Study of the Sign problem in canonical approach

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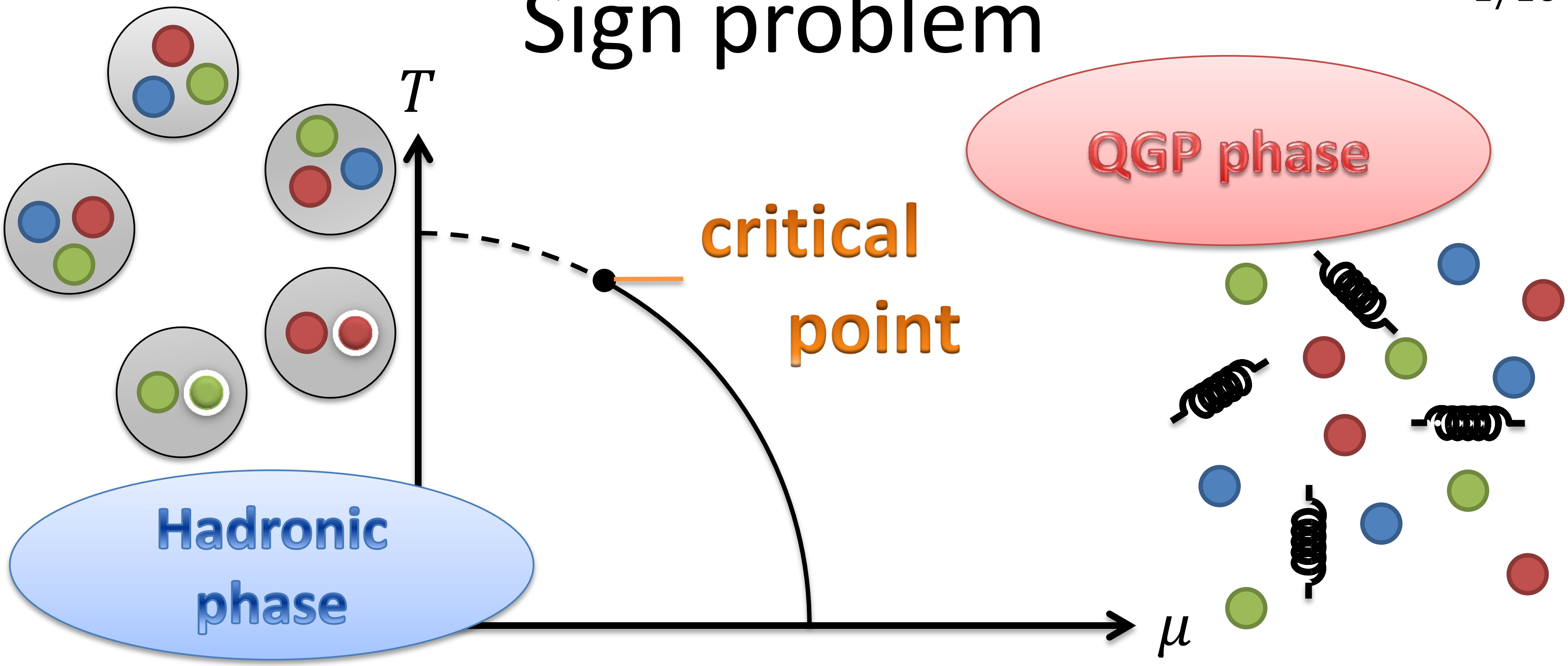
Sign problem



Sign problem

✓ Complex action

$$Z_{G.C.}(T,\mu;V) = \langle \det D(\mu) \rangle$$

$$Z_{G.C.} \in \mathbb{R}$$

$$\det D(\mu) \in \mathbb{C}$$

Solution ?

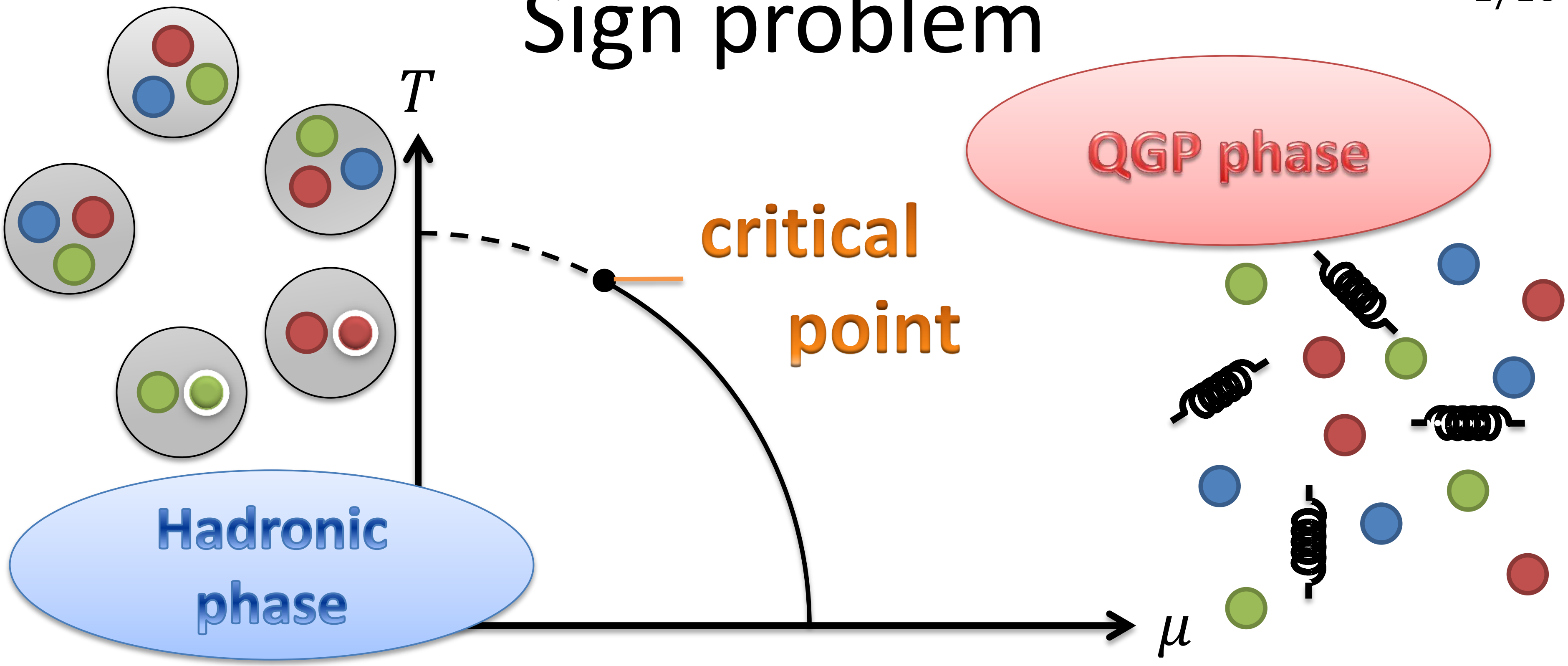
Reweighting

Complex Langevin

Lefschetz thimble

Canonical approach

Sign problem



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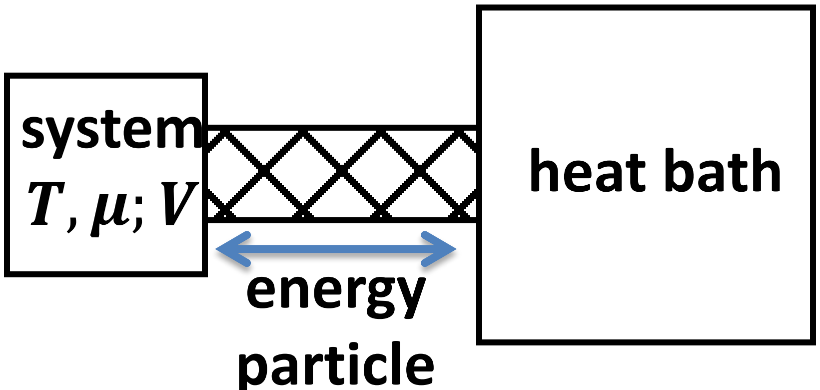
Canonical approach

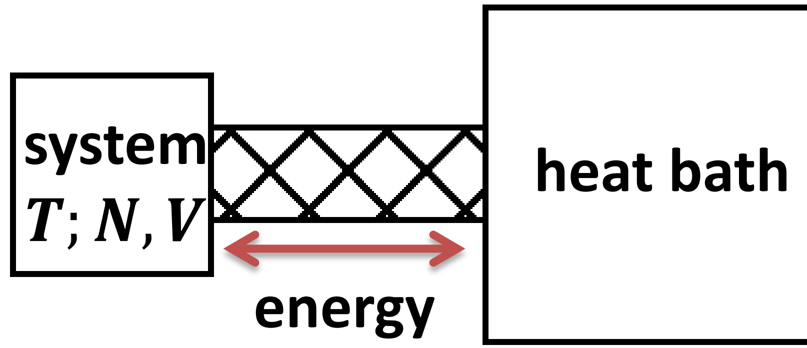
Canonical approach

A.Hasenfratz , D.Toussaint (1992)

