A Massive Momentum-Subtraction Scheme

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Motivation for a Mass Dependent Scheme

Lattice	$48^3 \times 96$ Physical	$64^3 \times 128$ Physical	$48^3 \times 96$ Fine
$1/a$ (GeV)	1.73	2.36	2.8
M_{D_l} (GeV)	1.42 – 1.68	1.49 – 2.12	1.51– 2.42

PDG $M_{D_l^{\pm}} = 1.86957(16)$ GeV, $M_{D_l^0} = 1.86480(14)$ GeV [1]

- massless quarks: $am \ll a\mu \ll \pi$
- Reduction in lattice artefacts when performing continuum extrapolation in a massive scheme, by potentially removing mass dependent $\mathcal{O}(a^2)$ terms

[1] see talk by **Tobi Tsang**, Friday, July 29, 13:00, arXiv: hep-lat/1602.04118v1 and hep-lat/1511.09328

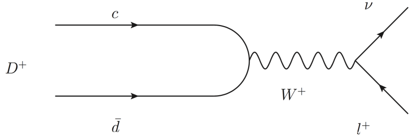
The Charm Project

Determine the decay constants f_D and f_{D_s} using

$$\langle 0 | A^{\mu}_{cq} | D_q(p) \rangle = f_{D_q} p^{\mu}_{D_q}$$

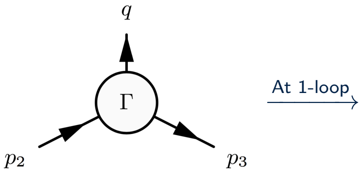
where $q = d, s$ and the axial current $A^{\mu}_{cq} = \bar{c}\gamma_{\mu}\gamma_5 q$.

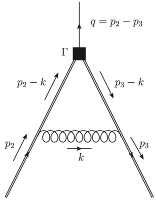
To obtain the decay constant, we need to renormalize the bare axial current.



Kinematics

- Symmetric Minkowski momentum $p_2^2 = p_3^2 = q^2 = -\mu^2$ with $\mu^2 > 0$
- Vertex $G_\Gamma^a(p_3, p_2) = \langle O_\Gamma^a(q)\bar{\psi}(p_3)\psi(p_2)\rangle$, fermion bilinear $O_\Gamma^a = \bar{\psi}\Gamma\tau^a\psi$
- Γ spans all the element of the basis of the Clifford algebra, $\Gamma = \mathrm{S}, \mathrm{P}, \mathrm{V}, \mathrm{A}, \mathrm{T}$
- Propagator $S(p) = \frac{i}{\not{p} - m - \Sigma(p) + i\epsilon}$
- Amputated vertex function $\Lambda_\Gamma^a(p_2, p_3) = S(p_3)^{-1} G_\Gamma^a(p_3, p_2) S(p_2)^{-1}$





Continuum Ward identities

- Consider chiral transformations with a regulator that does not break the symmetry, e.g. dim-reg
- Vector and axial transformations on $\bar{\psi}_i, \psi_j$ in the path integral imply:
    - Vector WI: $q \cdot \Lambda_{\mathrm{V}}^{a} = iS(p_2)^{-1} - iS(p_3)^{-1}$
    - Axial WI: $q \cdot \Lambda_{\mathrm{A}}^{a} = 2mi\Lambda_{\mathrm{P}}^{a} - \gamma_5 iS(p_2)^{-1} - iS(p_3)^{-1}\gamma_5$
- Flavor non-singlet $\tau^a = \sigma^+/2$

Renormalization

$$\psi_R = Z_q^{1/2}\psi\,, \quad S_R(p) = Z_q S(p)\,, \quad m_R = Z_m m$$

$$\left[\bar{\psi}\Gamma\psi\right]_R = Z_\Gamma \bar{\psi}\Gamma\psi\,, \quad A_R^\mu = Z_A A^\mu\,, \quad V_R^\mu = Z_V V^\mu$$

Renormalization of Λ_Γ: $\Lambda_{\Gamma,R} = \frac{Z_\Gamma}{Z_q}\Lambda_\Gamma$

- In general, $Z = Z(g, a\mu, am)$
- Regulator a
- Renormalization scale μ
- Renormalization constants are determined by imposing renormalization conditions. e.g. RI/SMOM [2].

