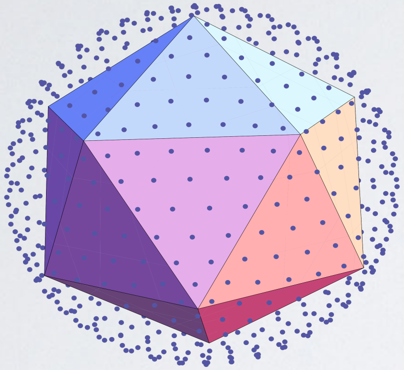
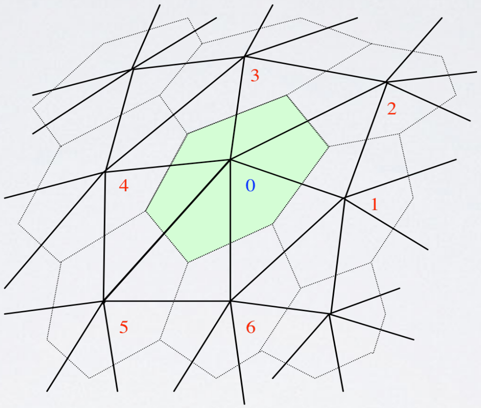
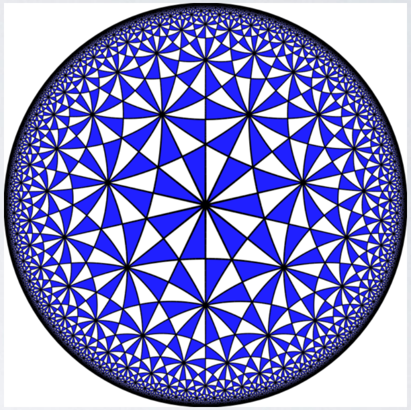
LATTICE QFT ON CURVED SPACE: FERMIONS & 2D CONFORMAL FIELD THEORY*







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Lattice 2016   Wednesday, 27 July 2016

