CHIRAL PERTURBATION AT FINITE VOLUME AND/OR WITH TWISTED BOUNDARY CONDITIONS

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Overview

Chiral Perturbation Theory

- ChPT = Effective field theory describing the lowest order pseudo-scalar representation
- or the (pseudo) Goldstone bosons from spontaneous breaking of chiral symmetry.
- The number of degrees of freedom depend on the case we look at
- Recent review of LECs:

  JB, Ecker,Ann.Rev.Nucl.Part.Sci. 64 (2014) 149 [arXiv:1405.6488]

Finite volume

- Lattice QCD calculates at different quark masses, volumes boundary conditions,. . .
- A general result by Lüscher: relate finite volume effects to scattering (1986)
- Chiral Perturbation Theory is also useful for this
- Start: Gasser and Leutwyler, Phys. Lett. B184 (1987) 83, Nucl. Phys. B 307 (1988) 763
  M_π, F_π, $\langle \bar{q}q \rangle$ one-loop equal mass case
- I will stay with ChPT and the p regime ($M_\pi L >> 1$)
- $1/m_\pi = 1.4$ fm
  may need to (and I will) go beyond leading $e^{-m_\pi L}$ terms
  "around the world as often as you like"
- Convergence of ChPT is given by $1/m_\rho \approx 0.25$ fm

Finite volume: selection of earlier ChPT results

- masses and decay constants for π, K, η one-loop
  Becirevic, Villadoro, Phys. Rev. D 69 (2004) 054010
- M_π at 2-loops (2-flavour)
  Colangelo, Haefeli, Nucl.Phys. B744 (2006) 14 [hep-lat/0602017]
- $\langle \bar{q}q \rangle$ at 2 loops (3-flavour)
  JB, Ghorbani, Phys. Lett. B636 (2006) 51 [hep-lat/0602019]
- Twisted mass at one-loop
  Colangelo, Wenger, Wu, Phys.Rev. D82 (2010) 034502 [arXiv:1003.0847]
- Twisted boundary conditions
  Sachrajda, Villadoro, Phys. Lett. B 609 (2005) 73 [hep-lat/0411033]

Papers

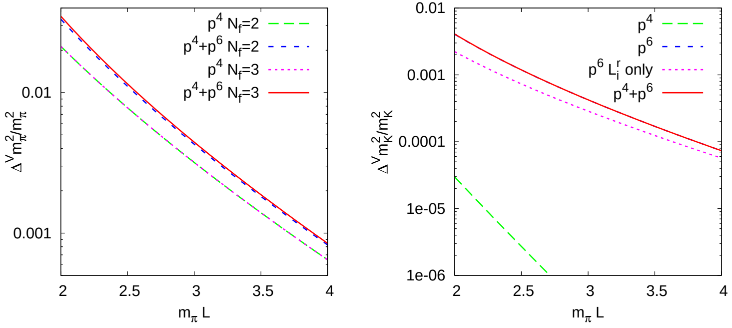
- Finite volume at two-loops (periodic)
  - Two-loop sunset integrals at finite volume, JB, Boström, Lähde, JHEP 1401(2014)019 [arXiv:1311.3531]
  - Finite Volume at two-loops in Chiral Perturbation Theory, JB, Rössler, JHEP 1501 (2015) 034 [arXiv:1411.6384]
  - Finite Volume for Three-Flavour Partially Quenched Chiral Perturbation Theory through NNLO in the Meson Sector, JB, Rössler, JHEP 1511 (2015) 097 [arXiv:1508.07238]
  - Finite Volume and Partially Quenched QCD-like Effective Field Theories, JB, Rössler, JHEP 1511 (2015) 017 [arXiv:1509.04082]
- Twisted boundary conditions
  - Masses, Decay Constants and Electromagnetic Form-factors with Twisted Boundary Conditions, JB, Relefors, JHEP 1405 (2014) 015 [arXiv:1402.1385]
  - The vector two-point function with twisted boundary conditions, JB, Relefors, to be published
  - $K_{\ell 3}$ wth staggered, finite volume and twisting, Bernard, JB, Gamiz, Relefors, to be published

Masses at two-loop order

- Sunset integrals at finite volume done

  JB, Boström and Lähde, JHEP 01 (2014) 019 [arXiv:1311.3531]
- Loop calculations:

  JB, Rössler, JHEP **1501** (2015) 034 [arXiv:1411.6384]



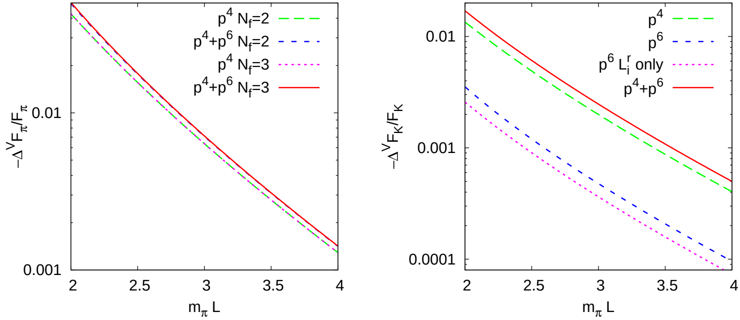
- Agreement for $N_f = 2, 3$ for pion
- K has no pion loop at LO

