Position-space approach to hadronic light-by-light scattering in the muon $g-2$ on the lattice

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Hadronic Light-by-Light Contribution

gyromagnetic moment: $\mu = g\frac{e}{2m}\mathbf{S}$

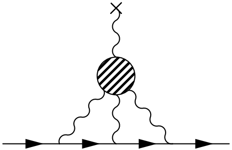
anomalous magnetic moment $a_\mu = \frac{g_\mu - 2}{2}$

contribution	$a_\mu[10^{-10}]$		reference
QED (leptons)	11 658 471.8853	$\pm$ 0.0036	Aoyama *et al* '12
HVP LO	690.75	$\pm$ 4.72	Jegerlehner and Szafron *et al* '11
HVP NLO	-10.03	$\pm$ 0.22	Jegerlehner and Szafron *et al* '11
HVP NNLO	1.24	$\pm$ 0.01	Kurz *et al* '14
HLBL LO	11.6	$\pm$ 4.0	Jegerlehner and Nyffeler '09
HLBL NLO	0.3	$\pm$ 0.2	Colangelo *et al* '14
EW	15.36	$\pm$ 0.10	Gnendiger *et al* '13
total	11 659 181.1	$\pm$ 6.2	
experimental	11 659 208.9	$\pm$ 6.3	Bennett *et al* '06

$\approx$ 3 standard deviations discrepancy $\rightarrow$ new physics?

State of the Art for HLbL

- not fully related to any cross section
- until now only model dependent estimates
- $\rightarrow$ large uncertainties



phenomenology: reduce model uncertainties for dominant contribution (π^0 , η , η' ; $\pi\pi$)

using experimental input $\rightarrow$ dispersion relations
Colangelo *et al* '14 '14 '15;
Pauk and Vanderhaeghen '14

lattice QCD

can provide a model independent first-principle estimate
only publications from one group so far: Blum *et al* '15
(talk by Luchang Jin earlier this session)

Lattice QCD: Current Approach

two independent developments by

- Blum *et al* '15
- our group (NA talk at DPG meeting March '15 and Green *et al* Lattice 2015 [arXiv:1510.08384], Asmussen *et al* in preparation)

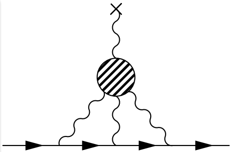
similarities

- get directly $F_2(q^2 = 0)$
- no cancellation of an $\mathcal{O}(\alpha^2)$ term
- position space
- perturbative treatment of the QED part

Our Approach

how

- QCD blob: lattice regularization
- everything else: position space perturbation theory in Euclidean formulation (most natural choice!)



strengths

- QED part computed in infinite volume in continuum
- no power law effects in the volume

challenges

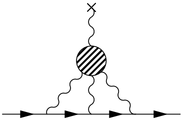
- need to calculate a four-point function

I will focus on the perturbative part.

Details of the calculation

In Euclidean space:

$$\langle \mu^-(p', s')|\, j_\rho(0)\, |\mu^-(p,s)\rangle = -\bar{u}^{s'}(p') \left[\gamma_\rho F_1(k^2) + \frac{\sigma_{\rho\tau} k_\tau}{2m} F_2(k^2)\right] u^s(p)$$



first steps

- start by the Feynman rules for QED
- project out $F_2(0)$ (cf. Kinoshita *et al* '70)
  $\Pi_{\mu\nu\lambda\rho}(q_1, q_2, k-q_1-q_2) = -k_\sigma \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k-q_1-q_2)$
  $F_2(0) = \frac{-i}{48m} \mathrm{Tr}\{[\gamma_\rho, \gamma_\tau](-i\not{p} + m)\Gamma_{\rho\tau}(p,p)(-i\not{p} + m)\}$
- on-shell muon momentum $p = im\hat{\epsilon}$ $(p^2 = -m^2)$
- Fourier transform

Vertex function in momentum space

vertex function

$$\Gamma_{\rho\sigma}(p',p) = -\,e^6 \int \frac{\mathrm{d}^4 q_1}{(2\pi)^4}\frac{\mathrm{d}^4 q_2}{(2\pi)^4}\frac{1}{q_1^2 q_2^2 (q_1+q_2-k)^2}$$

$$\frac{1}{(p'-q_1)^2+m^2}\frac{1}{(p'-q_1-q_2)^2+m^2}$$

$$\gamma_\mu(i\not{p}' - i\not{q}_1 - m)\gamma_\nu(i\not{p} - i\not{q}_1 - i\not{q}_2 - m)\gamma_\lambda$$

$$\frac{\partial}{\partial k_\rho}\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2)$$

$$\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = \int_{x1,x2,x3} e^{-i(q_1x_1+q_2x_2+q_3x_3)}\,\langle j_\mu(x_1)j_\nu(x_2)j_\lambda(x_3)j_\sigma(0)\rangle$$