*collaboration with G. Fleming, A. Gabarro, T. Ruben, C-I Tan and E. Weinberg

PLAN

Classical Simplicial Lattice Action

Piecewise Linear FEM/RC vs DEC

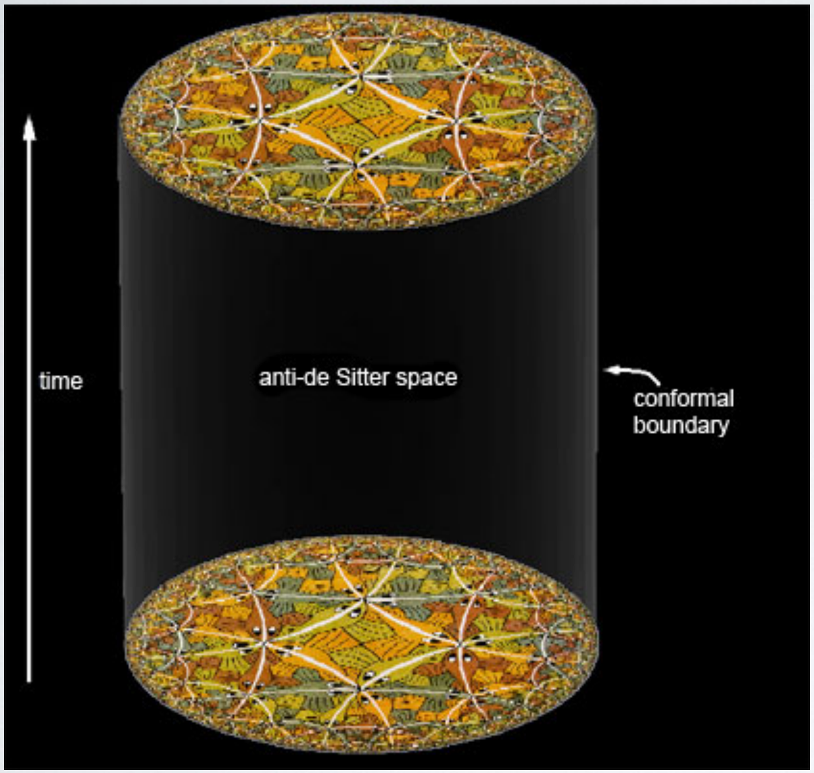
The Simplicial Dirac Solution

Testing on the 2D Ising Sphere

Quantum Simplicial Field Theory: Counter Terms

3D & 4D Coupled Dirac/Gauge/Scalar Lattice QFE

NOT ALL QUANTUM FIELDS LIVE ON FLAT RIEMANN MANIFOLDS

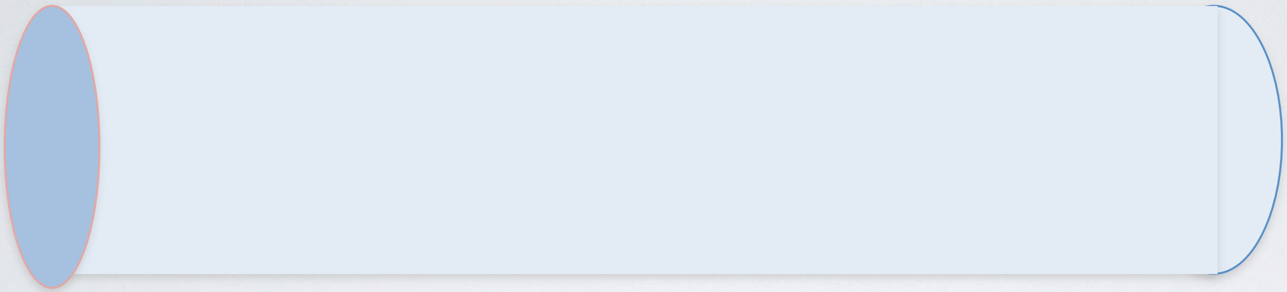


The Conformal Field Theory lives on the cylinder $\mathbb{R} \times \mathbb{S}^4$ on the boundary of AdS^5

Space-Time:Two Black Holes Merge

RADIAL QUANTIZATION OF CONFORMAL FIELD THEORY

$\mathbb{R} \times \mathbb{T}^3$ vs $\mathbb{R} \times \mathbb{S}^3$ ← Boundary of Global AdS5



← Isometries are the conformal group O(5,1)

On lattice scales exponentially

$$H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$$

$$1 < t < aL \implies 1 < \tau = log(r) < L$$

Applications:
(1) Near IR conformality — composite Higgs
(2) AdS/CFT weak-strong duality,
(3) CFT c-theorems, anomalies on sphere,
(4) Quantum Phase Transitions Critical Phenomena etc.

One Motivation:Radial Quantization for CFT

$$ds^2 = dx^\mu dx_\mu = e^{2\tau}[d\tau^2 + d\Omega^2]$$

Can drop Weyl factor!

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

Evolution: $H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$

$$\text{"}time\text{"} \quad \tau = log(r)\ , \quad \text{"}mass\text{"} \quad \Delta = d/2 - 1 + \eta$$

$$D \to x_\mu \partial_\mu = r\partial_r = \frac{\partial}{\partial \tau}$$

WHAT ABOUT LATTICE FIELDS ON RIEMANNIAN MANIFOLD?

Lattice Field theory on $\mathbb{R}^D$ Euclian Manifold provides a rigorous solutions to UV complete (renormalizable) quantum field theory.

Can this be generalized to a smooth Riemann manifold $(\mathcal{M}, g)$?

Renormalized perturbation theory does exist on $(\mathcal{M}, g)$

e.g. See Jack and Osborn "Background Field Calculations on Curved Space Time" NP 1984

IS THERE A SYSTEMATIC APPROACH ?

- **Topology:** The manifold $\mathcal{M}$ is replaced by a simplicial complex $\mathcal{M}_\sigma$ composed of elementary D-simplices, which is homeomorphic to the target manifold.

- **Geometry:** The metric on the target manifold $(\mathcal{M}, g)$ is approximated on the simplicial complex to form $(\mathcal{M}_\sigma, g_\sigma)$ by assigning lengths l_{ij} between neighboring sites and extending the metric with piecewise flat volumes into the interior of each simplex. This is the Regge calculus construction of the metric.

- **Hilbert Space:** The Hilbert space of continuum fields, $\phi(x)$, is truncated by expanding the fields in a finite element basis on each simplex, $\phi_\sigma(x) = E^i(x)\phi_i$, where the summation over i is implied and i runs over the vertices of each simplex.

$$(\mathcal{M}, g) \to (\mathcal{M}_\sigma, g_\sigma)$$

IS THERE A SYSTEMATIC APPROACH ?