[2] arXiv: hep-ph/0901.2599

RI/SMOM Conditions

$$\lim_{m_R \to 0} \frac{1}{12p^2} \mathrm{Tr}\left[iS_R(p)^{-1} \not{p} \right] \Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12m_R} \left\{ \mathrm{Tr}\left[-iS_R(p)^{-1} \right] \big|_{p^2=-\mu^2} - \frac{1}{2}\, \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{A},R})\, \gamma_5 \right] \big|_{\mathrm{sym}} \right\} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12q^2} \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{V},R}) \not{q} \right] \big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12q^2} \mathrm{Tr}\left[(q \cdot \Lambda_{\mathrm{A},R})\, \gamma_5 \not{q} \right] \big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12i} \mathrm{Tr}\left[\Lambda_{\mathrm{P},R} \gamma_5 \right] \big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to 0} \frac{1}{12} \mathrm{Tr}\left[\Lambda_{\mathrm{S},R} \right] \big|_{\mathrm{sym}} = 1$$

Tree level values

RC are consistent with trivial renormalizations at tree level e.g.

$$\lim_{m_R\to 0} \frac{1}{12p^2}\mathrm{Tr}\left[iS_R(p)^{-1}\ \not{p}\right]\Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R\to 0} \frac{1}{12p^2}Z_q^{-1}\mathrm{Tr}\left[iS(p)^{-1}\ \not{p}\right]\Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R\to 0} \frac{1}{12p^2}Z_q^{-1}\mathrm{Tr}\left[(\not{p}-m)\ \not{p}\right]\Big|_{p^2=-\mu^2} = 1$$

at tree level $Z_q = 1$, same for all the others.

This is a property we wish to preserve in the massive scheme

$Z_V = 1$ in SMOM

Bare Vector WI: $\quad q \cdot \Lambda_{\rm V} = iS(p_2)^{-1} - iS(p_3)^{-1}$

Rewriting in terms of renormalized quantities using,

$$S_R(p) = Z_q S(p) \quad \text{and} \quad \Lambda_{{\rm V},R} = \frac{Z_{\rm V}}{Z_q}\Lambda_{\rm V} \quad \Rightarrow$$

$$\frac{Z_q}{Z_{\rm V}}\ q \cdot \Lambda_{{\rm V},R} = i\ Z_q\ S_R(p_2)^{-1} - i\ Z_q\ S_R(p_3)^{-1}$$

multiplying by $\not{q}$ and taking trace, using

$$\lim_{m_R \to 0} \frac{1}{12p^2} \mathrm{Tr}\left[iS_R(p)^{-1}\ \not{p}\right]\Big|_{p^2=-\mu^2} = 1$$
$$\lim_{m_R \to 0} \frac{1}{12q^2} \mathrm{Tr}\left[(q \cdot \Lambda_{{\rm V},R})\ \not{q}\right]\Big|_{\rm sym} = 1$$

gives $\frac{Z_q}{Z_{\rm V}} = Z_q \quad \Rightarrow Z_{\rm V} = 1$

Heavy-Heavy RI/mSMOM Conditions

$$\lim_{m_R \to \bar{m}} \frac{1}{12p^2} \mathrm{Tr}\left[iS_R(p)^{-1} \not{p} \right]\Big|_{p^2=-\mu^2} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12m_R} \left\{ \mathrm{Tr}\left[-iS_R(p)^{-1} \right]\Big|_{p^2=-\mu^2} - \frac{1}{2}\,\mathrm{Tr}\left[\left(q \cdot \Lambda_{\mathrm{A},R} \right) \gamma_5 \right]\Big|_{\mathrm{sym}} \right\} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12q^2} \mathrm{Tr}\left[\left(q \cdot \Lambda_{\mathrm{V},R} \right) \not{q} \right]\Big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12q^2} \mathrm{Tr}\left[\left(q \cdot \Lambda_{\mathrm{A},R} - 2m_R i \Lambda_{\mathrm{P},R} \right) \gamma_5 \not{q} \right]\Big|_{\mathrm{sym}} = 1$$