Grand Canonical



Canonical



- ✓ **Both ensembles describe same thermodynamics**

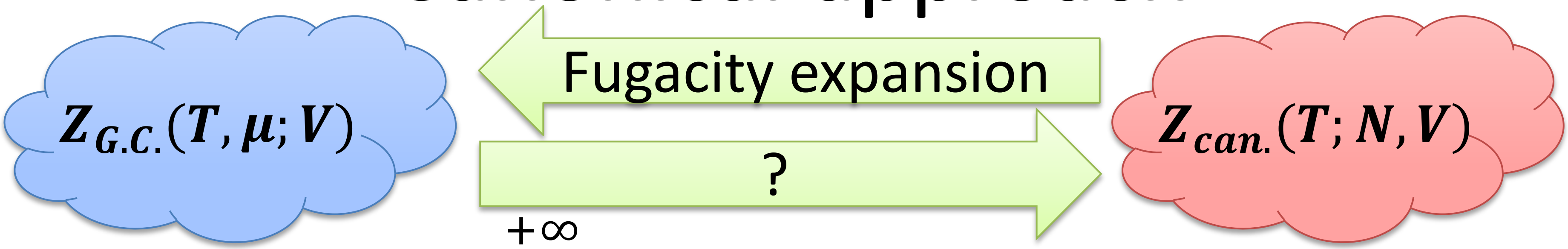
$$Z_{G.C.}(\mu) := \sum_{i,N} \langle E_i, N | e^{-(\hat{H}-\mu\hat{N})/T} | E_i, N \rangle$$

$$Z_{can.}(N) := \sum_{i} \langle E_i, N | e^{-\hat{H}/T} | E_i, N \rangle$$

- ✓ **Relation between the partition functions ?**
  - ➢ **Fugacity expansion**

$$Z_{G.C.}(\mu) = \sum_N Z_{can.}(N) \xi^N , \xi = e^{\frac{\mu}{T}}$$

Canonical approach

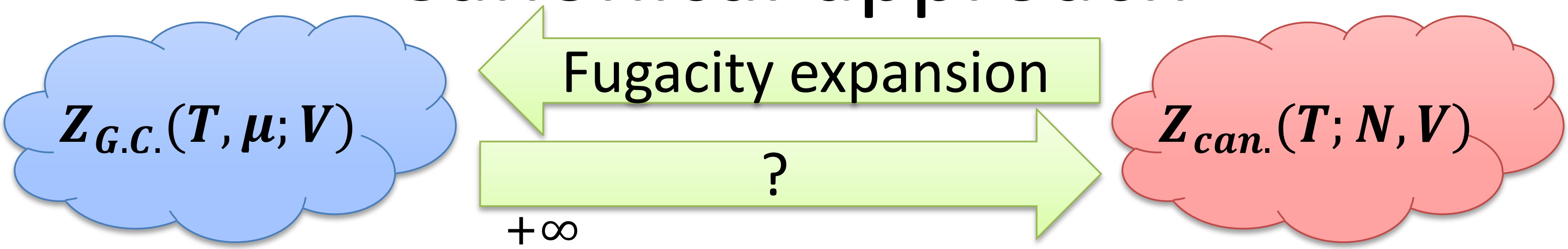


$$Z_{G.C.}(\mu) = \sum_{N=-\infty}^{+\infty} Z_{can.}(N)\xi^N \,, \xi = e^{\frac{\mu}{T}}$$

✓ **Regard as Laurent series with respect to fugacity**

$$Z_{can.}(N) = \frac{1}{2\pi i}\oint_C d\xi\, \xi^{-(N+1)} Z_{G.C.}(\xi)$$

Canonical approach



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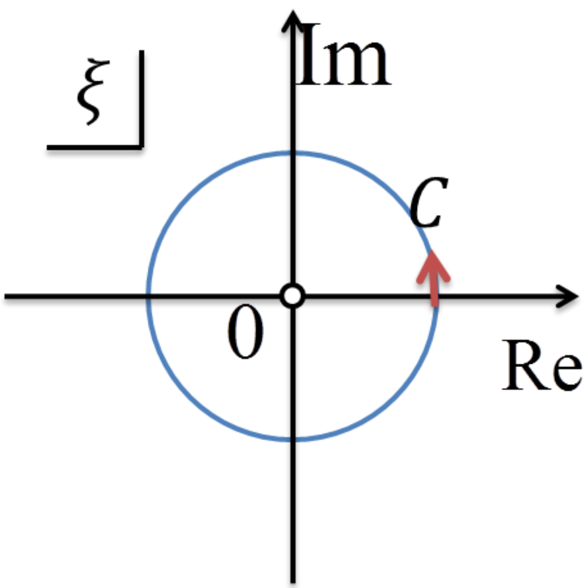
$$C: \xi = e^{i\frac{\mu}{T}} \, , \frac{\mu}{T} \in [-\pi, \pi)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\frac{\mu}{T}\, e^{-i\frac{\mu}{T}N} Z_{G.C.}(i\mu)$$



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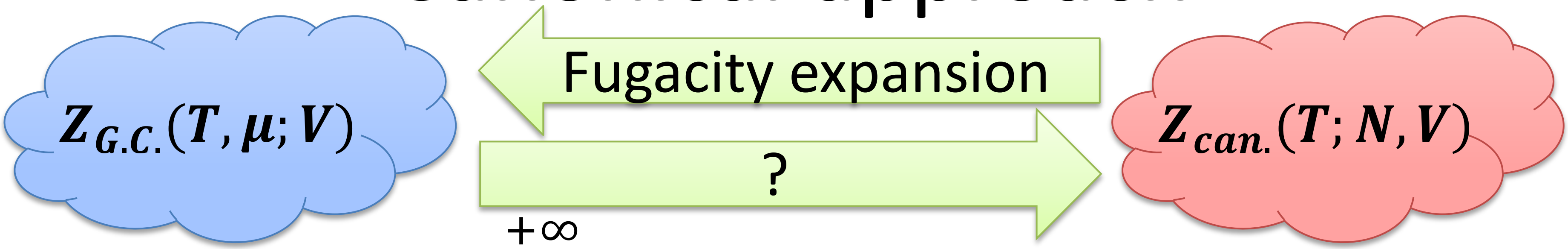

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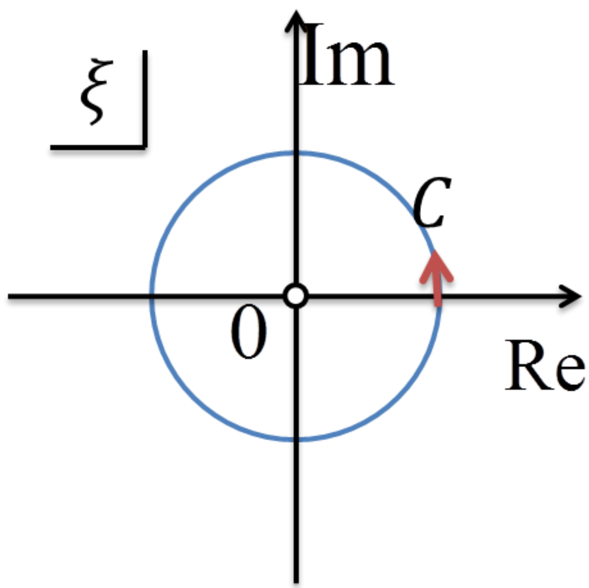
pure imaginary



✓ $Z_{can.}$ **is given by**

Fourier transformation

Sign problem in canonical approach

$$Z_{can}(N) = \langle Z_N \rangle$$

- ✓ $\boldsymbol{Z_{can.}} \in \mathbb{R}$
- ✓ $\boldsymbol{Z_N}$ **takes complex value** $\boldsymbol{Z_N = |Z_N| e^{i\theta_N}}$