- **Topology – Simplicial Complex:**
  Replace D-dimensional manifold $\mathcal{M}$ by a simplicial lattice $\mathcal{M}_\sigma$ composed of elementary D-simplicies

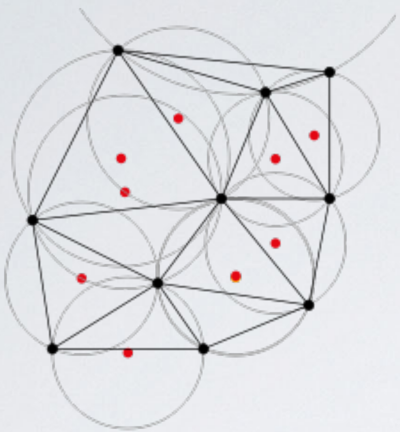
- **Geometry – Regge Calculus:**
  Riemann manifold with metric represented by edge lengths l_{ij} on lattice: $(\mathcal{M}, g) \implies (\mathcal{M}_\sigma, g_\sigma)$

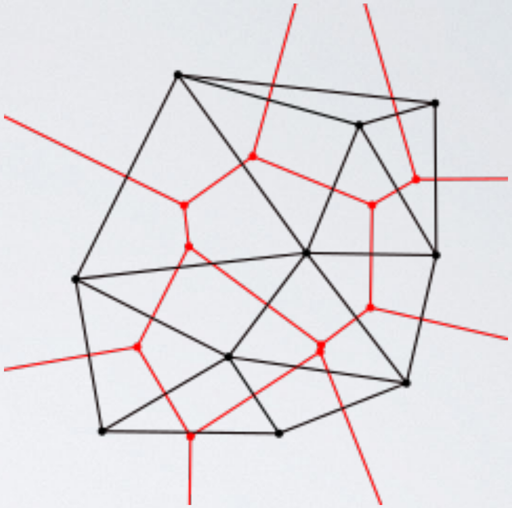
- **Truncated Hilbert Space – Finite Element Basis/DEC:**
  Define action replacing continuum fields, $\phi(x)$ expanding local basis on each cell $\phi_\sigma(x) = E^i(x)\phi_i$

- **Quantization – LFT Path Integral:**
  Monte Carlo Simulation of Path Integral with Quantum Finite Element (QFE) counter terms to control UV divergence.

$$(\mathcal{M}, g) \to (\mathcal{M}_\sigma, g_\sigma)$$

SIMPLICIAL TOPOLOGY





Set of points, lines, triangles, etc.

$$\sigma_0 \rightarrow \sigma_1 \rightarrow \sigma_2 \rightarrow \cdots \rightarrow \sigma_D$$

Boundary Matrix :

$$\partial \sigma_n(i_0, i_1, \cdots, i_n) = \sum_{k=0}^{n} (-1)^k \sigma_{n-1}(i_0, i_1, \cdots, \widehat{i}_k, \cdots, i_n) \; .$$

