θ-dependence of the massive Schwinger model

Eduardo Royo Amondarain

Departamento de Física Teórica
Universidad de Zaragoza

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Work in collaboration with:

V. Azcoiti (Universidad de Zaragoza)
G. Di Carlo (Laboratori Nazionali del Gran Sasso, INFN)
E. Follana (Universidad de Zaragoza)
A. Vaquero (Universita di Milano-Bicocca)
Outline

1. Motivation
2. Difficulties
3. Dealing with $\theta$-terms
   - First method
   - Second method
4. The massive Schwinger model with a $\theta$-term
   - Properties and formulation
   - Results
5. Conclusions and outlook
Motivation

- To understand the behaviour of QCD with a $\theta$-term.
- Strong CP problem: why $\theta$ is so small in nature?
- Relevant for axion physics: Peccei-Quinn mechanism.
- Possible extensions to other systems with a sign problem, such as finite density, condensed-matter models, etc.
Difficulties

- The main one: the (in)famous sign problem.

\[ \mathcal{L}_\theta^{Eucl} = i\theta q(x), \]

the \( \theta \)-term makes the Euclidean action complex, and the partition function cannot be interpreted as a probability distribution.

- Standard simulation algorithms fail. We cannot perform a MC at real-valued \( \theta \) (but we can at pure imaginary \( \theta \)).
Dealing with $\theta$-terms

- We use two different methods in order to avoid the sign problem.
- Both allow us to compute the expected value of observables like the topological charge $\langle q \rangle$, as a function of $\theta$.
- They need the **same input**: simulations at pure imaginary values of $\theta$.

New Proposal for Numerical Simulations of $\theta$-Vacuum-like Systems

V. Azcoiti,\(^1\) G. Di Carlo,\(^2\) A. Galante,\(^{2,3}\) and V. Laliena\(^1\)

\(^1\)Departamento de Física Teórica, Universidad de Zaragoza, Cl. Pedro Cerbuna 12, E-50009 Zaragoza, Spain
\(^2\)INFN, Laboratori Nazionali del Gran Sasso, 67010 Ascoli, L’Aquila, Italy
\(^3\)Dipartimento di Fisica dell’Università di L’Aquila, 67100 L’Aquila, Italy

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We propose a new approach to perform numerical simulations of $\theta$-vacuum-like systems, test it in two analytically solvable models, and apply it to $CP^3$. The main new ingredient in our approach is the method used to compute the probability distribution function of the topological charge at $\theta = 0$. We do not get unphysical phase transitions (flattening behavior of the free energy density) and reproduce the exact analytical results for the order parameter in the whole $\theta$ range within a few percent.

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First method (Azcoiti et al, 2002)

- Reconstruction of the probability distribution of the topological charge at $\theta = 0$.

\[ Z_V(\theta) = \sum_n p_V(n)e^{i\theta n} = \sum_x e^{-V f_V(x)}e^{i\theta V x}. \]

- In the infinite volume limit, the saddle point approximation gives:

\[ f'(x) = h. \]

$f'(x)$ can be recovered from simulations at imaginary values $\theta = -ih$, that are free from the sign problem.

- When the order parameter is not monotonous, the method fails: flattening.
The second method uses also $x(h)$, but starts by reformulating $Z$ as an even polynomial in $z := \cos(\theta/2)$.

$$Z_V(\theta) = \sum_{x_n} e^{-V f_V(x_n)} e^{i\theta V x_n} = \sum_{y_n \geq 0} G_V(y_n) \left( \cos^2 \frac{\theta}{2} \right)^V y_n.$$
Second method (Azcoiti et al, 2003)

\[ Z_V(\theta) = \sum_{x_n} e^{-Vf_V(x_n)} e^{i\theta Vx_n} = \sum_{y_n \geq 0} G_V(y_n) \left( \cos^2 \frac{\theta}{2} \right)^{V y_n} . \]

- The mean values of \( x_n \) and \( y_n \), \( x \) and \( y \), are related by

\[ y(z) = \frac{x(\theta)}{\tan \frac{\theta}{2}} \]

- Simulations at imaginary \( \theta = -ih \) give us access to the region \( z = \cosh \frac{h}{2} \geq 1 \), whereas the physical region is \( 0 \leq z \leq 1 \).
Second method (Azcoiti et al, 2003)

- Using the transformation

\[ y_\lambda(z) = y(e^{\lambda/2} z) \]

the quotient \( y_\lambda/y \) usually have a smooth dependence for small \( y \), allowing to reconstruct \( x(\theta) \).