Decay constants at two-loop order

- Sunset integrals at finite volume done

  JB, Boström and Lähde, JHEP 01 (2014) 019 [arXiv:1311.3531]
- Loop calculations:

  JB, Rössler, JHEP **1501** (2015) 034 [arXiv:1411.6384]



- Agreement for $N_f = 2, 3$ for pion
- K now has a pion loop at LO

Other p^6

Masses and decay constants at finite volume:

- Finite volume for PQ three flavour (all cases) JB, Rössler, JHEP **1511** (2015) 097, [arXiv:1508.07238]
- QCD-like theories, normal and PQ (one valence mass, one sea mass) JB, Rössler, JHEP **1511** (2015) 017, [arXiv:1509.04082]
  - $SU(N) \times SU(N)/SU(N)$
  - $SU(N)/SO(N)$ (including Majorana case)
  - $SU(2N)/Sp(2N)$
- If you want more graphs: look at the papers or play with the programs in CHIRON

Program availability

Making the programs more accessible for others to use:

- Two-loop results have very long expressions
- Many not published but available from http://www.thep.lu.se/∼bijnens/chpt/
- Many programs available on request from the authors
- Idea: make a more general framework
- CHIRON:

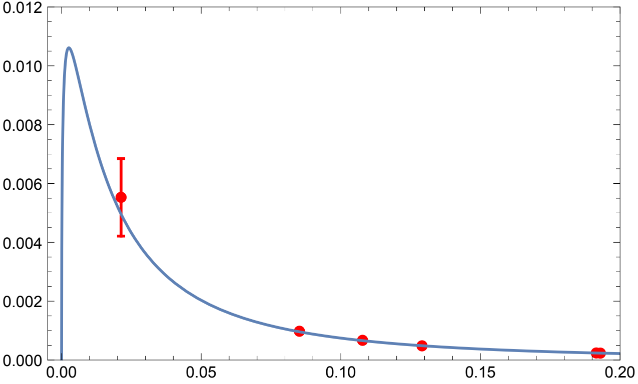
JB,
"CHIRON: a package for ChPT numerical results at two loops,"
Eur. Phys. J. C **75** (2015) 27 [arXiv:1412.0887]
http://www.thep.lu.se/∼bijnens/chiron/

Program availability: CHIRON

- Present version: 0.54
- Classes to deal with $L_i, C_i, L_i^{(n)}, K_i$, standardized in/output, changing the scale,...
- Loop integrals: one-loop and sunsetintegrals
- Included so far (at two-loop order):
  - Masses, decay constants and $\langle \bar{q}q \rangle$ for the three flavour case
  - Masses and decay constants at finite volume in the three flavour case
  - Masses and decay constants in the partially quenched case for three sea quarks
  - Masses and decay constants in the partially quenched case for three sea quarks at finite volume
- A large number of example programs is included
- Manual has already reached 94 pages
- I am continually adding results from my earlier work (remainder of this talk is being worked on)

Two-point: Why

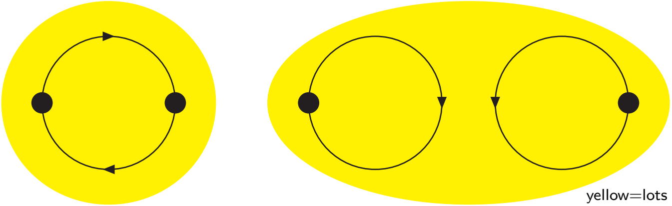
Muon: $a_\mu = (g-2)/2$ and $a_\mu^{\mathrm{LO,HVP}} = \int_0^\infty dQ^2 f\left(Q^2\right)\hat{\Pi}\left(Q^2\right)$



plot: $f\left(Q^2\right)\hat{\Pi}\left(Q^2\right)$ with $Q^2 = -q^2$ in GeV^2

Figure and data: Aubin, Blum, Chau, Golterman, Peris, Tu, Phys. Rev. D93 (2016) 054508 [arXiv:1512.07555]

Two-point: Connected versus disconnected



Connected Disconnected

- $\Pi_{ab}^{\mu\nu}(q) \equiv i \int d^4 x e^{iq\cdot x} \langle T(j_a^\mu(x) j_a^{\nu\dagger}(0))\rangle$
- $j_{\pi^+}^\mu = \bar{d}\gamma^\mu u$
- $j_u^\mu = \bar{u}\gamma^\mu u, \quad j_d^\mu = \bar{d}\gamma^\mu d, \quad j_s^\mu = \bar{s}\gamma^\mu s$
- $j_e^\mu = \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d - \frac{1}{3}\bar{s}\gamma^\mu s$
- Study in ChPT at one-loop:

  Della Morte, Jüttner, JHEP 1011 (2010) 154 [arXiv:1009.3783]