Vertex function in position space

vertex function

$$\Gamma_{\rho\sigma}(p,p) = -e^6 \int_{x,y} K_{\mu\nu\lambda}(x,y,p)\Pi_{\rho;\mu\nu\lambda\sigma}(x,y)$$

$$K_{\mu\nu\lambda}(x,y,p) = \gamma_\mu(i\not{p} + \not{\partial}^{(x)} - m)\gamma_\nu(i\not{p} + \not{\partial}^{(x)} + \not{\partial}^{(y)} - m)\gamma_\lambda \mathcal{I}(\hat{\epsilon},x,y)$$

$$\mathcal{I}(\hat{\epsilon},x,y) = \int_{q,k} \frac{1}{q^2k^2(q+k)^2}\frac{1}{(p-q)^2+m^2}\frac{1}{(p-q-k)^2+m^2} e^{-i(qx+ky)}$$

$$\Pi_{\rho;\mu\nu\lambda\sigma}(x,y) = \int_z iz_\rho \langle j_\mu(x)j_\nu(y)j_\sigma(z)j_\lambda(0)\rangle$$

- $\mathcal{I}$ is logarithmic infrared divergent for $p^2 = -m^2$ (introduce regulator)
- $K_{\mu\nu\lambda}$ is infrared finite

Details of the calculation

evaluating $\mathcal{I}(\hat{\epsilon}, x, y)$

$$\mathcal{I}(\hat{\epsilon}, x, y) = \int_{u,\text{IR-reg}} G_0(u-y) J(\hat{\epsilon}, u) J(\hat{\epsilon}, x-u)$$

$$J(\hat{\epsilon}, y) = \int_x G_0(x+y) e^{-m\hat{\epsilon}\cdot x} G_m(x)$$

Chebyshev expansion of J: $\quad J(\hat{\epsilon}, y) = \sum_{n\geq 0} z_n(y^2) U_n(\hat{\epsilon}\cdot\hat{y})$

U_n =Chebyshev polynomials of the second kind
(special case of the Gegenbauer polynomials)

$$G_m(x) = \frac{m}{4\pi^2|x|} K_1(m|x|) \quad (K_1 \text{ is a modified Bessel function})$$

z_n = linear combination of products of two modified Bessel functions.

HLbL Master Formula

master formula

$$a_\mu^{\mathrm{Hlbl}} = F_2(0) = \frac{me^6}{3} \int_y \int_x \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) i\Pi_{\rho;\mu\nu\lambda\sigma}(x,y)$$

- after contracting the Lorentz indices the integration reduces to a 3-dimentional integration over $x^2, y^2, x \cdot y$

QCD four-point function

$$i\Pi_{\rho;\mu\nu\lambda\sigma}(x,y) = -\int_z z_\rho \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \rangle$$

QED kernel function

$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$

- weights the position-space vertex
- averaged over the direction of the muon momentum
- we have computed it once and for all

Form Factors...

...or how to handle the Lorentz structure of $\bar{\mathcal{L}}$

tensor decomposition

$$\bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \langle \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(\hat{\epsilon},x,y)\rangle_{\hat{\epsilon}} = \sum_{A=I,II,III} \mathcal{G}^{A}_{\delta\rho\sigma\mu\alpha\nu\beta\lambda} T^{A}_{\alpha\beta\delta}(x,y)$$

$$\langle(\cdots)\rangle_{\hat{\epsilon}} = \frac{1}{2\pi^2}\int \mathrm{d}\Omega_{\epsilon}(\cdots)$$ average over the direction of the muon momentum

$\mathcal{G}^{I,II,III}_{\delta\rho\sigma\mu\alpha\nu\beta\lambda}$ =sums of products of Kronecker deltas

$$T^{I}_{\alpha\beta\delta}(x,y) = \partial^{(x)}_{\alpha}(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta})V_{\delta}(x,y)$$

$$T^{II}_{\alpha\beta\delta}(x,y) = m\partial^{(x)}_{\alpha}(T_{\beta\delta}(x,y) + \frac{1}{4}\delta_{\beta\delta}S(x,y))$$

$$T^{III}_{\alpha\beta\delta}(x,y) = m(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta})(T_{\alpha\delta}(x,y) + \frac{1}{4}\delta_{\alpha\delta}S(x,y))$$

Form Factors

scalar	$S(x,y) = \langle \mathcal{I} \rangle_{\hat{\epsilon}}$ (IR regulated)
vector	$V_\delta(x,y) = \langle \hat{\epsilon}_\delta \mathcal{I} \rangle_{\hat{\epsilon}}$
tensor	$T_{\beta\delta}(x,y) = \langle (\hat{\epsilon}_\delta \hat{\epsilon}_\beta - \frac{1}{4}\delta_{\delta\beta})\mathcal{I} \rangle_{\hat{\epsilon}}$