Sign problem in canonical approach

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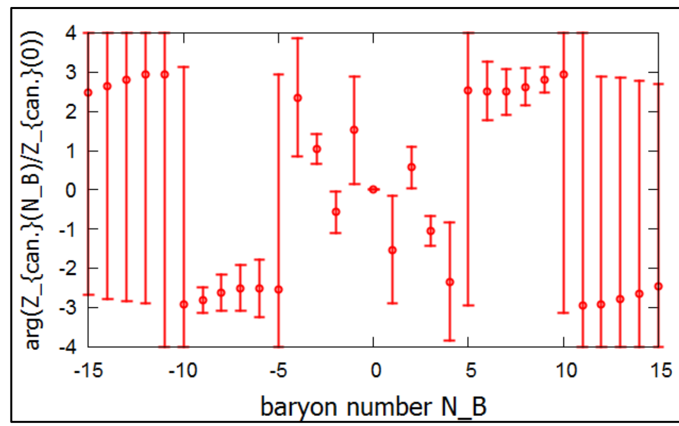
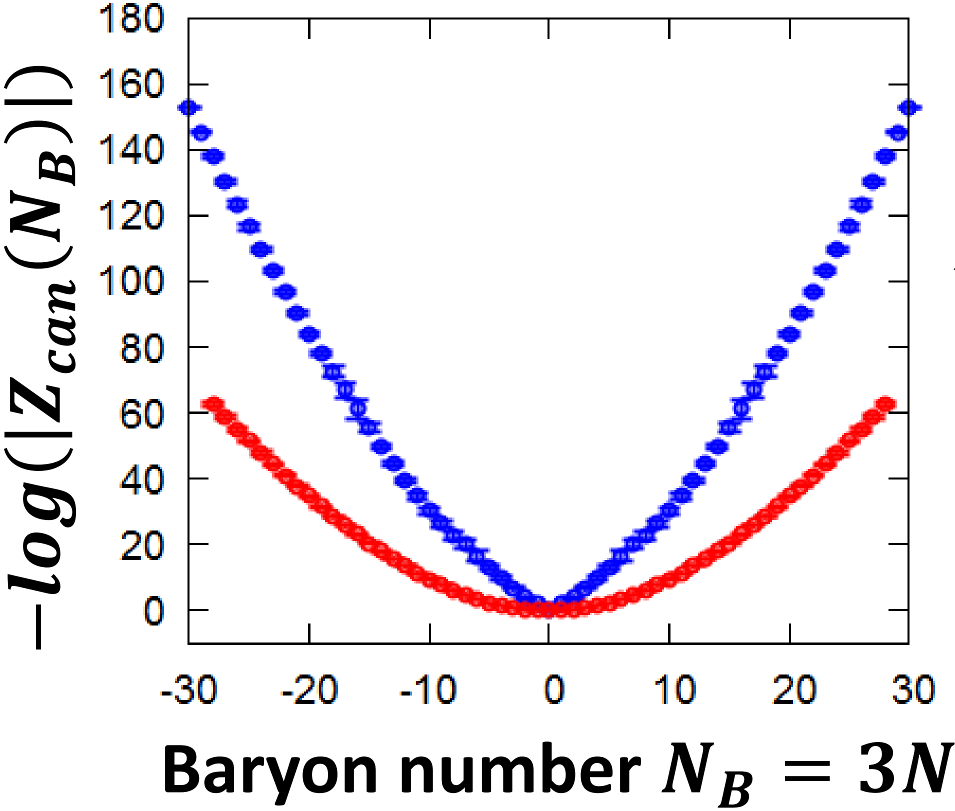
contents

- **The origin of the phase**
- N_B **dependence**
  - $\theta_N \propto N$
- T **dependence**
  - **high temperature → stable**
  - **low temperature → unstable**
- **How to understand these behavior ?**



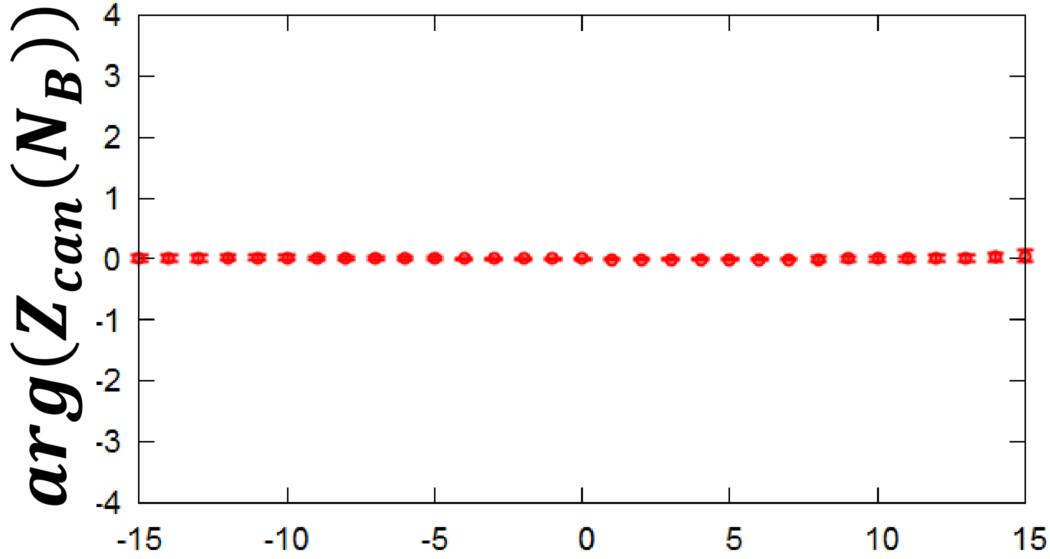
Numerical result of $Z_{can.}(N)$

- **Iwasaki – gauge action**
- **2-flavor Wilson – clover action**
- $\mathbf{8^3 \times 4}$ **Lattice , 100 conf.**
- $\boldsymbol{T_C \sim 220\mathrm{MeV}}$



✓ $|Z_{can}(N_B)|$ **is stable**

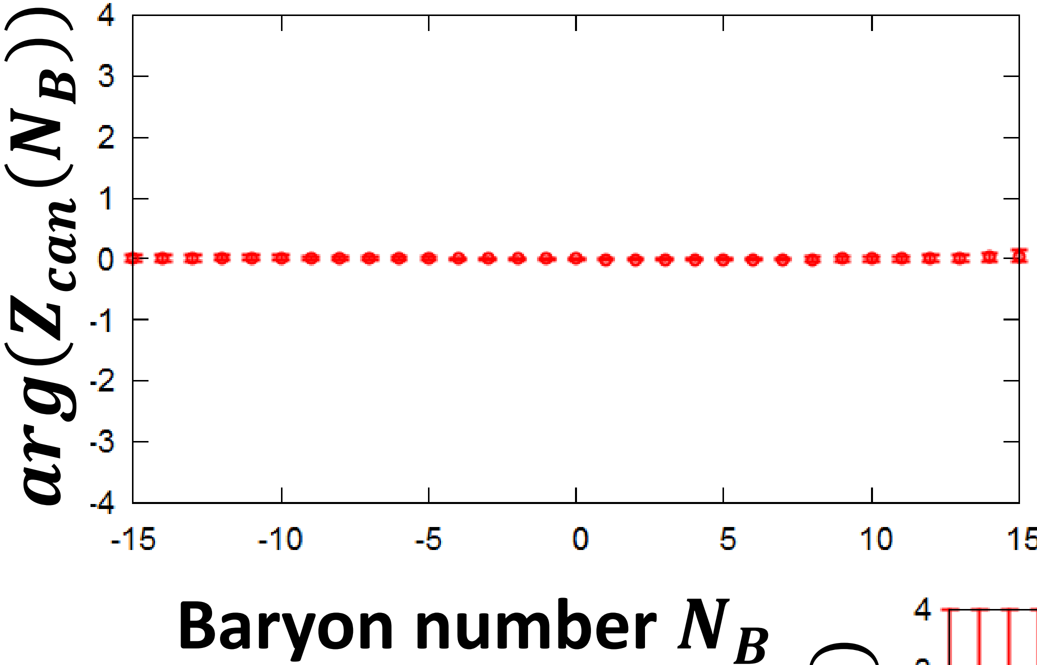
Numerical result of $Z_{can.}(N)$



Baryon number N_B

- ✓ $T = 1.68\,T_C$
- ✓ $arg(Z_{can})$ **is consistent with 0**

Numerical result of $Z_{can.}(N)$



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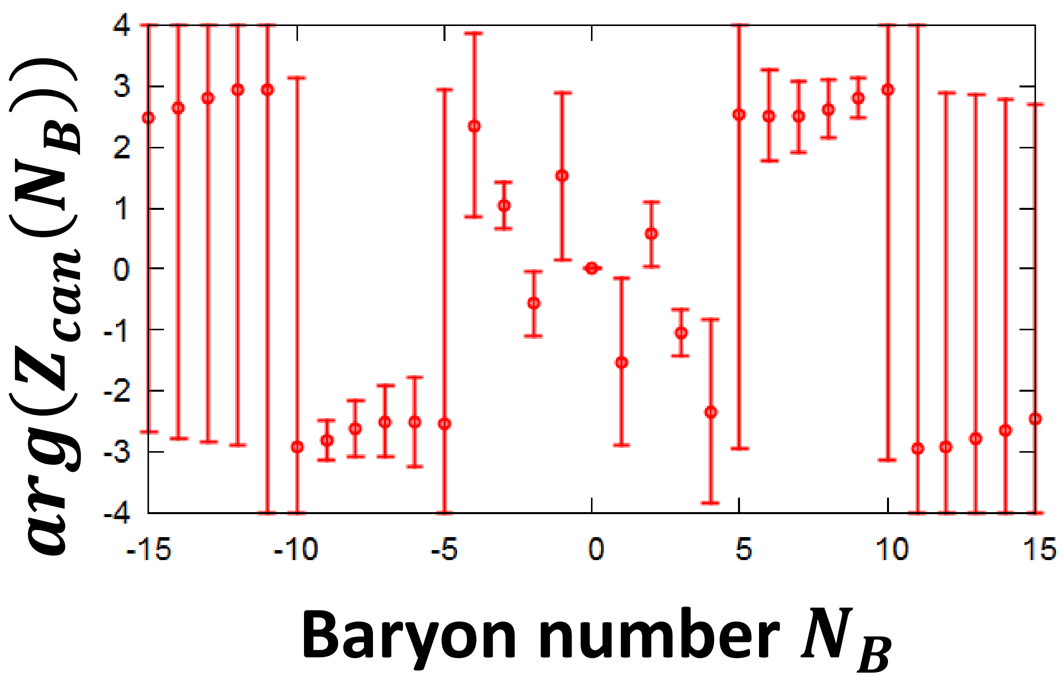

- ✓ $T = 0.813\, T_C$
- ✓ $|arg(Z_{can})| > \pi/2$
  - ➢ sign problem !