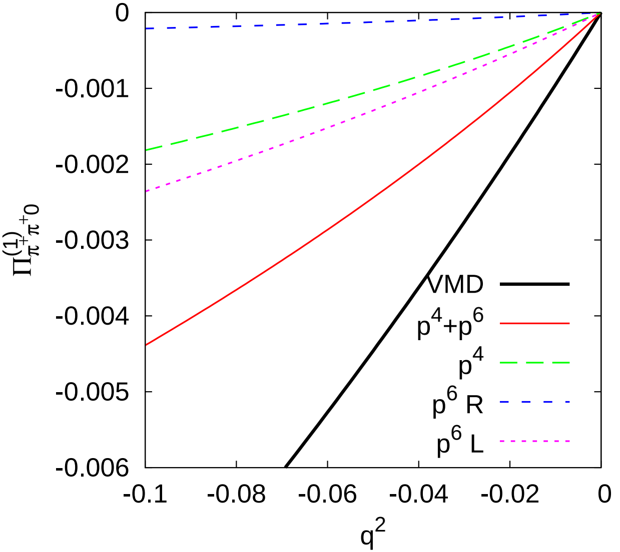
Two-point: Connected versus disconnected

- Include also singlet part of the vector current
- There are new terms in the Lagrangian
- p^4 only one more: $\langle L_{\mu\nu}\rangle\langle L^{\mu\nu}\rangle + \langle R_{\mu\nu}\rangle\langle R^{\mu\nu}\rangle$
- $\Longrightarrow$ The pure singlet vector current does not couple to mesons until p^6
- $\Longrightarrow$ Loop diagrams involving the pure singlet vector current only appear at p^8 (implies relations)
- p^6 (no full classification, just some examples)
  $\langle D_\rho L_{\mu\nu}\rangle\langle D^\rho L^{\mu\nu}\rangle + \langle D_\rho R_{\mu\nu}\rangle\langle D^\rho R^{\mu\nu}\rangle$,
  $\langle L_{\mu\nu}\rangle\left\langle L^{\mu\nu}\chi^\dagger U\right\rangle + \langle R_{\mu\nu}\rangle\left\langle R^{\mu\nu}\chi U^\dagger\right\rangle$,...
- Results at two-loop order, unquenched isospin limit

Two-point: Connected versus disconnected

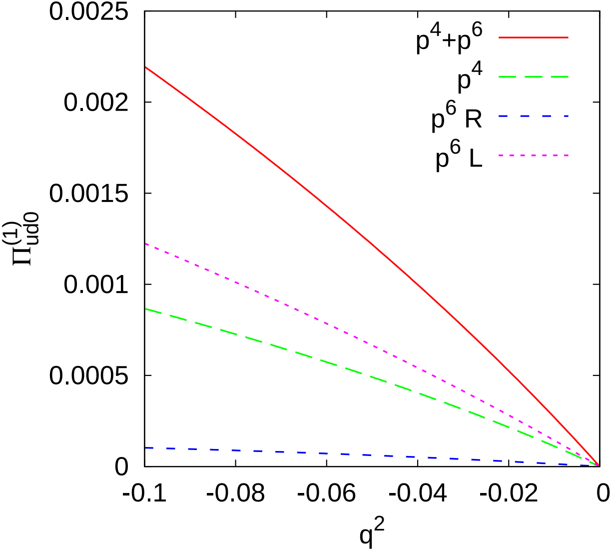
- $\Pi^{\mu\nu}_{\pi^+\pi^+}$: only connected
- $\Pi^{\mu\nu}_{ud}$: only disconnected
- $\Pi^{\mu\nu}_{uu} = \Pi^{\mu\nu}_{\pi^+\pi^+} + \Pi^{\mu\nu}_{ud}$
- $\Pi^{\mu\nu}_{ee} = \frac{5}{9}\Pi^{\mu\nu}_{\pi^+\pi^+} + \frac{1}{9}\Pi^{\mu\nu}_{ud}$
- Infinite volume (and the *ab* considered here):
  $\Pi^{\mu\nu}_{ab} = \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right)\Pi^{(1)}_{ab}$
- Large N_c + VMD estimate: $\Pi^{(1)}_{\pi^+\pi^+} = \dfrac{4F_\pi^2}{M_V^2 - q^2}$
- Plots on next pages are for $\Pi^{(1)}_{ab0}(q^2) = \Pi^{(1)}_{ab}(q^2) - \Pi^{(1)}_{ab}(0)$
- At p^4 the extra LEC cancels, at p^6 there are new LEC contributions, but no new ones in the loop parts

Two-point: Connected versus disconnected



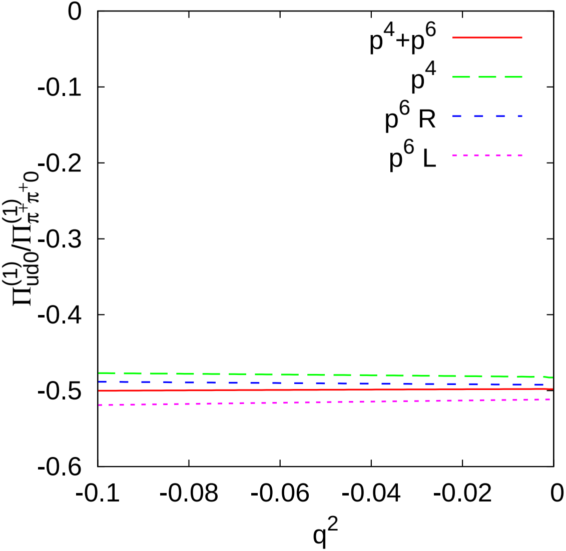
- Connected
- p^6 is large
- Due to the L_i^r loops

