The issue of the poles for lattice models

Comparison of CLE and reweighting for QCD

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1. poles and HDQCD
2. poles and full QCD
   [Aarts, Seiler, Sexty, Stamatescu in prep.]
3. Comparison of CLE and reweighting for full QCD
   [Fodor, Katz, Sexty, Torok (2015)]
Proof of convergence for CLE results

If there is fast decay \( P(x, y) \to 0 \) as \( x, y \to \infty \)

and a holomorphic action \( S(x) \)

then CLE converges to the correct result

[Aarts, Seiler, Stamatescu (2009)
Aarts, James, Seiler, Stamatescu (2011)]

Non-holomorphic action for nonzero density

\[
S = S_w[U_{\mu}] + \ln \det M(\mu)
\]

measure has zeros \((\det M = 0)\)
complex logarithm has a branch cut
meromorphic drift

[Mollgaard, Splittorff (2013), Greensite(2014)]

Incorporating poles to proof, investigations of toy models

[See Gert Aarts' talk]
Heavy Quark QCD at nonzero chemical potential (HDQCD)

Hopping parameter expansion of the fermion determinant
Spatial hoppings are dropped

\[ \text{Det } M(\mu) = \prod_x \det(1 + C P_x)^2 \det(1 + C' P_x^{-1})^2 \]

\[ P_x = \prod_\tau U_0(x + \tau a_0) \]

\[ C = [2 \kappa \exp(\mu)]^{N_\tau} \quad C' = [2 \kappa \exp(-\mu)]^{N_\tau} \]

\[ S = S_W[U_\mu] + \ln \text{Det } M(\mu) \]

Studied with reweighting

\[ R = e^{\sum_x C \text{Tr } P_x + C' \text{Tr } P^{-1}} \]

CLE study using gaugecooling

[De Pietri, Feo, Seiler, Stamatescu (2007)]
[Rindlisbacher, de Forcrand (2015)]

[Seiler, Sexty, Stamatescu (2012)]
[Aarts, Attanasio, Jäger, Sexty (2016)]
Critical chemical potential in HDQCD

\[ \langle \exp(2i\varphi) \rangle = \left( \frac{\text{Det } M(\mu)}{\text{Det } M(-\mu)} \right) \]

Phase average

\[ \text{Det } M(\mu) = \prod_x \det \left( 1 + 2\kappa e^{\mu} P_x \right)^2 \det \left( 1 + 2\kappa e^{-\mu} P_x^{-1} \right)^2 \]

\[ \det \left( 1 + CP \right) = 1 + C^3 + C \text{ Tr } P + C^2 \text{ Tr } P^{-1} \]

\[ \mu_c = -\ln \left( 2\kappa \right) \]

Hard sign problem \[ 1 < \mu < 1.8 \]

Except in the middle at half filling

\[ \mu_c = -\ln \left( 2\kappa \right) \]

At \( \mu_c \) only the second factor has an exponentially suppressed sign problem.
Do poles play a role in HDQCD?

Distribution around the zero of the determinant

Only gets close to the pole around $\mu_c$

Where it shows criticality

Otherwise the pole is outside of the distribution

Worst case for poles:
zero temperature lattice
Distribution of the local determinants on the complex plane

\[ \mu = 1.3 \]

Well separated from poles

Exact results

\[ \mu = 1.425 = \mu_c \]

Distribution close to real axis, but “touches” pole

Very faint “whiskers”
Similar to the toy model case
Negligible contribution to averages
Conclusion for HDQCD

Results are unaffected by poles almost everywhere

Near the critical chemical potential
we have indications that results are probably OK
affected by a negligibly small contamination

Phase diagram mapped out with complex Langevin

[Aarts, Attanasio, Jäger, Sexty arxiv:1606.05561]
[See Felipe Attanasio's talk]
Full QCD and the issue of poles

Unimproved staggered and Wilson fermions with CLE

[Sexty (2014), Aarts, Seiler, Sexty, Stamatescu (2015)]

\[ S_{\text{eff}} = S_g(U) - N_f \ln \det M(U) \]
\[ = S_g(U) - N_f \sum_i \ln \lambda_i(U) \]