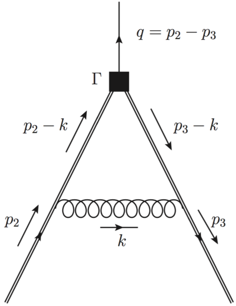
$$\lim_{m_R \to \bar{m}} \frac{1}{12i} \mathrm{Tr}\left[\Lambda_{\mathrm{P},R} \gamma_5 \right]\Big|_{\mathrm{sym}} = 1$$

$$\lim_{m_R \to \bar{m}} \frac{1}{12} \mathrm{Tr}\left[\Lambda_{\mathrm{S},R} \right] - \frac{1}{6q^2} \mathrm{Tr}\left[2im_R \Lambda_{P,R} \gamma_5 \not{q} \right]\Big|_{\mathrm{sym}} = 1$$

Check at 1-loop in perturbation theory using dim reg

Dimensional Regularization, $D = 4 - 2\epsilon$

$$\Lambda_\Gamma^{(1)} = -ig^2 C_2(F) \int_k \frac{\gamma_\mu [\not{p}_2 - \not{k} + m]\Gamma[\not{p}_3 - \not{k} + m]\gamma^\mu}{k^2[(p_2 - k)^2 - m^2][(p_3 - k)^2 - m^2]}$$



Results at 1-loop

$$Z_q = 1 + \frac{\alpha}{4\pi} C_2(F) \left(\frac{1}{\epsilon} - \gamma_E + 1 - \frac{m^2}{\mu^2} - \frac{m^4}{\mu^4} \ln\left(\frac{m^2}{m^2+\mu^2}\right) - \ln\left(\frac{m^2+\mu^2}{\tilde{\mu}^2}\right)\right)$$

$$\Lambda_{\mathrm{V}}^{(1)\sigma}(p_2,p_3) = \frac{\alpha}{4\pi} C_2(F) \Big[A_{\mathrm{V}} \frac{1}{\mu^2} \; (i\epsilon^{\sigma\rho\alpha\beta}\gamma_\rho\gamma^5 p_{3\alpha} p_{2\beta}) + B_{\mathrm{V}}\gamma^\sigma + C_{\mathrm{V}} \frac{1}{\mu^2} (p_2^\sigma \; \not{p}_2 + p_3^\sigma \; \not{p}_3) + D_{\mathrm{V}} \frac{1}{\mu^2} (p_2^\sigma \; \not{p}_3 + p_3^\sigma \; \not{p}_2) + E_{\mathrm{V}} \frac{1}{\mu} (p_2^\sigma + p_3^\sigma) \Big]$$

$$A_{\mathrm{V}} = \frac{4}{3} \left[\left(\frac{1}{2} - \frac{m^2}{\mu^2}\right) C_0 + \left(1 + \frac{m^2}{\mu^2}\right) \log\left(\frac{m^2}{m^2+\mu^2}\right) - \sqrt{1 + 4\frac{m^2}{\mu^2}} \log\left(\frac{\sqrt{1+4\frac{m^2}{\mu^2}} - 1}{\sqrt{1+4\frac{m^2}{\mu^2}} + 1}\right)\right]$$

$$B_{\mathrm{V}} = \frac{1}{\epsilon} - \gamma_E + \frac{1}{3} \Big[-C_0 \left(1 - 4\frac{m^2}{\mu^2} - 2\frac{m^4}{\mu^4}\right) + 2\left(3 - \frac{m^2}{\mu^2}\right)\frac{m^2}{\mu^2} \log\left(\frac{m^2}{m^2+\mu^2}\right) + \left(1 - 4\frac{m^2}{\mu^2}\right) \log\left(\frac{m^2}{\tilde{\mu}^2}\right) -4\left(1 - \frac{m^2}{\mu^2}\right) \log\left(\frac{m^2+\mu^2}{\tilde{\mu}^2}\right) - \left(1 - 2\frac{m^2}{\mu^2}\right) \sqrt{1 + 4\frac{m^2}{\mu^2}} \log\left(\frac{\sqrt{1+4\frac{m^2}{\mu^2}} - 1}{\sqrt{1+4\frac{m^2}{\mu^2}} + 1}\right)\Big]$$