Two-point: Connected versus disconnected



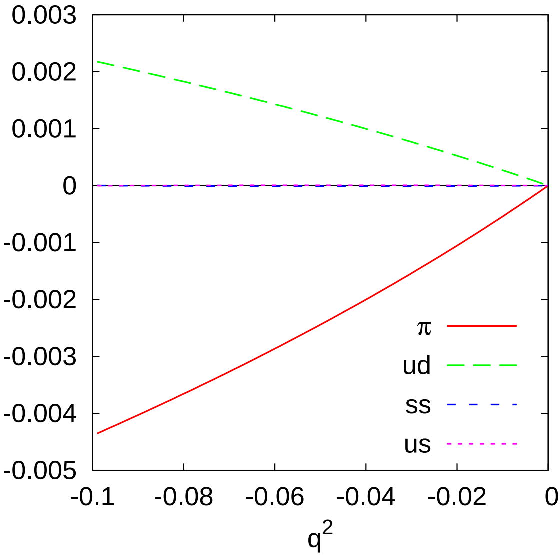
- Disconnected
- p^6 is large
- Due to the L_i^r loops
- about $-\frac{1}{2}$ connected
- $-\frac{1}{10}$ is from $\Pi^{(1)}_{ee} = \frac{5}{9}\Pi^{(1)}_{\pi^+\pi^+} + \frac{1}{9}\Pi^{(1)}_{ud}$

Two-point: Connected versus disconnected



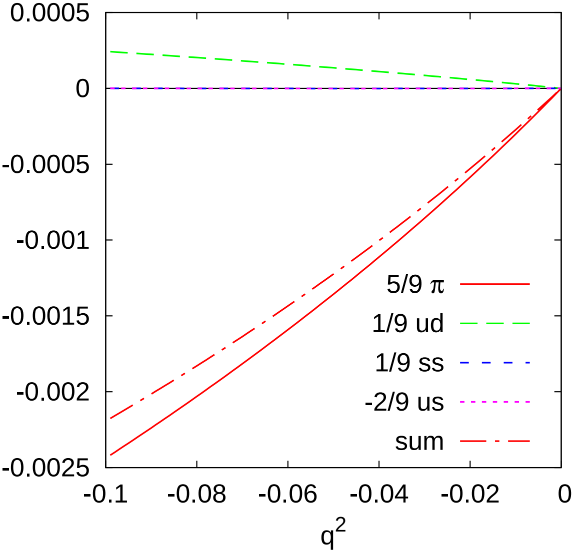
- p^4 and p^6 pion part exactly $-\frac{1}{2}$
- not true for unsubtracted at p^4(LEC)
- not true for pure LEC at p^6

Two-point: Including strange



- π connected u,d
- ud disconnected u,d
- ss strange current
- us mixed strange-u,d
- strange part is very small: $q^2 = 0$ subtraction (only kaon loops) p^4 and p^6 cancel largely

Two-point: with strange, electromagnetic current



- π connected u,d
- *ud* disconnected u,d
- *ss* strange current
- *us* mixed s–u,d
- new p^6 LEC cancels

Twisted boundary conditions

- On a lattice at finite volume $p^i = 2\pi n^i/L$: very few momenta directly accessible
- Put a constraint on certain quark fields in some directions: $q(x^i + L) = e^{i\theta_q^i} q(x^i)$
- Then momenta are $p^i = \theta^i/L + 2\pi n^i/L$. Allows to map out momentum space on the lattice much better

  Bedaque,. . .
- Small note:
  - Beware what people call momentum:
    is θ^i/L included or not?
  - Reason: a colour singlet gauge transformation
    $G_\mu^S \to G_\mu^S - \partial_\mu \epsilon(x)$, $q(x) \to e^{i\epsilon(x)} q(x)$, $\epsilon(x) = -\theta_q^i x^i/L$
  - Boundary condition
    Twisted $\Leftrightarrow$ constant background field+periodic

Twisted boundary conditions: Drawbacks

Drawbacks:

- Box: Rotation invariance $\rightarrow$ cubic invariance
- Twisting: reduces symmetry further

Consequences:

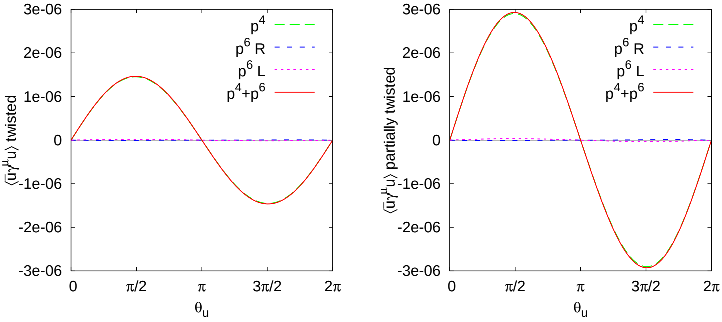
- $m^2(\vec{p}^2) = E^2 - \vec{p}^2$ is not constant
- There are typically more form-factors
- In general: quantities depend on more (all) components of the momenta
- Charge conjugation involves a change in momentum