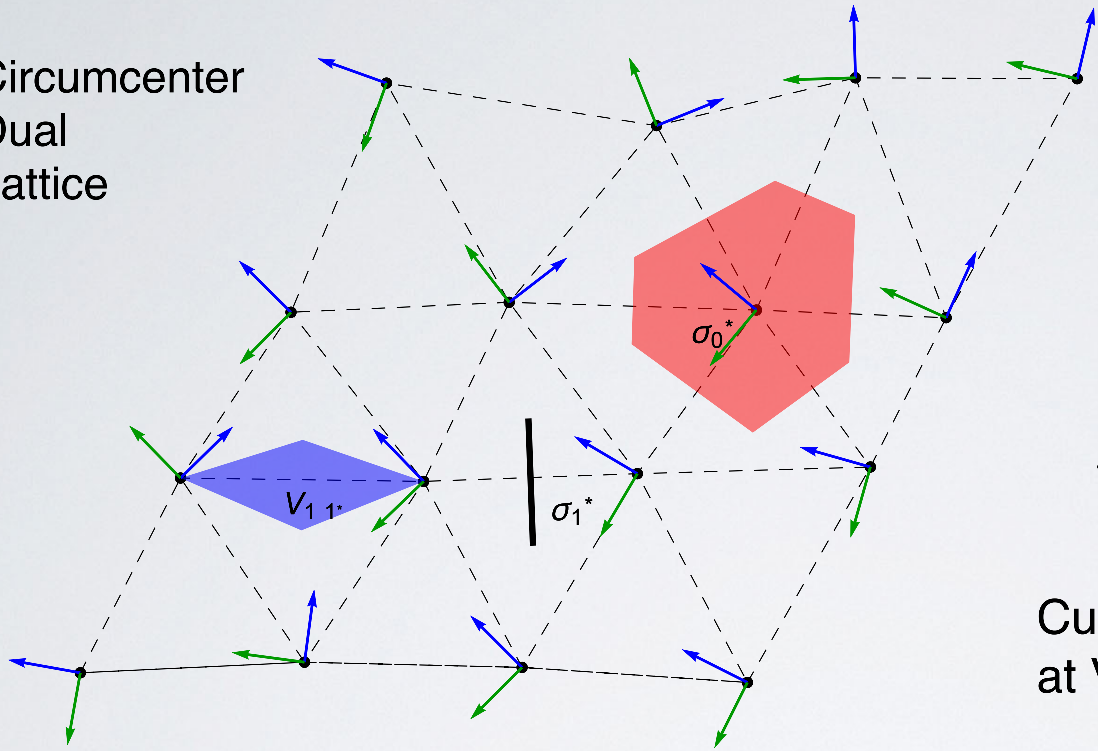
Co- Boundary Matrix : ∂^T

Dual Simplex: $\sigma_0^* \leftarrow \sigma_1^* \leftarrow \cdots \leftarrow \sigma_D^*$

Beautiful Discrete Topology with chains, duality, homology sequences etc. NO metric necessary

REGGE CALCULUS MANIFOLD

Circumcenter
Dual
Lattice



$$V_{nn'} = \int \sigma_n \wedge \sigma^*_{n'}$$
$$= \frac{n!(D-n)!}{D!} |\sigma_n||\sigma^*_{n'}|$$

$$g_{\mu\nu}(x) \implies g_\sigma(y)$$

Curvature is concentrated
at Vertex in 2D or Hinge D > 2

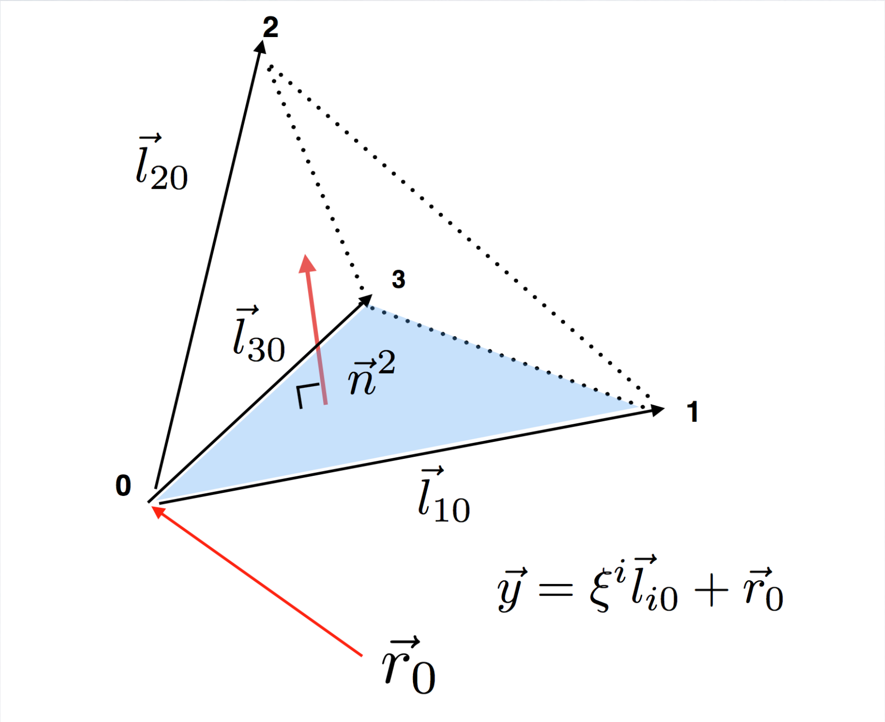
This is just piece-wise
linear FEM of metric
field to express
Einstein Action