Results at 1-loop

$$C_{\mathrm{V}} = -\frac{2}{3}\left[\left(1-\frac{m^2}{\mu^2}\right)\frac{m^2}{\mu^2}\log\left(\frac{m^2}{m^2+\mu^2}\right)+\left(1-2\frac{m^2}{\mu^2}\right)\sqrt{1+4\frac{m^2}{\mu^2}}\log\left(\frac{\sqrt{1+4\frac{m^2}{\mu^2}}-1}{\sqrt{1+4\frac{m^2}{\mu^2}}+1}\right)\right.$$
$$\left.+\left(2-\frac{m^2}{\mu^2}\right)-2C_0\frac{m^2}{\mu^2}\left(1+\frac{m^2}{\mu^2}\right)-\left(1-4\frac{m^2}{\mu^2}\right)\log\left(\frac{m^2}{\tilde{\mu}^2}\right)+\left(1-4\frac{m^2}{\mu^2}\right)\log\left(\frac{m^2+\mu^2}{\tilde{\mu}^2}\right)\right]$$

$$D_{\mathrm{V}} = \frac{2}{3}\left[(1+C_0)\left(1-2\frac{m^2}{\mu^2}\right)-2\left(1+\frac{m^2}{\mu^2}\right)\frac{m^2}{\mu^2}\log\left(\frac{m^2}{m^2+\mu^2}\right)\right]$$

satisfies bare WI, **and** ... $Z_V = 1$!
Similarly for Z_A and all other identities.

In particular no μ dependence for the renormalization constant of Noether currents.

Heavy-Light RI/mSMOM Conditions

$$\lim_{\substack{m_R \to 0 \\ M_R \to \bar{m}}} \frac{1}{12q^2} \mathrm{Tr} \left[\left(q \cdot \Lambda_{\mathcal{V},R} - (M_R - m_R) \Lambda_{\mathcal{S},R} \right) \not{q} \right] \big|_{\mathrm{sym}} = \lim_{\substack{m_R \to 0 \\ M_R \to \bar{m}}} \frac{1}{12q^2} \mathrm{Tr} \left[\left(i\zeta^{-1} S_{H,R}(p_2)^{-1} - i\zeta S_{l,R}(p_3)^{-1} \right) \not{q} \right]$$

$$\lim_{\substack{m_R \to 0 \\ M_R \to \bar{m}}} \frac{1}{12q^2} \mathrm{Tr} \left[\left(q \cdot \Lambda_{\mathcal{A},R} - (M_R + m_R) i \Lambda_{\mathcal{P},R} \right) \gamma_5 \not{q} \right] \big|_{\mathrm{sym}} = \lim_{\substack{m_R \to 0 \\ M_R \to \bar{m}}} \frac{1}{12q^2} \mathrm{Tr} \left[\left(-i\gamma^5 \zeta^{-1} S_{H,R}(p_2)^{-1} - i\zeta S_{l,R}(p_3)^{-1} \gamma^5 \right) \gamma_5 \not{q} \right]$$

$$\lim_{\substack{m_R \to 0 \\ M_R \to \bar{m}}} \frac{1}{12i} \mathrm{Tr} \left[\Lambda_{\mathcal{P},R} \gamma_5 \right] \big|_{\mathrm{sym}} = \lim_{\substack{m_R \to 0 \\ M_R \to \bar{m}}} \left\{ \frac{1}{12(M_R + m_R)} \left\{ \mathrm{Tr} \left[-i\zeta^{-1} S_{H,R}(p)^{-1} \right] \Big|_{p^2 = -\mu^2} - \frac{1}{2} \mathrm{Tr} \left[\left(q \cdot \Lambda_{\mathcal{A},R} \right) \gamma_5 \right] \big|_{\mathrm{sym}} \right\} + \frac{1}{12(M_R + m_R)} \left\{ \mathrm{Tr} \left[-i\zeta S_{l,R}(p)^{-1} \right] \Big|_{p^2 = -\mu^2} - \frac{1}{2} \mathrm{Tr} \left[\left(q \cdot \Lambda_{\mathcal{A},R} \right) \gamma_5 \right] \big|_{\mathrm{sym}} \right\} \right\}.$$

where M and m refer to heavy and light quark masses respectively and $\zeta = \frac{\sqrt{Z_l}}{\sqrt{Z_H}}$.