Two-point function: twisted boundary conditions

JB, Relefors, JHEP 05 (201)4 015 [arXiv:1402.1385]

- $\int_V \frac{d^d k}{(2\pi)^d} \frac{k_\mu}{k^2 - m^2} \neq 0$
- $\langle \bar{u}\gamma^\mu u \rangle \neq 0$
- $j^\mu_{\pi^+} = \bar{d}\gamma^\mu u$
  satisfies $\partial_\mu \left\langle T(j^\mu_{\pi^+}(x) j^{\nu\dagger}_{\pi^+}(0)) \right\rangle = \delta^{(4)}(x) \left\langle \bar{d}\gamma^\nu d - \bar{u}\gamma^\nu u \right\rangle$
- $\Pi^{\mu\nu}_a(q) \equiv i \int d^4 x e^{iq\cdot x} \left\langle T(j^\mu_a(x) j^{\nu\dagger}_a(0)) \right\rangle$
  Satisfies WT identity. $q_\mu \Pi^{\mu\nu}_{\pi^+} = \left\langle \bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d \right\rangle$
- ChPT at one-loop satisfies this
  see also Aubin et al, Phys.Rev. D88 (2013) 7, 074505 [arXiv:1307.4701]
- two-loop in partially quenched: JB, Relefors, in preparation
  satisfies the WT identity (as it should)

$\langle \bar{u}\gamma^{\mu} u \rangle$



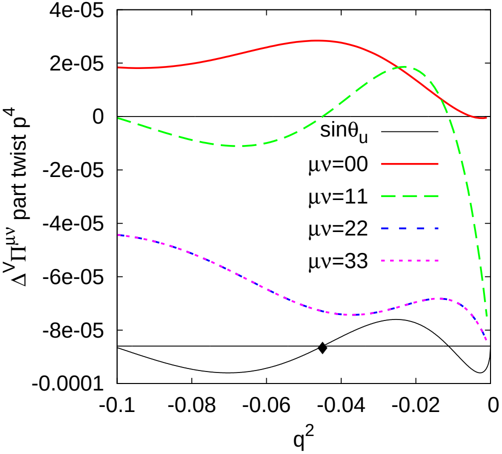
Fully twisted　　　　Partially twisted

$\theta_u = (0, \theta_u, 0, 0)$, all others untwisted

$m_\pi L = 4$　　　　(ratio at p^4=2 up to kaon loops)

For comparison: $\langle \bar{u}u \rangle^V \approx -2.4\ 10^{-5}\ \text{GeV}^3$
$\langle \bar{u}u \rangle \approx -1.2\ 10^{-2}\ \text{GeV}^3$

Two-point: partially twisted, one-loop



$q = \left(0, \sqrt{-q^2}, 0, 0\right)$

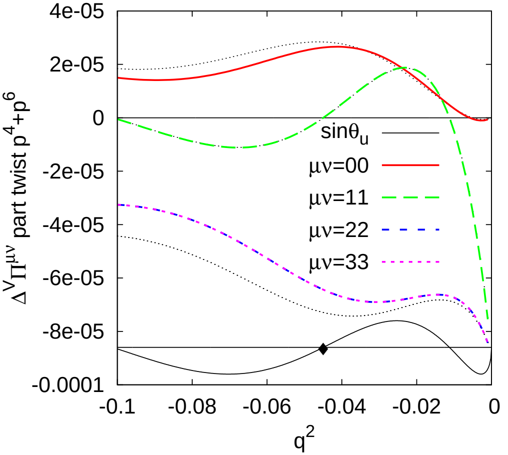
$\Pi^{22} = \Pi^{33}$
$\vec{\theta}_u = L\, q$
$m_{\pi 0} L = 4$
$m_{\pi 0} = 0.135$ GeV

$-q^2 \Pi^{(1)}_{\mathrm{VMD}} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2}$
$\approx 5\text{e-}3 \cdot \frac{q^2}{0.1}$

diamond: periodic
Note: $\Pi^{\mu\nu}(0) \neq 0$

Correction is at the % level

Two-point: partially twisted, with two-loop



$q = \left(0, \sqrt{-q^2}, 0, 0\right)$

$\Pi^{22} = \Pi^{33}$

$\vec{\theta}_u = L\,q$

$m_{\pi 0} L = 4$

$m_{\pi 0} = 0.135$ GeV

$$-q^2 \Pi^{(1)}_{\mathrm{VMD}} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2}$$

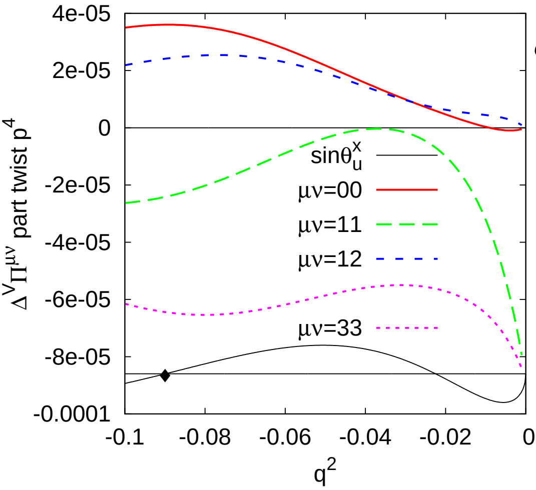
$$\approx 5\text{e-}3 \cdot \frac{q^2}{0.1}$$

diamond: periodic

Note: $\Pi^{\mu\nu}(0) \neq 0$

Correction from two loop is reasonable (thin lines are p^4)

