θ-dependence of the massive Schwinger model

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Outline

Motivation

- To understand the behaviour of QCD with a θ-term.
- Strong CP problem: why θ is so small in nature?
- Relevant for axion physics: Peccei-Quinn mechanism.
- Possible extensions to other systems with a sign problem, such as finite density, condensed-matter models, etc.

Difficulties

- The main one: the (in)famous sign problem.

$$\mathcal{L}_\theta^{Eucl} = i\theta q(x),$$

  the θ-term makes the Euclidean action complex, and the partition function cannot be interpreted as a probability distribution.

- Standard simulation algorithms fail. We cannot perform a MC at real-valued θ (but we can at pure imaginary θ).

Dealing with θ-terms

- We use two different methods in order to avoid the sign problem.
- Both allow us to compute the expected value of observables like the topological charge $< q >$, as a function of θ.
- They need the **same input**: simulations at pure imaginary values of θ.

New Proposal for Numerical Simulations of θ-Vacuum-like Systems

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We propose a new approach to perform numerical simulations of θ-vacuum-like systems, test it in two analytically solvable models, and apply it to CP^3. The main new ingredient in our approach is the method used to compute the probability distribution function of the topological charge at $\theta = 0$. We do not get unphysical phase transitions (flattening behavior of the free energy density) and reproduce the exact analytical results for the order parameter in the whole θ range within a few percent.

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First method (Azcoiti et al, 2002)

- Reconstruction of the probability distribution of the topological charge at $\theta = 0$.

$$Z_V(\theta) = \sum_n p_V(n) e^{i\theta n} = \sum_x e^{-V f_V(x)} e^{i\theta V x}.$$

- In the infinite volume limit, the saddle point approximation gives:

$$f'(x) = h.$$

$f'(x)$ can be recovered from simulations at imaginary values $\theta = -ih$, that are free from the sign problem.

- When the order parameter is not monotonous, the method fails: flattening.

Second method (Azcoiti et al, 2003)

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θ-vacuum systems via real action simulations

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Abstract

Inspired by the results of the Ising model within an imaginary external magnetic field, we introduce a transformation in quantum systems with a θ-vacuum term that amounts to a rescaling of $z = \cos\frac{\theta}{2}$. Making use of this transformation we are able to determine the order parameter as a function of θ. The approach is successfully tested in models with both broken and unbroken CP symmetry at $\theta = \pi$.

- The second method uses also $x(h)$, but starts by reformulating Z as an even polynomial in $z := \cos(\theta/2)$.

$$Z_V(\theta) = \sum_{x_n} e^{-Vf_V(x_n)} e^{i\theta V x_n} = \sum_{y_n \geq 0} G_V(y_n) \left(\cos^2 \frac{\theta}{2}\right)^{V y_n}.$$

Second method (Azcoiti et al, 2003)

$$Z_V(\theta) = \sum_{x_n} e^{-V f_V(x_n)} e^{i\theta V x_n} = \sum_{y_n \geq 0} G_V(y_n) \left(\cos^2 \frac{\theta}{2}\right)^{V y_n} .$$

- The mean values of x_n and y_n, x and y, are related by

$$y(z) = \frac{x(\theta)}{\tan \frac{\theta}{2}}$$

- Simulations at imaginary $\theta = -ih$ give us access to the region $z = \cosh \frac{h}{2} \geq 1$, whereas the physical region is $0 \leq z \leq 1$.

Second method (Azcoiti et al, 2003)

- Using the transformation

$$y_\lambda(z) = y(e^{\lambda/2} z)$$

the quotient y_λ / y usually have a smooth dependence for small y, allowing to reconstruct $x(\theta)$.