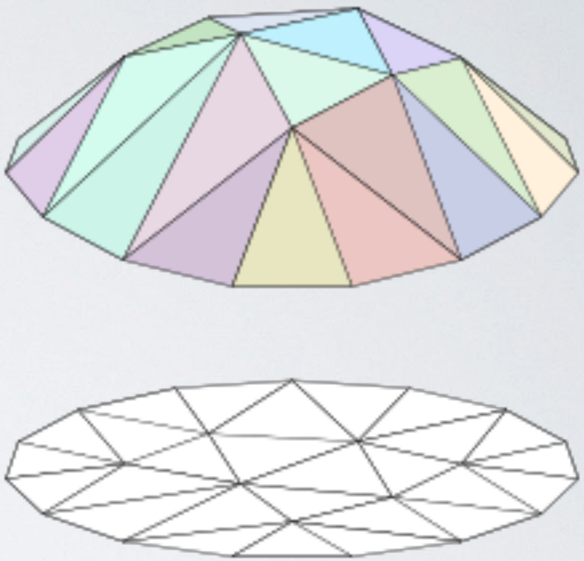
Discrete Metric Data $\{l_{ij} = |\sigma_1(i,j)|\}$

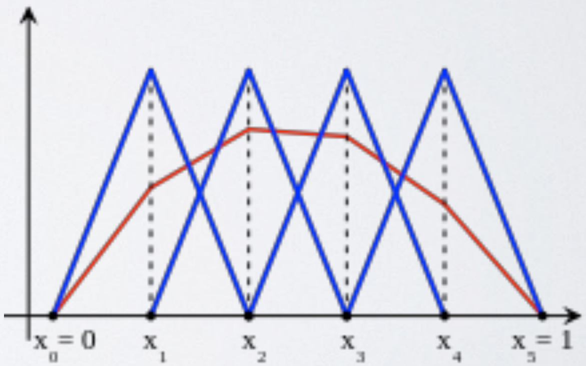
$$\text{dAction} = \text{deficit angle} \times |\sigma^*_{D-2}|$$

Hilbert Space: Finite Element Interpolates

$$\phi(x) \to \phi_\sigma(x) = \sum_i W^i(x)\phi_i$$

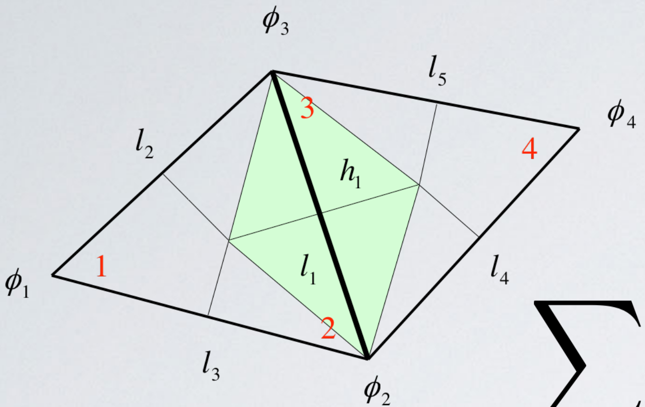






RCB, M. Cheng and G.T. Fleming,
"Improved Lattice Radial Quantization" PoS LATTICE2013 (2013) 335

REGGE CALCULUS FORMULATION



LINEAR FEM/ REGGE CALCULUS *

$$\sum_{\triangle_{kij}} \sqrt{g(k)} g^{ij}(k) \frac{(\phi_k - \phi_i)(\phi_k - \phi_j)}{l_{ki} l_{kj}}$$

Only for D = 2

$$\text{Delaunay Link Area:} \quad A_d = h_1 l_1$$

$$FEM : A_d \frac{(\phi_2 - \phi_3)^2}{l_1^2}$$

DISCRETE EXTERIOR CALCULUS

$$\nabla^2 \phi \to (\delta d + d\delta)\phi$$

$$\delta \equiv *d*$$

$$\langle \mathcal{M} | d\omega \rangle = \langle \partial \mathcal{M} | \omega \rangle$$

SIMPLICIAL DIRAC EQUATION

$$S = \frac{1}{2}\int d^D x \sqrt{g}\bar{\psi}[\mathbf{e}^\mu(\partial_\mu - \frac{i}{4}\boldsymbol{\omega}_\mu(x)) + m]\psi(x)$$

$\mathbf{e}^\mu(x) \equiv e^\mu_a(x)\gamma^a$ Vierbein & Spin connection*

$\boldsymbol{\omega}_\mu(x) \equiv \omega^{ab}_\mu(x)\sigma_{ab}$, $\sigma_{ab} = i[\gamma_a, \gamma_a]/2$

Tetrad Postulate $\partial_\mu \mathbf{e}^\nu + \Gamma^\nu_{\mu,\lambda}\mathbf{e}^\lambda = i[\boldsymbol{\omega}_\mu, \mathbf{e}^\nu]$.