Two-point: partially twisted, one-loop



$$q = \left(0, \frac{\sqrt{-q^2}}{\sqrt{2}}, \frac{\sqrt{-q^2}}{\sqrt{2}}, 0\right)$$

$\Pi^{11} = \Pi^{22}$

$\vec{\theta}_u = L\, q$

$m_{\pi 0} L = 4$

$m_{\pi 0} = 0.135$ GeV

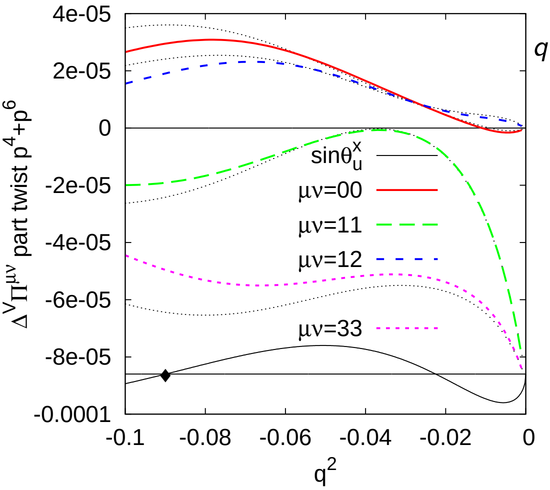
$$-q^2 \Pi^{(1)}_{\mathrm{VMD}} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2} \approx 5\text{e-}3 \cdot \frac{q^2}{0.1}$$

diamond: periodic

Note: $\Pi^{\mu\nu}(0) \neq 0$

Correction is at the % level

Two-point: partially twisted, one-loop



$$q = \left(0, \frac{\sqrt{-q^2}}{\sqrt{2}}, \frac{\sqrt{-q^2}}{\sqrt{2}}, 0\right)$$

$\Pi^{11} = \Pi^{22}$

$\vec{\theta}_u = L\,q$

$m_{\pi 0} L = 4$

$m_{\pi 0} = 0.135$ GeV

$$-q^2 \Pi^{(1)}_{\mathrm{VMD}} = \frac{-4q^2 F_\pi^2}{M_V^2 - q^2} \approx 5\text{e-}3 \cdot \frac{q^2}{0.1}$$

diamond: periodic

Note: $\Pi^{\mu\nu}(0) \neq 0$

Two loop correction again reasonable (thin lines are p^4)

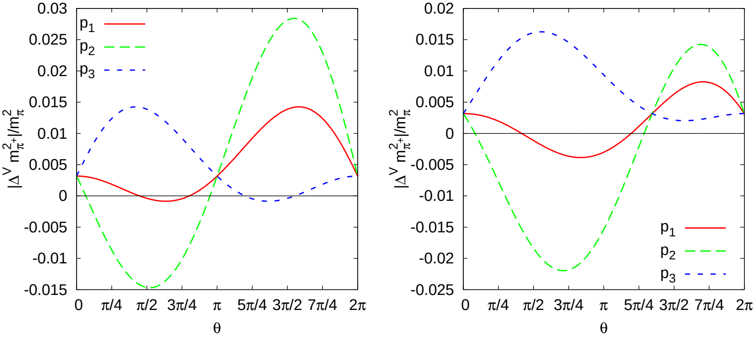
$K_{\ell 3}$: Twisting and finite volume

- There are more form-factors since Lorentz-invariance and even cubic symmetry is broken
- Masses become twist and volume dependent
- All these need to be remembered in the Ward identities
- Masses needed when checking Ward identities
- For unquenched twisted masses, decay constants and electromagnetic form-factor (see there for earlier work):
  JB, Relefors, JHEP 05 (2014) 015 [arXiv:1402.1385]
- Partial twisting and quenching, staggered: masses and $K_{\ell 3}$
  Bernard, JB, Gamiz, Relefors, in preparation

Partial twisting: masses

Bernard, JB, Gamiz, Relefors, in preparation

$m_\pi L = 3$, $\vec{\theta}_u = (\theta, 0, 0)$, $\vec{\theta}_d = \vec{\theta}_s = \vec{\theta}_{d\mathrm{sea}} = \vec{\theta}_{s\mathrm{sea}} = 0$



$\vec{\theta}_{u\mathrm{sea}} = 0$ $\vec{\theta}_{u\mathrm{sea}} = (\pi/3, 0, 0)$

$\vec{p}_1 = (\theta, 0, 0)/L$, $\vec{p}_2 = (\theta + 2\pi, 0, 0)/L$, $\vec{p}_3 = (\theta - 2\pi, 0, 0)/L$,