- Defining the exponent

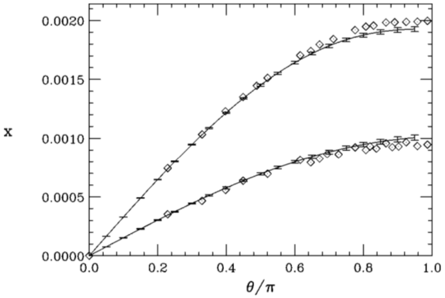
$$\gamma_\lambda := \frac{2}{\lambda} \log \left(\frac{y_\lambda}{y} \right) \ (y \to 0)$$

and computing it from MC simulations at imaginary θ, we obtain the behaviour of $x(\theta)$ as $\theta \to \pi$,

$$x(\theta) \propto (\pi - \theta)^{\gamma - 1} \ (\theta \to \pi)$$

Applying the methods

- Both methods have been applied successfully to a wide variety of models (Ising with an imaginary field, CP^1, CP^3, CP^9)



Massive Schwinger model with a θ-term

Toy model for QCD:

- A model with fermions, confining.
- Has a non-trivial topology, axial anomaly through a non vanishing value of the chiral condensate in the chiral limit (1-flavour).
- Its continuum action is

$$S = \int d^2x \{ \bar{\psi}\gamma_\mu \left(\partial_\mu + iA_\mu\right)\psi + m\bar{\psi}\psi + \frac{1}{4e^2} F_{\mu\nu}^2 + \frac{i\theta}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu} \}.$$

Massive Schwinger model with a θ-term

- At large fermion mass, it tends to pure gauge two-dimensional electrodynamics (exactly solvable) $\rightarrow$ spontaneous symmetry breaking.
- At small fermion mass, no symmetry breaking:

$$\langle q \rangle = m\Sigma \sin\theta + O(m^2)$$

- A critical point separating large and small fermion masses is expected. (Coleman, Ann. of Phys.101, (1976) 239; Hamer et al., Nucl.Phys.B208 (1982) 413; Byrnes et al., Phys.Rev.D 66 (2002) 013002; Shimizu et al., Phys.Rev.D 90 (2014) 014508).

Massive Schwinger model: lattice formulation

Compact formulation for the gauge fields,

$$U_{n\mu} \equiv U_\mu(n) = e^{i\varphi_{n\mu}}; \quad \varphi \in [-\pi, \pi],$$

and the usual Wilson gauge action with staggered fermions,

$$S = \frac{1}{2} \sum_{n,\mu} \eta_\mu(n) \bar{\chi}(n) \{ U_\mu(n) \chi(n+\mu) - U_\mu^\dagger(n-\mu) \chi(n-\mu) \}$$
$$+ m \sum_n \bar{\chi}(n) \chi(n) - \beta \sum_n \mathrm{Re}\,(U_{\Box n}) - i\theta \sum_n q(n),$$

where $U_{\Box n}$ is the plaquette variable $U_1(n)U_2(n+\hat{1})U_1^\dagger(n+\hat{2})U_2^\dagger(n)$, and the local topological charge $q(n)$ is defined as

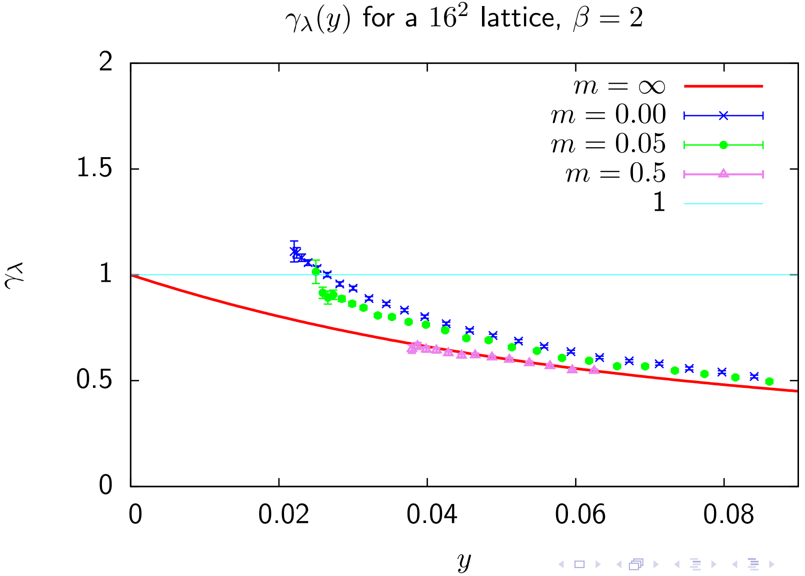
$$q(n) = \frac{1}{2\pi} \{ [\varphi_1(n) + \varphi_2(n+\hat{1}) - \varphi_1(n+\hat{2}) - \varphi_2(n)] \mod 2\pi \}$$