Complex phase of $Z_{can.}(N)$

$$Z_{can.}(N) = \frac{3}{2\pi}\int_{-\pi/3}^{\pi/3} d\frac{\mu}{T}\, e^{-i\frac{\mu}{T}N}\, Z_{G.C.}(i\mu)$$

- ✓ $\boldsymbol{Z_{G.C.}(i\mu) = \langle det(D(i\mu))\rangle} \in \mathbb{R}$
- ✓ $\boldsymbol{\langle det(D(i\mu))\rangle \neq \langle det(D(-i\mu))\rangle}$ **makes phase?**

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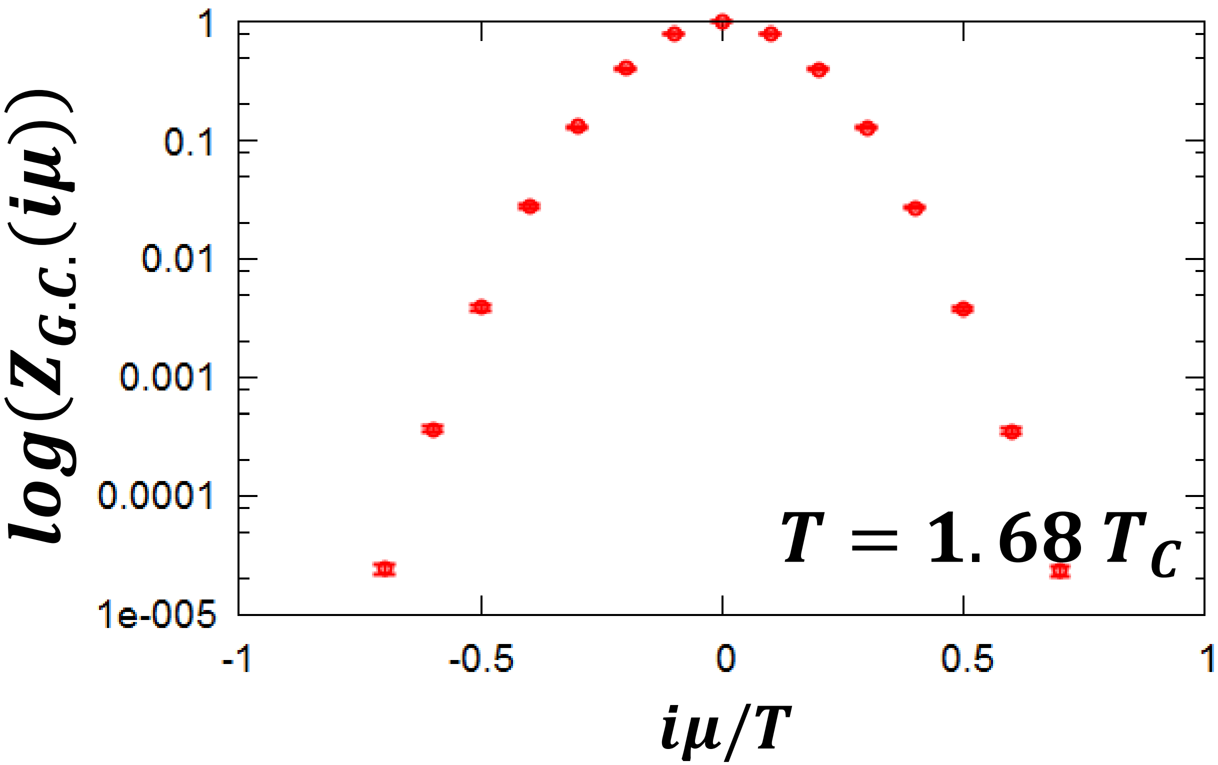

$$\boldsymbol{Z_{G.C.}(i\mu/T) \sim e^{-a\left(\frac{i\mu}{T}\right)^2 + b\frac{i\mu}{T}}}$$

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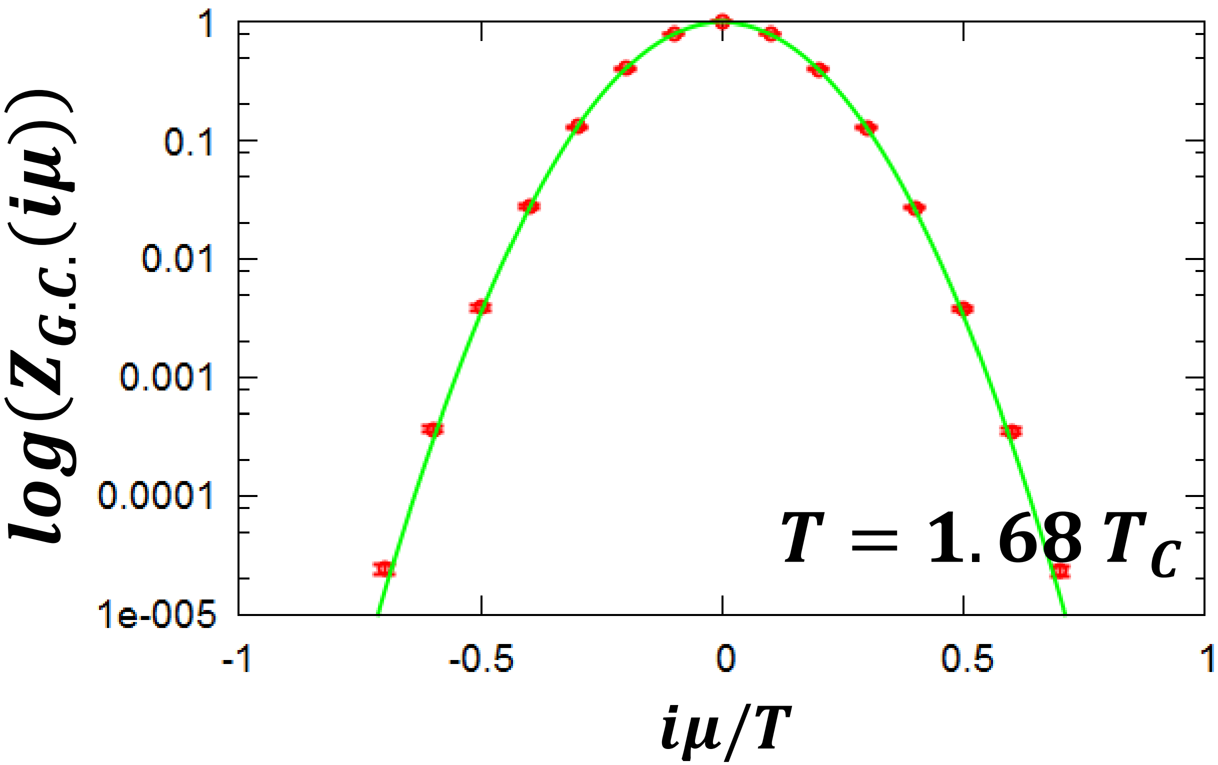

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Complex phase of $Z_{can.}(N)$

$$Z_{can.}(N) \sim (real\ const.) \int_{-\pi/3}^{\pi/3} dx\ e^{-iNx} e^{-ax^2+bx}$$

$$= (const.) e^{\frac{(b-iN)^2}{4a}} \int_{-\pi/3}^{\pi/3} dx\ e^{-a\left(x-\frac{b-iN}{2a}\right)^2}$$

phase !

~Gaussian int.

✓ $\boldsymbol{arg(Z_{can}(N)) \sim -\frac{b}{2a}N}$

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phase !