Drift term of fermions

\[ K_f = N_f \sum_i \frac{D \lambda_i(U)}{\lambda_i(U)} \]

Poles can be an issue if eigenvalue density around zero is not vanishing

Total phase of the determinant is sum of all the phases

→ Sign problem can still be hard
Spectrum of the Dirac operator above the deconfinement transition

- 12^3\times 4 lattice, \(\beta=5.3\), \(m=0.05\), \(N_F=4\), \(\mu/T=0.4\), free fermions
- 8^3\times 4 lattice, \(\beta=5.3\), \(m=0.05\), \(N_F=4\), \(\mu/T=1.2\), free fermions
- 12^3\times 4 lattice, \(\beta=5.3\), \(m=0.05\), \(N_F=4\), \(\mu/T=2.0\), free fermions
- 12^3\times 4 lattice, \(\beta=5.3\), \(m=0.05\), \(N_F=4\), \(\mu/T=3.2\), free fermions
The phase of the determinant

Langevin time evolution

At high temperatures, eigenvalue density is zero at the origin. Even tough the sign problem can be hard.

At low temperatures, non-zero eigenvalue density is expected (Banks-Casher relation). Can we deal with it?

Histogram

Conclusions for full QCD

At high temperatures, eigenvalue density is zero at the origin. Even tough the sign problem can be hard.

At low temperatures, non-zero eigenvalue density is expected (Banks-Casher relation). Can we deal with it?
Reweighting

\[
\langle F \rangle_\mu = \frac{\int DU \, e^{-S_E} \det M (\mu) F}{\int DU \, e^{-S_E} \det M (\mu)} = \frac{\int DU \, e^{-S_E} R \frac{\det M (\mu)}{R} F}{\int DU \, e^{-S_E} R \frac{\det M (\mu)}{R}}
\]

\[
= \frac{\langle F \, \det M (\mu) / R \rangle_R}{\langle \det M (\mu) / R \rangle_R}
\]

\[
R = \det M (\mu = 0), |\det M (\mu)|, \text{etc.}
\]

\[
\left\langle \frac{\det M (\mu)}{R} \right\rangle_R = \frac{Z (\mu)}{Z_R} = \exp \left( -\frac{V}{T} \Delta f (\mu, T) \right)
\]

\[
\Delta f (\mu, T) = \text{free energy difference}
\]

Exponentially small as the volume increases \( \langle F \rangle_\mu \to 0/0 \)

Reweighting works for large temperatures and small volumes

Sign problem gets hard at \( \mu / T \approx 1 \)
Comparison with reweighting for full QCD

[Fodor, Katz, Sexty, Török 2015]

Reweighting from ensemble at

\[ R = \text{Det} M (\mu = 0) \]
Overlap problem

Histogram of weights
Relative to the largest weight in ensemble

Average becomes dominated by very few configurations
Sign problem

Sign problem gets hard around $\mu/T \approx 1 - 1.5$

\[
\langle \exp(2i\varphi) \rangle = \frac{\det M(\mu)}{\det M(-\mu)}
\]
Comparisons as a function of beta

Similarly to HDQCD
Cooling breaks down at small beta

at $N_T=4$ breakdown at $\beta=5.1 - 5.2$

At larger $N_T$?
Comparisons as a function of beta

\[ N_T = 6 \]

\[ N_T = 8 \]

Breakdown prevents simulations in the confined phase for staggered fermions with \( N_T = 4, 6, 8 \)

Two ensembles:

\[ m_\pi \approx 4.8 T_c \]
\[ m_\pi \approx 2.3 T_c \]
Conclusions

Zeroes of the measure can affect validity of CLE if prob. density around them is non-vanising.

In HDQCD poles only have a negligible effect around critical chemical potential, otherwise exact.
In full QCD high temperature simulations are OK.
Low temperatures?

Comparison of reweighting with CLE they agree where both works.
Reliability can be judged independent of the other method.
Low temperature phase not yet reached with $N_T = 8$. 