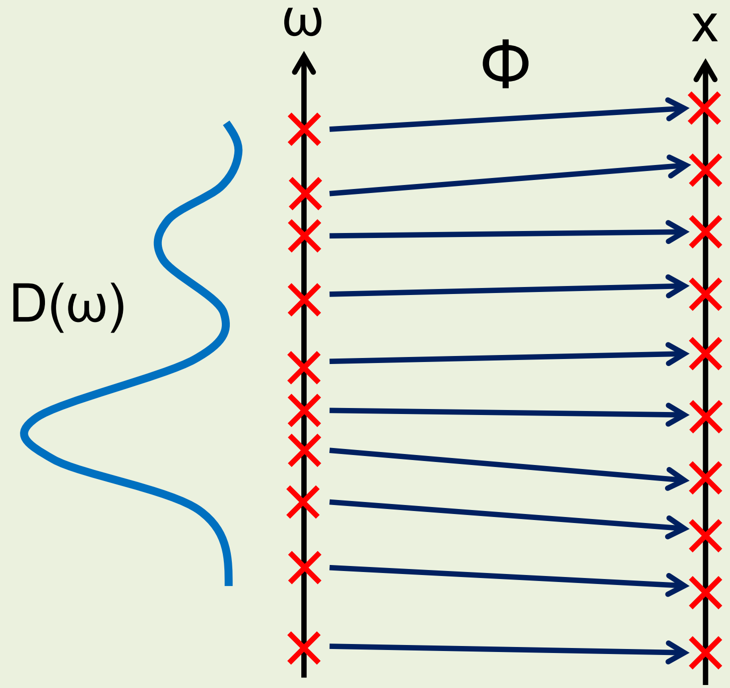
Update schemes which change the number of the basis are also possible.

Stochastic method: default model

K.S.D. Beach, arXiv:cond-mat/0403055

$$x \equiv \phi(\omega) = \frac{1}{\mathcal{N}} \int_{-\infty}^{\omega} d\omega' D(\omega')$$

$D(\omega)$: Default model (prior information)



$$G(\tau) = \int d\omega \; A(\omega)\tilde{K}(\omega,\tau)$$

$$= \int dx \; n(x)\tilde{K}(\phi^{-1}(x),\tau)$$

$$n(x) \equiv \frac{A(\phi^{-1}(x))}{D(\phi^{-1}(x))}$$

$$\bar{G}(\tau_0) - \int dx \; n(x) = 0$$

$$\langle A(\omega)\rangle_\alpha = \langle n(\phi(\omega))\rangle_\alpha D(\omega)$$

Stochastic method: comparison with MEM(1)

Maximum entropy method (MEM)

$$F \equiv \chi^2/2 - \alpha S \qquad S \equiv -\int d\omega\, A(\omega) \ln\left(\frac{A(\omega)}{D(\omega)}\right)$$: entropy

Minimizing F

$$\left.\frac{\delta F}{\delta A}\right|_{A=\bar{A}} = 0 \Rightarrow \bar{A}(\omega)$$: the most likely solution

Stochastic method

K.S.D. Beach, arXiv:cond-mat/0403055

$$H[n] \equiv \chi^2 = \int dx\, \epsilon(x) n(x) + \frac{1}{2}\int dx dy V(x,y) n(x) n(y)$$: Hamiltonian

Mean filed treatment

$$(n(x) - \bar{n}(x))(n(y) - \bar{n}(y)) \approx 0 \Rightarrow \bar{A}(\omega) = \bar{n}(\phi(\omega)) D(\omega)$$

Stochastic method: comparison with MEM(1)

Maximum entropy method (MEM)

$$F \equiv \chi^2/2 - \alpha S \qquad S \equiv -\int d\omega\; A(\omega) \ln\left(\frac{A(\omega)}{D(\omega)}\right)$$: entropy

Minimizing F

$$\left.\frac{\delta F}{\delta A}\right|_{A=\bar{A}} = 0 \quad \Rightarrow \quad \bar{A}(\omega)$$: the most likely solution

Equivalent!