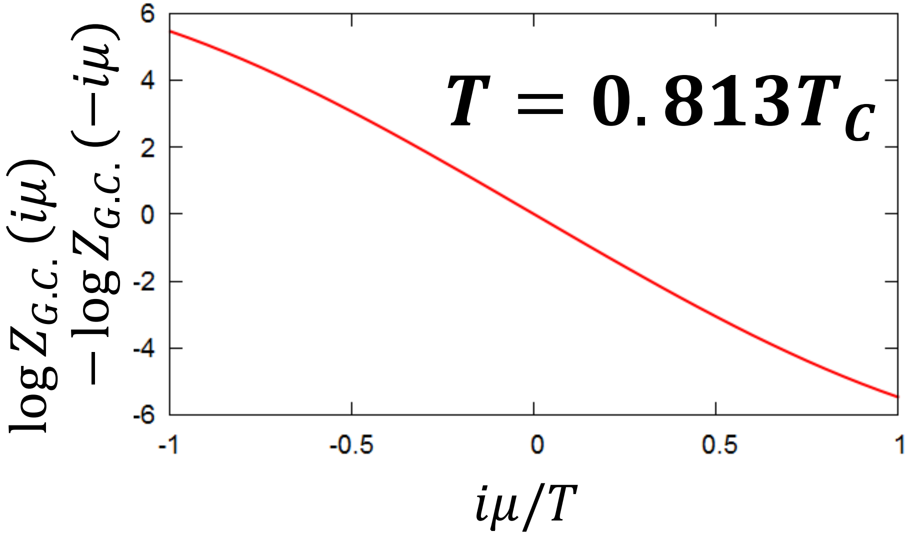
~Gaussian int.

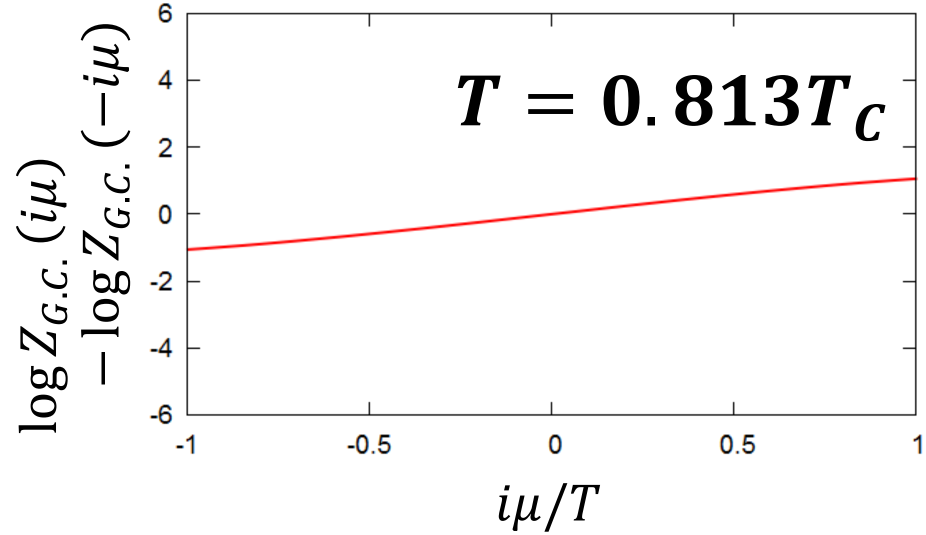
- ✓ $\boldsymbol{arg(Z_{can}(N)) \sim -\frac{b}{2a}N}$
- ✓ **Comparison with the previous work**
  - ➢ Li, X. Meng, A. Alexandru, K. F. Liu (2008)

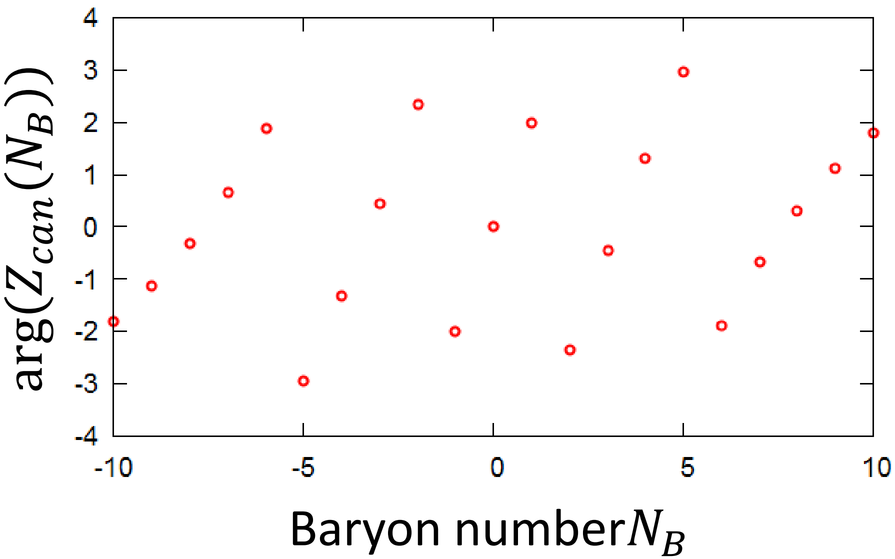
    $\arg(Z_{can}(N)) \sim -N * \arg(W_1)$

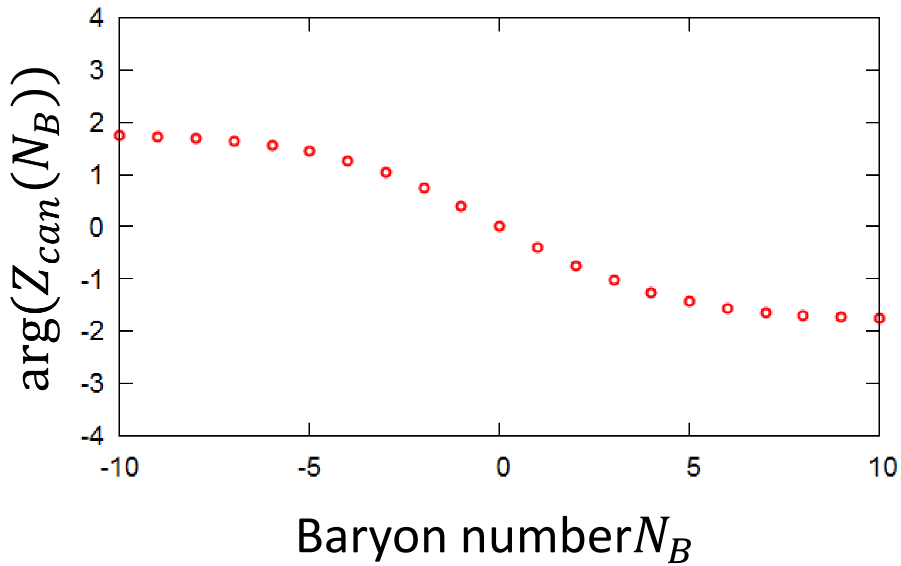
    W_1: coefficient of winding number expansion

N_B dependence of the phase





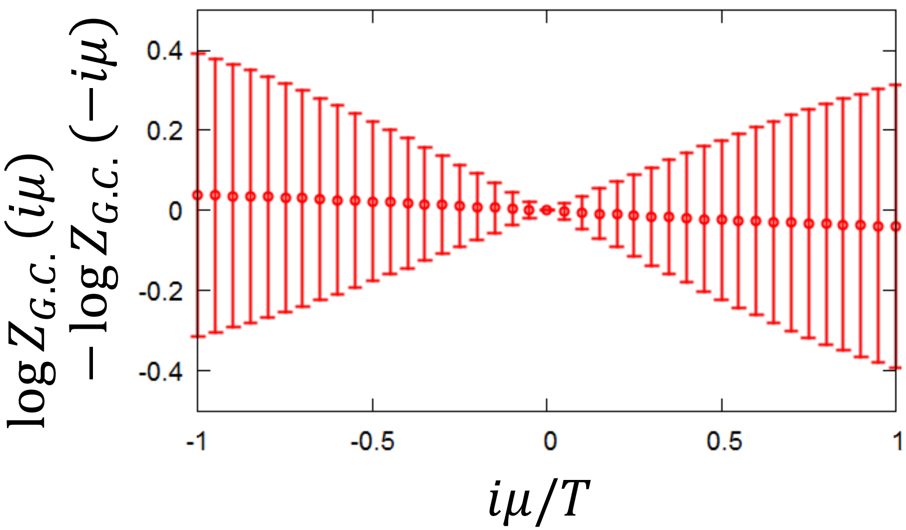




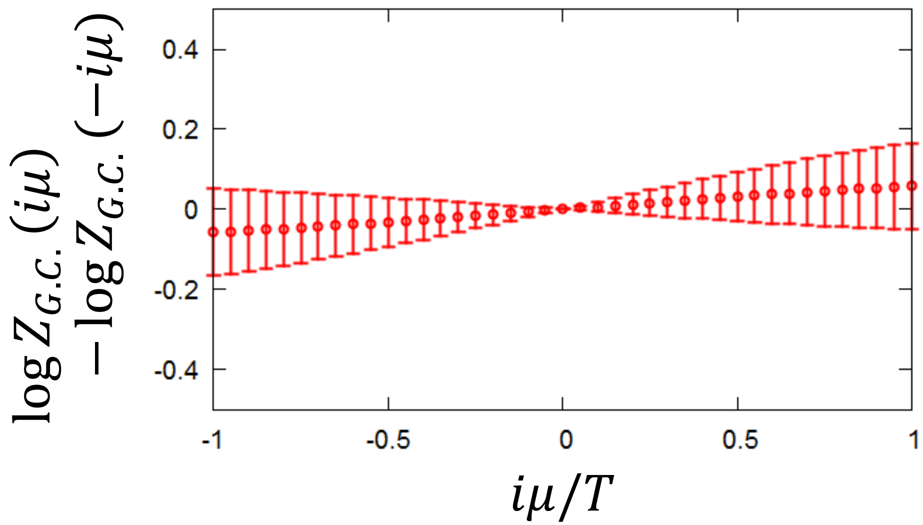
✓ $arg(Z_{can}(N))$ **on each conf. →** $\theta_N \propto (-N)$