(1) New spin structure "knows" about intrinsic geometry

(2) Need to avoid simplex curvature singularities at sites.

(3) Spinors rotations: Spin(D) is double of Lorentz O(D).

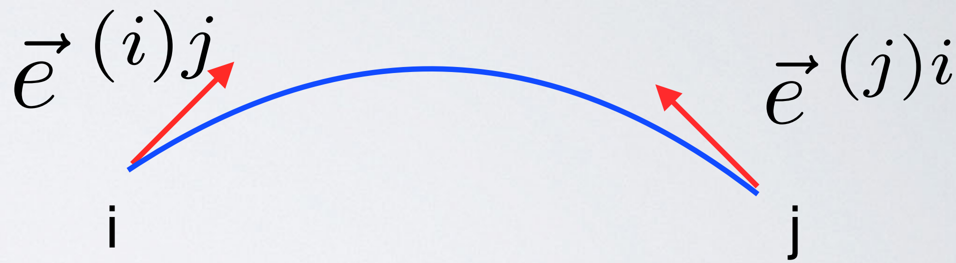
e.g. D = 2 as $\theta \to 2\pi$ $e^{i(\theta/2)\sigma_3/2} \to -1$

Continuum Acton

$$S = \int d^D x \sqrt{g}\, \bar{\psi}[\mathbf{e}^{\mu}(\partial_{\mu} - i\boldsymbol{\omega}_{\mu}(x)) + m]\psi(x) \, ,$$

Tetrad Hypothesis

$$e_a^{(i)j}\gamma^a\Omega_{ij} + \Omega_{ij}e_a^{(j)i}\gamma^a = 0$$



Simplicial Lattice Action

$$S_{naive} = \frac{1}{2}\sum_{\langle i,j \rangle} \frac{V_{ij}}{l_{ij}}[\bar{\psi}_i \vec{e}^{\,(i)j} \cdot \vec{\gamma}\Omega_{ij}\psi_j - \bar{\psi}_j\Omega_{ji}\vec{e}^{\,(i)j} \cdot \vec{\gamma}\psi_i] + \frac{1}{2}mV_i\bar{\psi}_i\psi_i$$

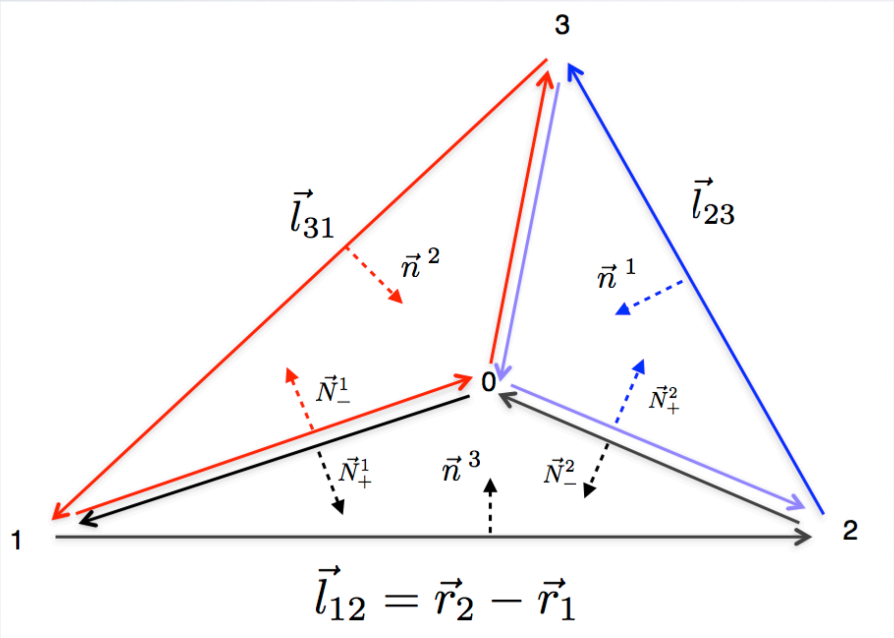
$$\psi_i \to \Lambda_i\psi \quad , \quad \bar{\psi}_j \to \bar{\psi}_j\Lambda_j^{\dagger} \quad , \quad \mathbf{e}^{(i)j} \to \Lambda_i\mathbf{e}^{(i)j}\Lambda_i^{\dagger} \quad , \quad \Omega_{ij} \to \Lambda_i\Omega_{ij}\Lambda_j^{\dagger}$$