Stochastic method

K.S.D. Beach, arXiv:co[illegible]mat/0403055

$$H[n] \equiv \chi^2 = \int dx\; \epsilon(x) n(x) + \frac{1}{2}\int dxdyV(\cdots y) n(x) n(y)$$: Hamiltonian

Mean filed treatment

$$(n(x) - \bar{n}(x))(n(y) - \bar{n}(y)) \approx 0 \quad \Rightarrow \quad \bar{A}(\omega) = \bar{n}(\phi(\omega)) D(\omega)$$

Stochastic method: comparison with MEM(2)

S. Fuchs *et al.*, PRE81, 056701 (2010)

MEM

$$P[A|\bar{G}] = \frac{P[\bar{G}|A]P[A]}{P[\bar{G}]}$$

- Prior probability

$$P[A] \propto \exp(\alpha S)$$

- Likelihood function

$$P[\bar{G}|A] \propto \exp\left(-\chi^2/2\right)$$

- Posterior probability

$$P[A|\bar{G}] \propto e^{-F}$$

$$\Rightarrow \max P[A|\bar{G}] \leftrightarrow \min F \leftrightarrow \frac{\delta F}{\delta A} = 0$$

Stochastic method

$$P[n|\bar{G}] = \frac{P[\bar{G}|n]P[n]}{P[\bar{G}]}$$

- Prior probability

$$P[n] \propto \Theta[n]\delta\left(\int dx\, n(x) - \bar{G}(\tau_0)\right)$$

- Likelihood function

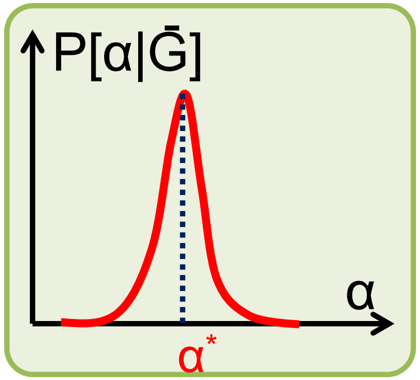
$$P[\bar{G}|n] \propto \exp\left(-\chi^2/2\alpha\right)$$

- Posterior probability

$$P[n|\bar{G}] = \Theta[n]\delta\left(\int dx\, n(x) - \bar{G}(\tau_0)\right) e^{-\chi^2/2\alpha}$$

$$\Rightarrow \langle n(x)\rangle_\alpha = \int \mathcal{D}n\, n(x) P[n|\bar{G}]$$

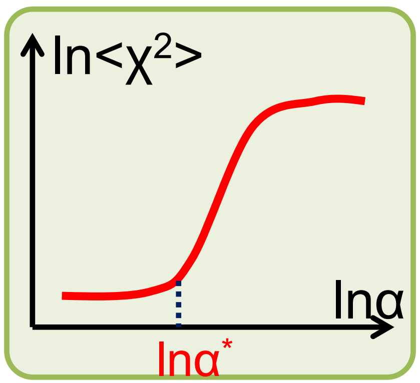
Stochastic method: eliminating α



(a) By using the posterior probability P[α|Ḡ]

$$P[\alpha|\overline{G}] \propto P[\alpha] \int \mathcal{D}A\, e^{-\chi^2/2\alpha} \quad P[\alpha] = 1,\ 1/\alpha$$

Choosing α at the peak location of P[α|Ḡ] or
Taking average $\langle\langle A(\omega)\rangle\rangle \equiv \int d\alpha\ \langle A(\omega)\rangle_\alpha P[\alpha|\bar{G}]$



(b) By using the log-log plot of α vs $<\chi^2>$

Flat region at large α: default model dominant
Crossover region: both χ^2-fitting and the default model are important
Flat region at small α: χ^2-fitting dominant, overfitting

Choosing α at the kink of ln$<\chi^2>$

K.S.D. Beach, arXiv:cond-mat/0403055