Massive Schwinger model: MC simulation

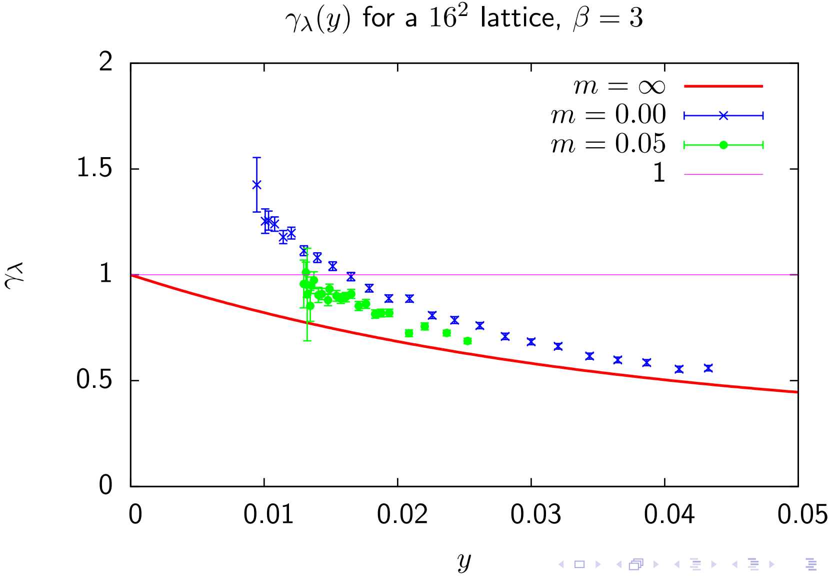
- Standard Metropolis algorithm.
- Sweep: every link in the lattice is updated sequentially.
- The fermionic determinant is computed at each update.

- Simulations at $\beta \in \{2, 3, 4\}$ and $m \in \{0, 0.05, 0.5, \infty\}$.
- $\approx 10^6$ measurements per point.
- No sign of topological freezing.