T dependence of the phase

$$Z_{G.C.}(i\mu) \sim e^{-a\left(i\frac{\mu}{T}\right)^2 + b\left(i\frac{\mu}{T}\right)}$$



$T = 0.813\, T_C$

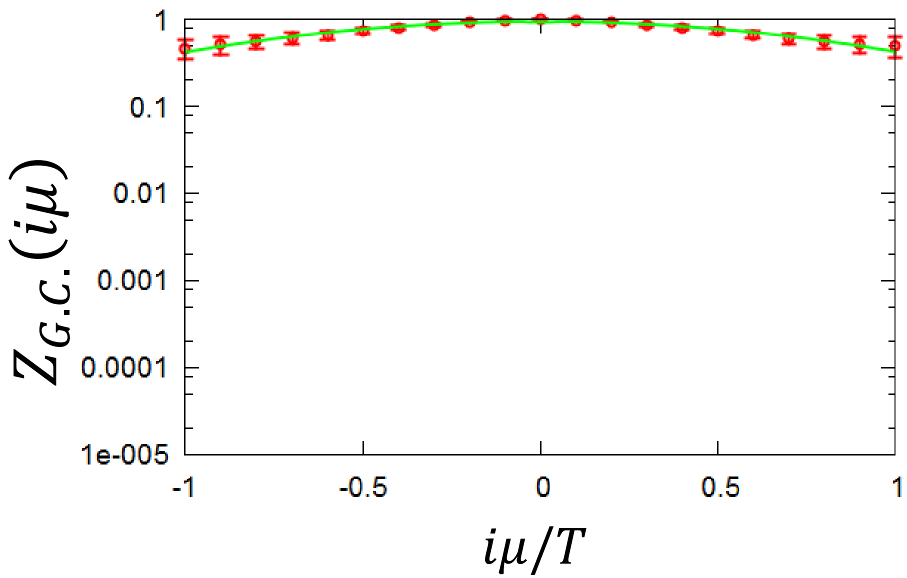


$T = 1.68\, T_C$

- ✓ **b takes almost the same order**
- ✓ **fluctuation at low T is slightly large**

T dependence of the phase

$$Z_{G.C.}(i\mu) \sim e^{-a\left(i\frac{\mu}{T}\right)^2 + b\left(i\frac{\mu}{T}\right)}$$



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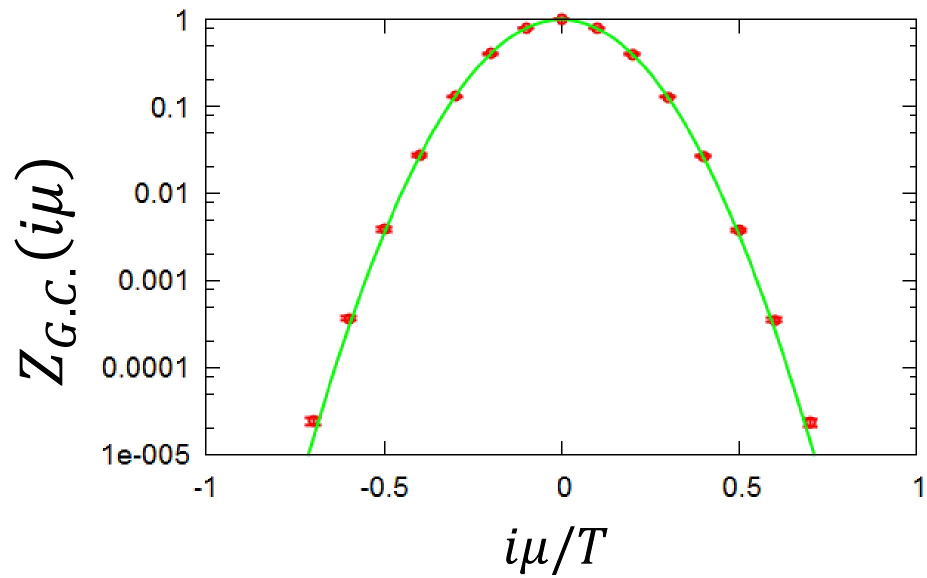
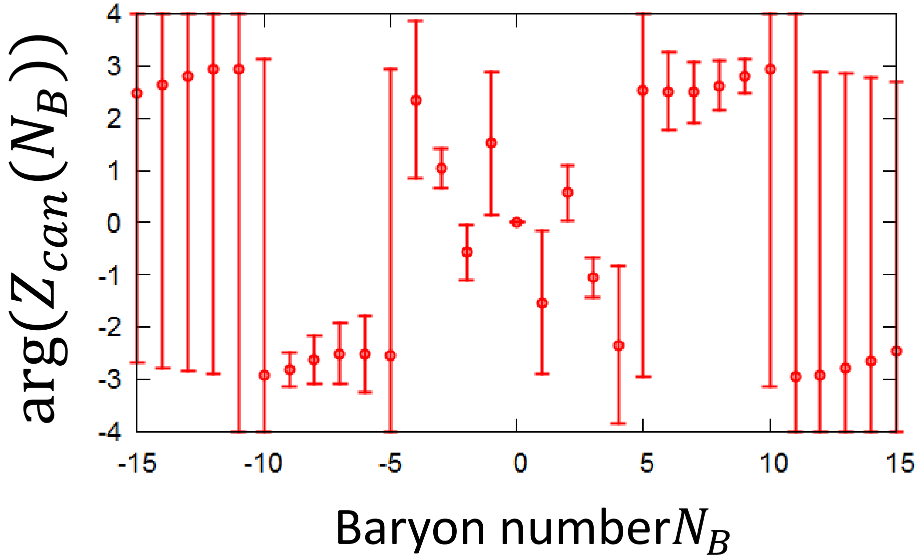


$T = 1.68\, T_C$

- ✓ $a_{0.831T_C} \sim 0.806$
- ✓ $a_{1.68T_C} \sim 22.7$
- ✓ $a_{highT} \gg a_{lowT}$