6 form factors

$$S(x,y) = g^{(0)}$$

$$V_\delta(x,y) = x_\delta g^{(1)} + y_\delta g^{(2)}$$

$$T_{\alpha\beta}(x,y) = (x_\alpha x_\beta - \frac{x^2}{4}\delta_{\alpha\beta}) l^{(1)} + (y_\alpha y_\beta - \frac{y^2}{4}\delta_{\alpha\beta}) l^{(2)} + (x_\alpha y_\beta + y_\alpha x_\beta - \frac{x \cdot y}{2}\delta_{\alpha\beta}) l^{(3)}$$

all form factors depend on $x^2, y^2, x \cdot y$

Example: Form Factor $g^{(2)}$

$$g^{(2)}(x^2, x\cdot y, y^2) = \frac{1}{8\pi y^2 |x| \sin^3\beta}\int_0^\infty du\, u^2 \int_0^\pi d\phi_1$$

$$\left\{2\sin\beta + \left(\frac{y^2+u^2}{2|u||y|} - \cos\beta\cos\phi_1\right)\frac{\log\chi}{\sin\phi_1}\right\}\sum_{n=0}^{\infty}$$

$$\left\{z_n(|u|)z_{n+1}(|x-u|)\left[|x-u|\cos\phi_1 \frac{U_n}{n+1} + (|u|\cos\phi_1 - |x|)\frac{U_{n+1}}{n+2}\right]\right.$$

$$\left. + z_{n+1}(|u|)z_n(|x-u|)\left[(|u|\cos\phi_1 - |x|)\frac{U_n}{n+1} + |x-u|\cos\phi_1\frac{U_{n+1}}{n+2}\right]\right\}$$

where

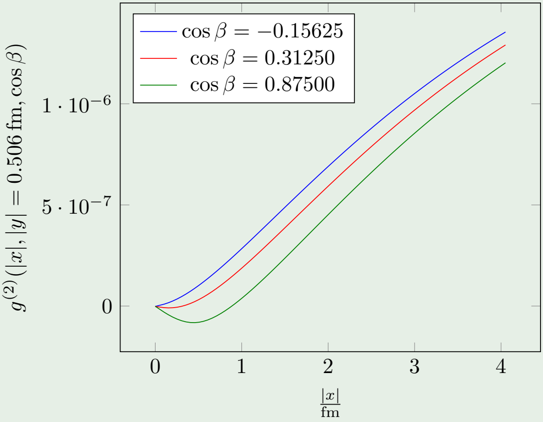
$$x\cdot y = |x||y|\cos\beta\,, \quad |x-u| = \sqrt{|x|^2 + |u|^2 - 2|x||u|\cos\phi_1}$$

$$\chi = \frac{y^2+u^2-2|u||y|\cos(\beta-\phi)}{y^2+u^2-2|u||y|\cos(\beta+\phi)}\,, \quad U_n = U_n\left(\frac{|x|\cos\phi_1 - |u|}{|u-x|}\right)$$

z_n =linear combination of products of two modified Bessel functions.

Form factors

Example $g^{(2)}(|x|, |y|, \cos\beta)$

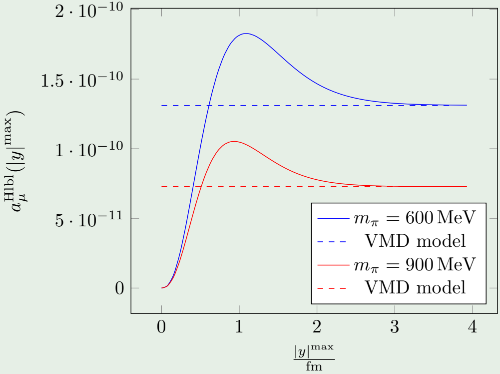


all 6 form factors

- computed to about 5 digits precision
- stored once and for all

Numerical test: pion-pole contribution to a_μ^{Hlbl}

pion-pole contribution



3d integration

- $\int_y \to 2\pi^2 \int_0^\infty \mathrm{d}|y||y|^3$
- $\int_x \to 4\pi \int_0^\infty \mathrm{d}|x|\,|x|^3 \int_0^\pi \mathrm{d}\beta \sin^2\beta$

cutoff for x integration

- $|x|^{\max} = 4.05\,\text{fm}$

result for VMD model from momentum-space representation

Lattice aspects

large volume needed

$|y|^{\text{max}} \gtrsim 2 - 3\,\text{fm}$ needed even for

$m_\pi = 600 - 900\,\text{MeV}$

remember we do not have power law effects(!)

computational cost (fully connected contribution)

- With the help of sequential propagators, the computation is arranged so that the d^4x integral can be evaluated at the sink
- If the 1-dim. integral over $|y|$ is done with N evaluations of the integrand:
  - (1+N) forward propagators
  - 6(1+N) sequential propagators

Conclusions and Outlook

so far:

- explicit formula for a_μ^{Hlbl}
- kernel function multiplying the position-space correlation function
  $\rightarrow$ stored on disc (form factors), ready to be used
- verified the kernel function

what next?

- calculate the four-point correlation function on the lattice
- work out the Hlbl contribution to a_μ^{Hlbl}

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Thank you for your attention!