Lattice Regularization

Lattice WI for chiral symmetry

$$\nabla_\mu^* \langle A_\mu^a(x)\psi(y)\bar{\psi}(z)\rangle = 2m\langle P^a(x)\psi(y)\bar{\psi}(z)\rangle + \text{contact terms}$$
$$+\langle X^a(x)\psi(y)\bar{\psi}(z)\rangle$$

- X^a explicit chiral symmetry breaking by lattice regulator
- Reproduces usual continuum result when regulator is removed
  $\Rightarrow \; X^a(x) = aO_5^a(x)$
- Renormalize operators, $O_5^a(x)$ mixed with lower-dim operators
- Testa[3]: power divergencies do not contribute to the anomalous dimensions
  $\Rightarrow \; A_{R,\mu} = Z_A\,(g, am)\,A_\mu$

$$O_{5R}^a(x) = Z_5\left[O_5^a(x) + \underbrace{\frac{\bar{m}}{a}P^a(x) + \frac{Z_A - 1}{a}\nabla_\mu^* A_\mu^a(x)}_{\frac{\tilde{Z}}{a}\tilde{O}(x)}\right]$$

[3] arXiv: hep-th/9803147

Summary

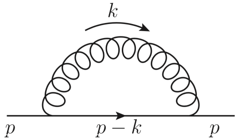
- Generalised SMOM to non-vanishing fermion mass
- Derived non-perturbatively, checked at 1-loop in perturbation theory
- Both for heavy-heavy and heavy-light vertex functions such $Z_{V,A}^{\text{cons}} = 1$
- Obtain $Z_{V,A}^{\text{local}}$ by taking ratios of vertex function with appropriate projectors
- Numerical implementation and tests will be performed on renormalizing matrix elements used to obtain decay constants and form factors in semi-leptonics

Backup Slides

Finiteness of the ζ ratio

$$\zeta = \frac{\sqrt{Z_l}}{\sqrt{Z_H}}$$

- BPHZ theorem: Remove all the divergences of a graph, G, using **local subtractions** only $\Longrightarrow$
  coeffs. multiplying the divergent part are local.
- Possible structure of the coeffs: 1, $\underbrace{p^2/m^2}_{\text{IR div}}$, $\underbrace{m^2/p^2}_{\text{non-local}}$, $\underbrace{\ln\left(\frac{m^2}{p^2}\right)}_{\text{non-local}}$



Example: $Z_A = 1$ for Heavy-Heavy Vertex

Bare axial WI:

$$q \cdot \Lambda_{\mathrm{A}} = 2mi\Lambda_{\mathrm{P}} - \gamma_5 iS(p_2)^{-1} - iS(p_3)^{-1}\gamma_5$$

Rewriting in terms of renormalized quantities

$$\frac{1}{Z_{\mathrm{A}}} q \cdot \Lambda_{\mathrm{A},R} - \frac{1}{Z_m Z_{\mathrm{P}}} 2m_R i\Lambda_{\mathrm{P},R} = -\left\{\gamma_5 iS_R(p_2)^{-1} + iS_R(p_3)^{-1}\gamma_5\right\}$$

Example: $Z_A = 1$ for Heavy-Heavy Vertex

1. Trace with $\gamma^5 \not{q}$

$$(Z_\mathrm{A} - 1) = \left(1 - \frac{Z_A}{Z_m Z_\mathrm{P}}\right) C_{mP}\,,$$

$$C_{m\mathrm{P}} = \lim_{m_R \to \bar{m}} \frac{1}{12q^2} \mathrm{Tr}\left[2im_R \Lambda_{\mathrm{P},R} \gamma_5 \not{q}\right]\Big|_\mathrm{sym}$$

2. Trace with γ^5

$$(Z_\mathrm{A} - 1) C_{qA} = -2Z_\mathrm{A}\left(1 - \frac{1}{Z_m Z_\mathrm{P}}\right)\,,$$

$$C_{q\mathrm{A}} = \lim_{m_R \to \bar{m}} \frac{1}{12m_R} \mathrm{Tr}\left[q \cdot \Lambda_{\mathrm{A},R} \gamma_5\right]\Big|_\mathrm{sym}$$

Together give $Z_A = 1$ and $Z_m Z_p = 1$.