T dependence of the phase

$$arg(Z_{can}(N)) \sim -\frac{b}{2a}N$$



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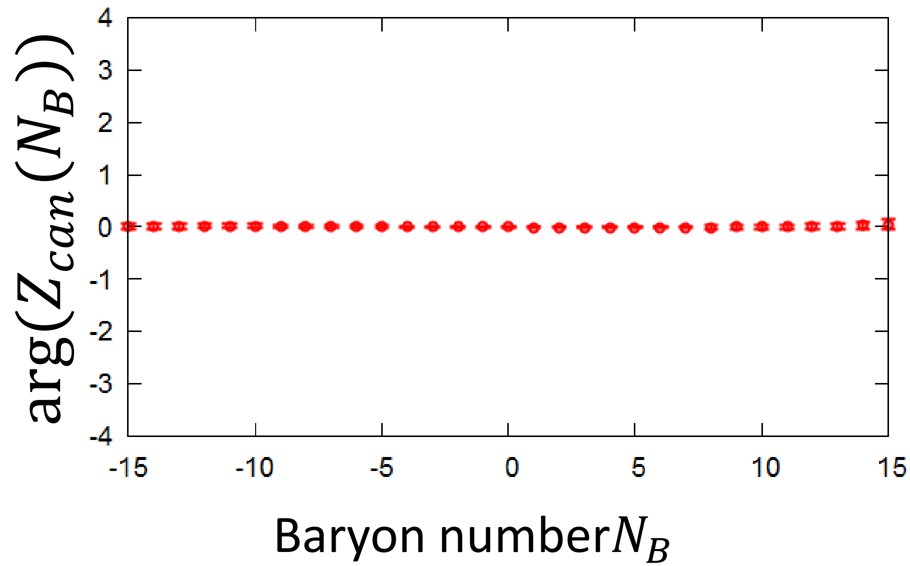
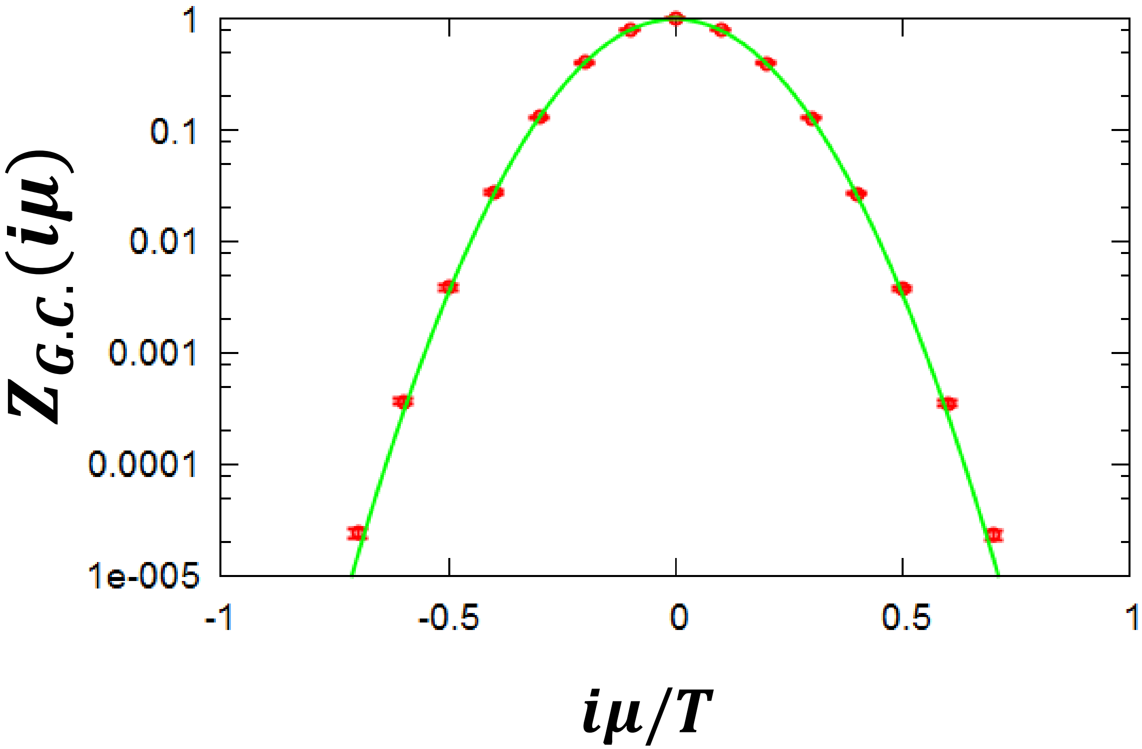


$T = 1.68\, T_C$

- ✓ $a_{low\,T} \ll a_{high\,T}$
- ✓ **fluctuation of the θ_N increases at low T**

To suppress the complex phase

$$arg(Z_{can}(N)) \sim -\frac{b}{2a}N$$



$$Z_{G.C.}(i\mu/T) \sim e^{-a\left(\frac{i\mu}{T}\right)^2+b\frac{i\mu}{T}}$$

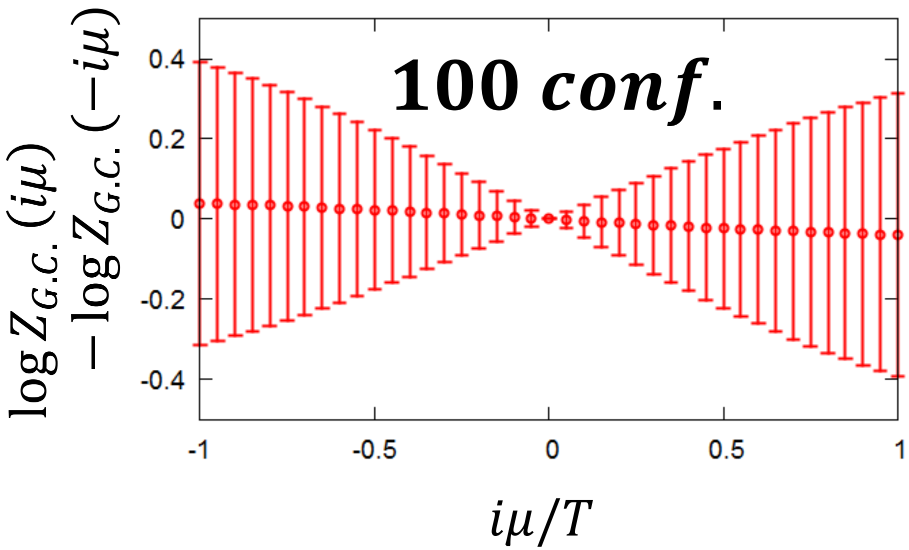
- a : **Gaussian slope**
- b : **asymmetric part**