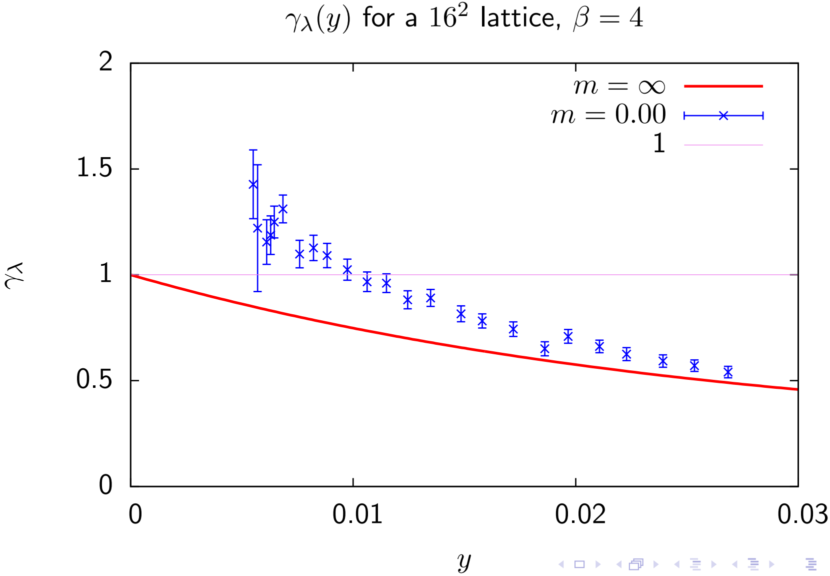
Results: $\beta = 2$



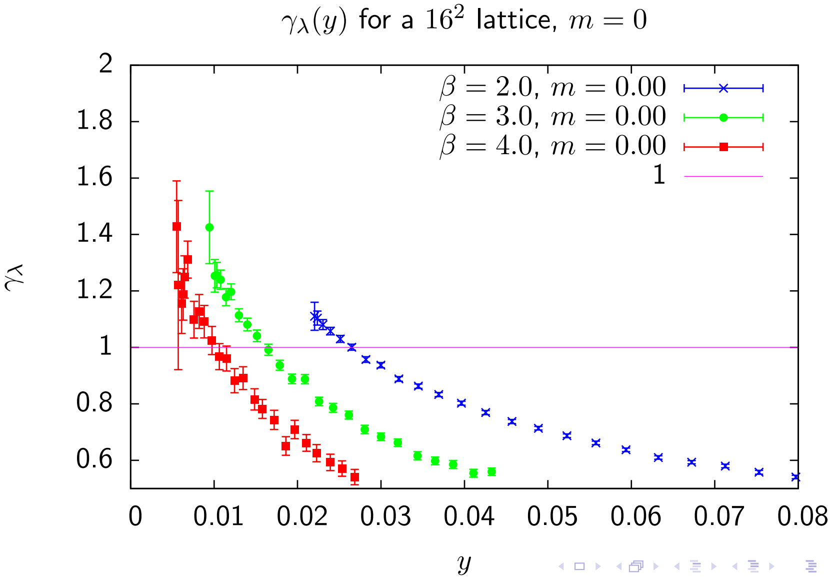
Results: $\beta = 3$



Results: $\beta = 4$

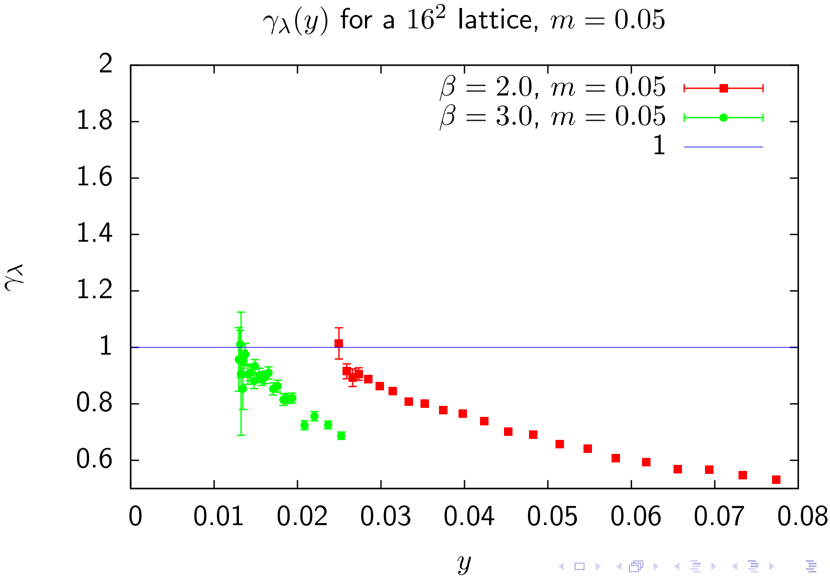


Results: $m = 0$



$\gamma_\lambda(y)$ for a 16^2 lattice, $m = 0$
$\beta = 2.0,\ m = 0.00$
$\beta = 3.0,\ m = 0.00$
$\beta = 4.0,\ m = 0.00$
1
γ_λ
y
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08

Results: $m = 0.05$



Conclusions and outlook

- There are methods that can treat systems with a θ-like term. They have been tested in a wide variety of models (Ising, CP^3, CP^9).
- We have applied them to a toy model of QCD, the Schwinger model with a θ-term, obtaining results compatible with previous work.

- The methods described here should be applicable to QCD with a θ-term.
- We are starting simulations, first in quenched QCD, to test concrete implementations of both the dynamics and the topological charge operator.
- One likely problem is the topological freezing at small a ($< .05$ fm).