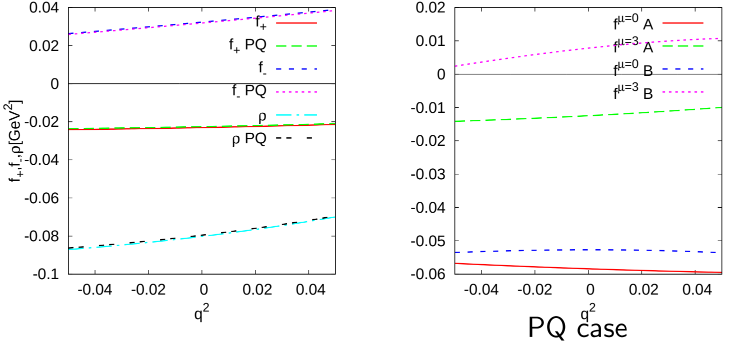
$K_{\ell 3}$

- $q = p - p'$
  $\langle \pi^-(p') | \bar{s}\gamma_\mu u(0) | K^0(p) \rangle = f_+(p_\mu + p'_\mu) + f_- q_\mu + h_\mu \,.$
- $\langle \pi^-(p') | (m_s - m_u)\bar{s} u(0) | K^0(p) \rangle = \rho \,.$
- Ward identity: $(p^2 - p'^2) f_+ + q^2 f_- + q^\mu h_\mu = \rho$
- ChPT:
  - p^4 Isopin conserving and breaking Gasser, Leutwyler, 1985
  - p^6 Isospin conserving JB, Talavera, 2003
  - p^6 Isospin breaking JB, Ghorbani, 2007
  - p^4 partially quenched, staggered Bernard, JB, Gamiz, 2013
  - p^4 Finite volume Ghorbani, Ghorbani, 2013 ($q^2 = 0$)
  - p^4 Finite volume, twisted, partially quenched, staggered Bernard, JB, Gamiz, Relefors, in preparation
  - Rare decays: p^4 Mescia, Smith 2007, p^6 JB, Ghorbani, 2007
- Split in f_+, f_- and h_μ not unique

$K_{\ell 3}$

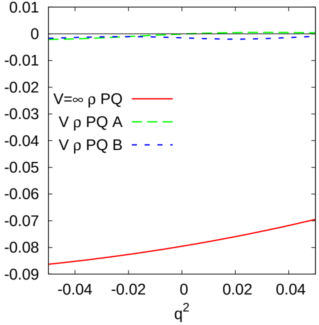
- Masses: finite volume masses with twist effect included.
- $p = \left(\sqrt{m_K^2(\vec{p}) + \vec{p}^2}, \vec{p}\right)$
- $p' = \left(\sqrt{m_\pi^2(\vec{p}') + \vec{p}'^2}, \vec{p}'\right)$
- q^2 calculated with m_K^2 and m_π^2 at $V = \infty$ will also have volume corrections (small effect)
- First: Twisting and partially quenched
- Second: Staggered as well

$K_{\ell 3}$: infinite volume

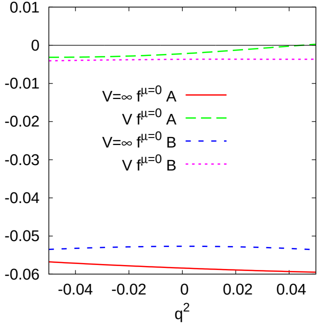


- The components are the well defined ones at finite volume
- plots: p^4 (neglecting the $L_9^r q^2$ term)
- Valence masses with $m_\pi = 135$ GeV and $m_K = 0.495$ GeV
- PQ case with $\hat{m}_{\text{sea}} = 1.1\hat{m}$, $m_{s\text{sea}} = 1.1 m_s$.
- case A: $\vec{p} = 0$,   case B: $\vec{p}' = 0$

$K_{\ell 3}$



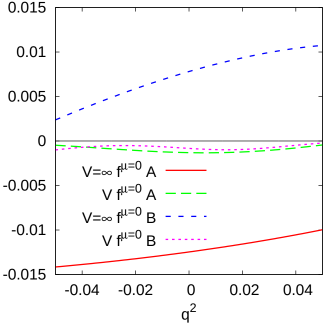
ρ



$\mu = 0$

$\rho_\infty \approx 0.23\ \text{GeV}^2$
$m_\pi L = 3$

$K_{\ell 3}$



$\mu = 3$

Calculate the volume corrections for exactly what you did

What do you calculate on the lattice?