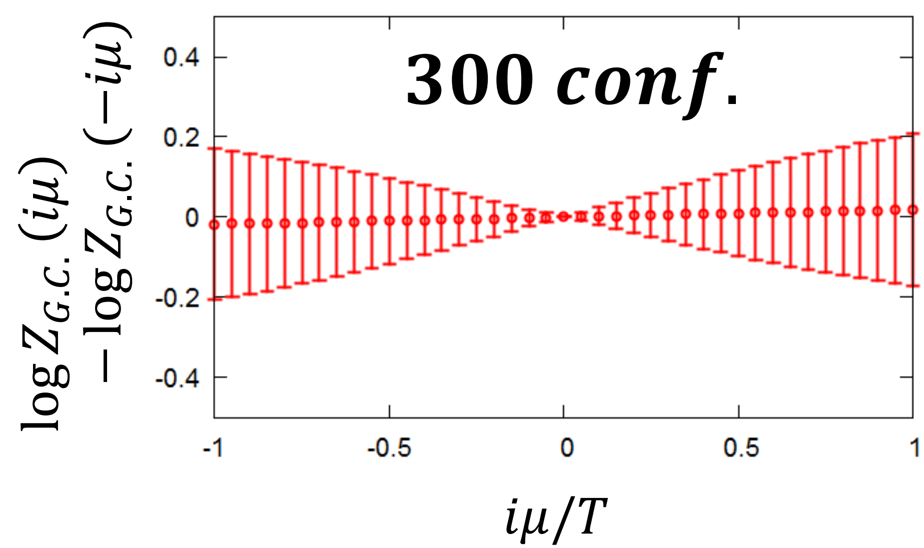
✓ **We try to suppress** b **!**

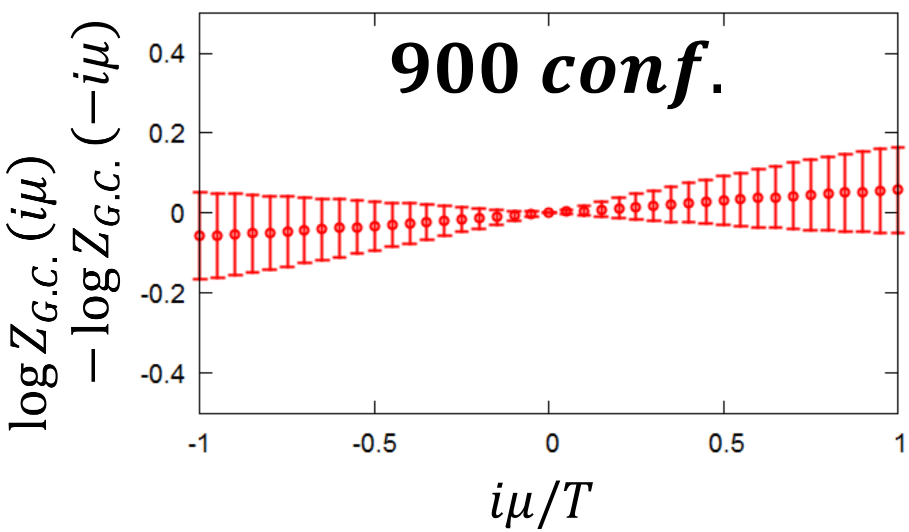
✓ **naive idea : increase conf.#**

To suppress the complex phase

$T = 0.813\, T_C$

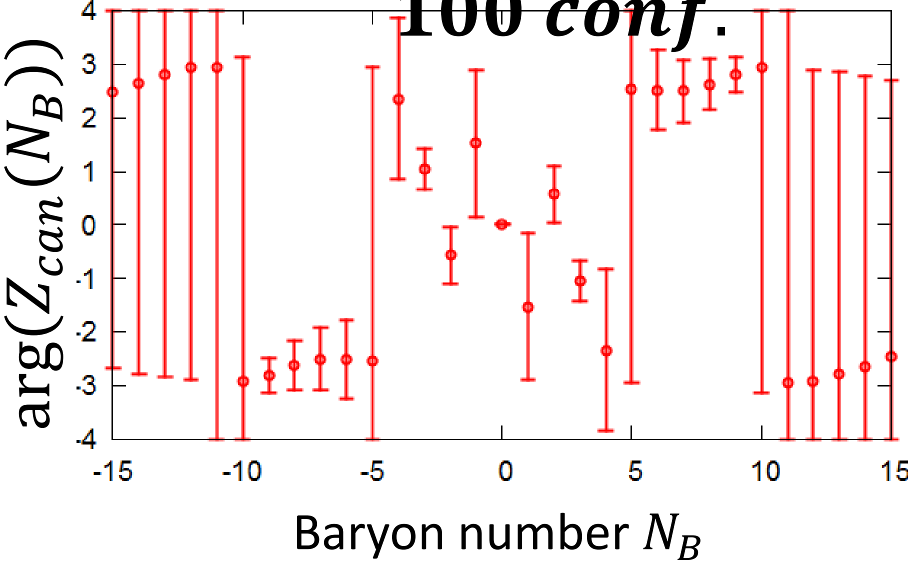






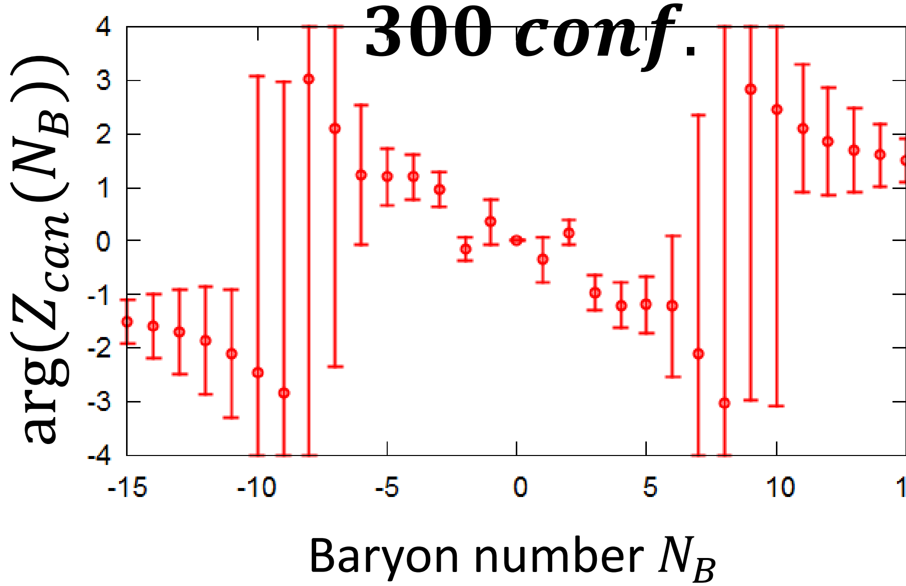
To suppress the complex phase

$100\ conf.$

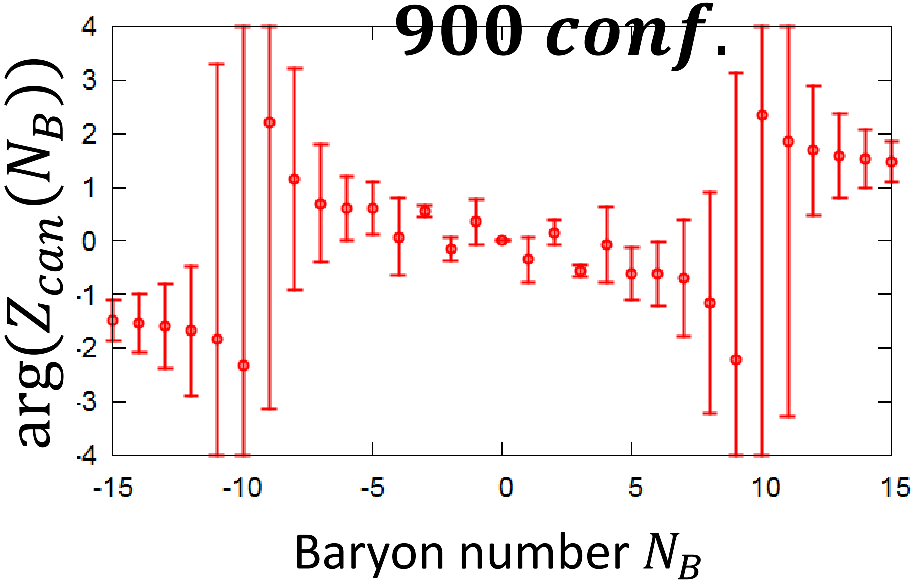


$T = 0.813\ T_C$

$300\ conf.$



$900\ conf.$



✓ $|arg(Z_{can}(N_B))| < \pi/2$ **in 1σ for $|N_B| \leq 7$**

Summary

- Sign problem emerges as a complex phase in the canonical approach. $Z_{can}(N) = \langle Z_N e^{i\theta_N} \rangle$
- In some approximation, $\theta_N \sim -\frac{b}{2a} N$.
- N dependence is consistent with previous work and the numerical results.
- T dependence can be understood from the property of $Z_{G.C.}(i\mu)$ (Gaussian slope a)
- We suppress θ_N by increasing the configurations.

  $-900\ conf., T = 0.813\ T_C \rightarrow \left|\theta_{N_B}\right| < \pi/2(|N_B| \leq 7)$

back up

Winding number expansion

Li, X. Meng, A. Alexandru,K. F. Liu (2008)

$$Z_{can.}(T;N,V) = \left\langle \frac{1}{2\pi}\int_{-\pi}^{\pi} d\frac{\mu}{T}\, e^{-i\frac{\mu}{T}N}\, \frac{Det\{D(i\mu)\}}{Det\{D(0)\}} \right\rangle_g$$

- ✓ **discrete Fourier transf. is expensive!**
- ✓ **low cost calculation of $\mathbf{Det}(\boldsymbol{D}(\boldsymbol{i\mu}))$**

$$Det\{D(\mu)\} = Det\{1-\kappa Q(\mu)\} = e^{Tr\{\log(1-\kappa Q)\}}$$

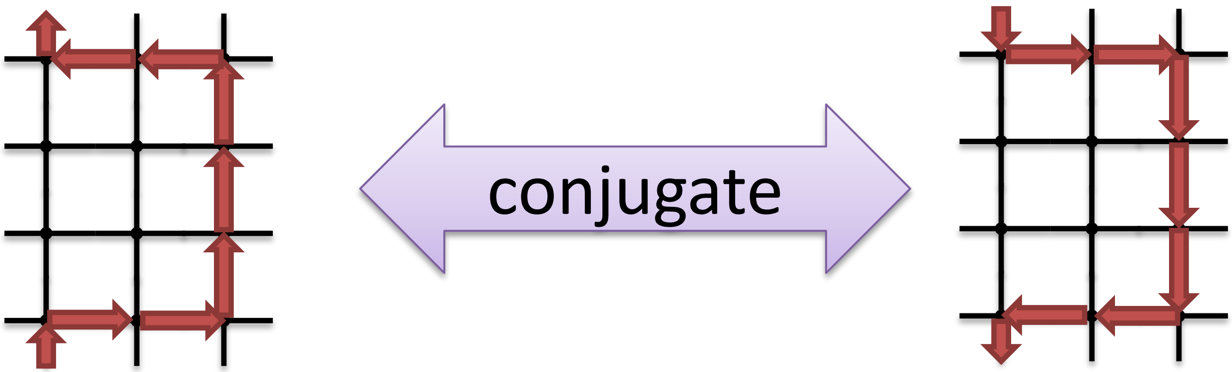
κ : hopping parameter, Q : hopping term

➢ **Winding number expansion method**

$$Tr\{\log(1-\kappa Q)\} = \sum_k W_k e^{\frac{\mu}{T}k}$$

- ✓ $\{\boldsymbol{W_k}\}$ **does not depend on $\boldsymbol{\mu}$**

$W_k \ni$



$\in W_{-k}$

✓ $\det D(i\mu) = \exp\left\{\sum_k W_k e^{i\frac{\mu}{T}k}\right\}$ is real

Winding number expansion

Li, X. Meng, A. Alexandru,K. F. Liu (2008)

Canonical partition function	← Fourier trans.	Grand canonical partition function

$$Z_{can.}(T;n,V) = \left\langle \frac{1}{2\pi}\int_{-\pi}^{\pi} d\frac{\mu}{T}\, e^{-i\frac{\mu}{T}n}\, \frac{Det\{D(i\mu)\}}{Det\{D(0)\}} \right\rangle_g$$

sign real , positive

$$\gamma_5 D(\mu)\gamma_5 = D(-\mu^*)^\dagger$$

calculate $Det\{D(i\mu)\}$ at **low cost** !

$$Det\{D(\mu)\} = Det\{1-\kappa Q(\mu)\} = e^{Tr\{\log(1-\kappa Q)\}}$$

κ : hopping parameter

$$Tr\{\log(1-\kappa Q)\} = -\sum_n \frac{\kappa^n}{n} Tr\{Q^n\}$$

$$= \sum_k W_k e^{\frac{\mu}{T}k}$$

k ; winding number