- Defining the exponent

\[ \gamma_\lambda := \frac{2}{\lambda} \log \left( \frac{y_\lambda}{y} \right) \quad (y \to 0) \]

and computing it from MC simulations at imaginary \( \theta \), we obtain the behaviour of \( x(\theta) \) as \( \theta \to \pi \),

\[ x(\theta) \propto (\pi - \theta)^{-1} \quad (\theta \to \pi) \]
Applying the methods

- Both methods have been applied successfully to a wide variety of models (Ising with an imaginary field, $CP^1, CP^3, CP^9$)
Massive Schwinger model with a $\theta$-term

Toy model for QCD:

- A model with fermions, confining.
- Has a non-trivial topology, axial anomaly through a non vanishing value of the chiral condensate in the chiral limit (1-flavour).
- Its continuum action is

$$S = \int d^2x \left\{ \overline{\psi} \gamma_\mu (\partial_\mu + i A_\mu) \psi + m \overline{\psi} \psi + \frac{1}{4e^2} F_{\mu\nu}^2 + \frac{i \theta}{4\pi} \epsilon_{\mu\nu} F_{\mu\nu} \right\}.$$
Massive Schwinger model with a $\theta$-term

- At large fermion mass, it tends to pure gauge two-dimensional electrodynamics (exactly solvable) → spontaneous symmetry breaking.
- At small fermion mass, no symmetry breaking:

  \[
  \langle q \rangle = m\Sigma \sin \theta + O(m^2)
  \]

Massive Schwinger model: lattice formulation

Compact formulation for the gauge fields,

\[ U_{n\mu} \equiv U_\mu(n) = e^{i\varphi_{n\mu}}; \quad \varphi \in [-\pi, \pi], \]

and the usual Wilson gauge action with staggered fermions,

\[
S = \frac{1}{2} \sum_{n,\mu} \eta_\mu(n) \bar{\chi}(n) \{ U_\mu(n) \chi(n + \mu) - U_\mu^\dagger(n - \mu) \chi(n - \mu) \} \\
+ m \sum_n \bar{\chi}(n) \chi(n) - \beta \sum_n \text{Re}(U_{\square n}) - i\theta \sum_n q(n),
\]

where \( U_{\square n} \) is the plaquette variable \( U_1(n)U_2(n + \hat{1})U_1^\dagger(n + \hat{2})U_2^\dagger(n) \), and the local topological charge \( q(n) \) is defined as

\[
q(n) = \frac{1}{2\pi} \left\{ \left[ \varphi_1(n) + \varphi_2(n + \hat{1}) - \varphi_1(n + \hat{2}) - \varphi_2(n) \right] \mod 2\pi \right\}
\]
Massive Schwinger model: MC simulation

- Standard Metropolis algorithm.
- Sweep: every link in the lattice is updated sequentially.
- The fermionic determinant is computed at each update.

- Simulations at $\beta \in \{2, 3, 4\}$ and $m \in \{0, 0.05, 0.5, \infty\}$.
- $\approx 10^6$ measurements per point.
- No sign of topological freezing.
Results: $\beta = 2$

$\gamma_{\lambda}(y)$ for a $16^2$ lattice, $\beta = 2$
Results: $\beta = 3$

$\gamma(y)$ for a $16^2$ lattice, $\beta = 3$

$m = \infty$

$m = 0.00$

$m = 0.05$

$\theta$ massive Schwinger
Results: $\beta = 4$

$\gamma_\lambda(y)$ for a $16^2$ lattice, $\beta = 4$
Results: $m = 0$

$\gamma(\lambda) \text{ for a } 16^2 \text{ lattice, } m = 0$

$\beta = 2.0, m = 0.00$
$\beta = 3.0, m = 0.00$
$\beta = 4.0, m = 0.00$
Results: $m = 0.05$

$\gamma_\lambda(y)$ for a $16^2$ lattice, $m = 0.05$

$\beta = 2.0$, $m = 0.05$

$\beta = 3.0$, $m = 0.05$
There are methods that can treat systems with a $\theta$-like term. They have been tested in a wide variety of models (Ising, $CP^3$, $CP^9$).

We have applied them to a toy model of QCD, the Schwinger model with a $\theta$-term, obtaining results compatible with previous work.

The methods described here should be applicable to QCD with a $\theta$-term.

We are starting simulations, first in quenched QCD, to test concrete implementations of both the dynamics and the topological charge operator.

One likely problem is the topological freezing at small $a$ ($< .05$ fm).