- Want $f_+(0)$ at infinite volume and physical masses
- WT identity: $(p^2 - p'^2) f_+ + q^2 f_- + q_\mu h^\mu = \rho$
- Assume calculation at physical masses
- All parts in the WTI at fixed $\vec{p}, \vec{p}'$ have finite volume corrections: p^2, p'^2, q^2, f_-, $q^\mu h_\mu$ and ρ
- Can use WTI at finite volume and then extrapolate f_+ or extrapolate ρ and then use WTI

MILC lattices and numbers Preliminary

a(fm)	m_l/m_s	L(fm)	m_π(MeV)	m_K(MeV)	$m_\pi L$
0.15	0.035	4.8	134	505	3.25
0.12	0.2	2.9	309	539	4.5
	0.1	2.9	220	516	3.2
	0.1	3.8	220	516	4.3
	0.1	4.8	220	516	5.4
	0.035	5.7	135	504	3.9
0.09	0.2	2.9	312	539	4.5
	0.1	4.2	222	523	4.7
	0.035	5.6	129	495	3.7
0.06	0.2	2.8	319	547	4.5
	0.035	5.5	134	491	3.7

Results: $\vec{\theta}_u = (0,\theta,\theta,\theta)$ (staggered)

Finite volume part of WI divided by $m_K^2 - m_\pi^2$:

$$\frac{\Delta^V m_K^2 - \Delta^V m_\pi^2}{m_K^2 - m_\pi^2} + \Delta^V f_+(0) + \frac{q_\mu h^\mu}{m_K^2 - m_\pi^2} = \frac{\Delta^V \rho}{m_K^2 - m_\pi^2}$$

m_π	$m_\pi L$	"mass"	"f_+"	"h_μ"	"ρ"
134	3.25	0.00000	−0.00042	0.00007	−0.00036
309	4.5	0.00013	−0.00003	−0.00041	−0.00031
220	3.2	0.00054	−0.00048	−0.00084	−0.00077
220	4.3	−0.00007	−0.00009	−0.00005	−0.00021
220	5.4	−0.00005	−0.00003	0.00001	−0.00006
135	3.9	−0.00006	−0.00020	0.00005	−0.00021
312	4.5	0.00047	0.00023	−0.00068	−0.00001
222	4.7	−0.00000	0.00018	−0.00003	0.00014
129	3.7	−0.00013	−0.00004	0.00009	−0.00007
319	4.5	0.00052	0.00037	−0.00081	0.00008
134	3.7	−0.00016	0.00045	0.00013	0.00043

Results: $\vec{\theta}_u = (0, \theta, 0, 0)$ (staggered)

Finite volume part of WI divided by $m_K^2 - m_\pi^2$:

$$\frac{\Delta^V m_K^2 - \Delta^V m_\pi^2}{m_K^2 - m_\pi^2} + \Delta^V f_+(0) + \frac{q_\mu h^\mu}{m_K^2 - m_\pi^2} = \frac{\Delta^V \rho}{m_K^2 - m_\pi^2}$$

m_π	$m_\pi L$	"mass"	"f_+"	"h_μ"	"ρ"
134	3.25	−0.00003	−0.00066	0.00008	−0.00061
309	4.5	−0.00030	−0.00017	−0.00002	−0.00049
220	3.2	−0.00078	−0.00105	0.00036	−0.00148
220	4.3	−0.00033	−0.00034	0.00018	−0.00049
220	5.4	−0.00008	−0.00010	0.00003	−0.00015
135	3.9	−0.00002	−0.00032	0.00001	−0.00033
312	4.5	−0.00019	0.00002	−0.00009	−0.00026
222	4.7	−0.00024	−0.00018	0.00017	−0.00025
129	3.7	−0.00003	−0.00050	−0.00001	−0.00054
319	4.5	−0.00026	0.00013	−0.00012	−0.00025
134	3.7	−0.00005	−0.00058	0.00001	−0.00062

Results: $\vec{\theta}_u = (0, \theta, 0, 0)$ (not staggered)

Finite volume part of WI divided by $m_K^2 - m_\pi^2$:

$$\frac{\Delta^V m_K^2 - \Delta^V m_\pi^2}{m_K^2 - m_\pi^2} + \Delta^V f_+(0) + \frac{q_\mu h^\mu}{m_K^2 - m_\pi^2} = \frac{\Delta^V \rho}{m_K^2 - m_\pi^2}$$

m_π	$m_\pi L$	"mass"	"f_+"	"h_μ"	"ρ"
134	3.25	−0.00049	−0.00124	0.00037	−0.00137
309	4.5	−0.00033	0.00014	−0.00004	0.00022
220	3.2	−0.00113	0.00077	0.00067	0.00031
220	4.3	−0.00062	−0.00011	0.00046	−0.00027
220	5.4	−0.00014	−0.00011	0.00010	−0.00016
135	3.9	0.00004	−0.00045	−0.00008	−0.00049
312	4.5	0.00031	0.00015	−0.00009	−0.00025
222	4.7	−0.00037	−0.00015	0.00027	−0.00025
129	3.7	−0.00000	−0.00066	−0.00005	−0.00071
319	4.5	−0.00031	0.00015	−0.00011	−0.00027
134	3.7	−0.00007	−0.00064	0.00001	−0.00070

Conclusions

- Showed you results for:
  - Masses and decay constants at finite volume at two-loops for many cases (two and three flavour, partially quenched and QCDlike models)
  - Hadronic vacuum polarization: vector two-point function
    - Connected versus disconnected at two-loops
    - Connected: twisting and finite volume at two-loops
  - $K_{\ell 3}$ twisted and staggered at one-loop
    - The WI are satisfied very exactly (note rounding)
    - The corrections are small for present lattices ($< 0.1\%$)
- Be careful: ChPT must exactly correspond to your lattice calculation
- Programs available (for published ones) via CHIRON
  Those for this talk: sometime later this year