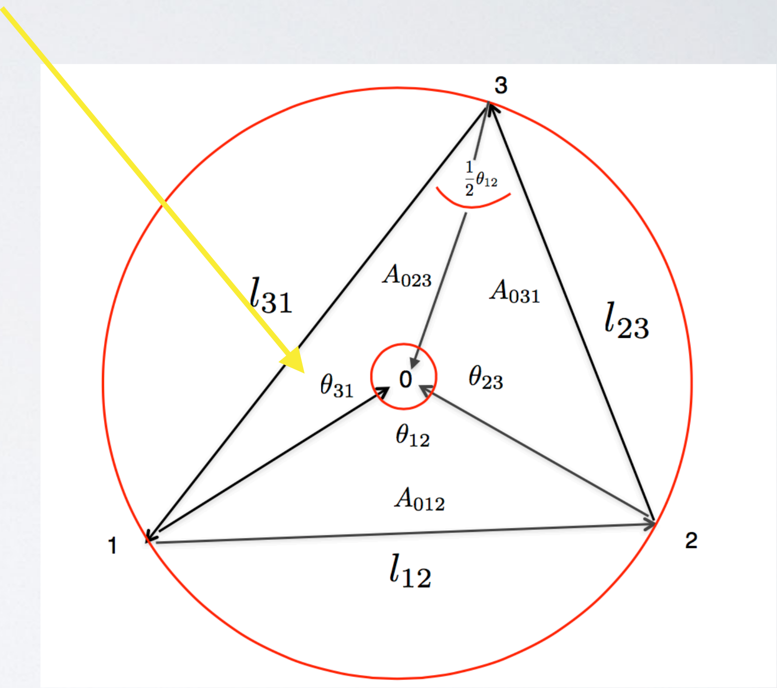
COMMENT: NOT USING LINEAR FEM

$$S_{linear} = \frac{A_{123}}{6} \sum_{\langle i,,j \rangle} \bar{\psi}_i (\vec{n}^{\,j} - \vec{n}^{\,i}) \cdot \vec{\sigma} \psi_j$$

New Dirac Element is 3 linear elements meeting ghost sites at Circumcenter

$$\phi_0 = c_1 \phi_1 + c_2 \phi_2 + c_3 \phi_3$$





$$c_k = \frac{4A_{0ij}}{l_{ij}^2} \frac{4A_{0ik}}{l_{ik}^2} = \cot(\theta_{ik}/2) \cot(\theta_{jk}/2)$$

Sort of ? $\sqrt{\delta d + d\delta} = \delta + d$

WILSON/CLOVER TERM

$$[\gamma_\mu(\partial_\mu - iA_\mu)]^2 = (\partial_\mu - iA_\mu)^2 - \frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} \ ,$$

$$\boldsymbol{D}_\mu = \partial_\mu - i\boldsymbol{\omega}_\mu$$

$$[\mathbf{e}_a^\mu(\partial_\mu - i\boldsymbol{\omega}_\mu)]^2 = \frac{1}{\sqrt{g}}\boldsymbol{D}_\mu\sqrt{g}g^{\mu\nu}\boldsymbol{D}_\nu - \frac{1}{2}\sigma^{ab}e_a^\mu e_b^\nu \boldsymbol{R}_{\mu\nu}$$

$$S_{Wilson} = \frac{r}{2}\sum_{\langle i,j\rangle}\frac{aV_{ij}}{l_{ij}^2}(\bar{\psi}_i - \bar{\psi}_j\Omega_{ji})(\psi_i - \Omega_{ij}\psi_j)$$

Construction Procedure for Discrete Spin connection

(1) Assume Elements with Spherical Triangles (i,j,k) or boundaries give by geodesics on an 2D manifold

(Angles at each vertex add to 2 pi exactly)

(2) Calculate discrete "curl" around the triangle

$$\Omega_{ij}\Omega_{jk}\Omega_{ki} = e^{i(2\pi - \delta_\triangle)\sigma_3/2}$$

(3) Fix $\Omega_{ij} \to \pm\Omega_{ij}$ so $\delta_\triangle \sim A_{ijk}/4\pi R$

Sphere: or any manifold with this topology has a unique lattice spin connection up to gauge Lorentz transformation on spinors

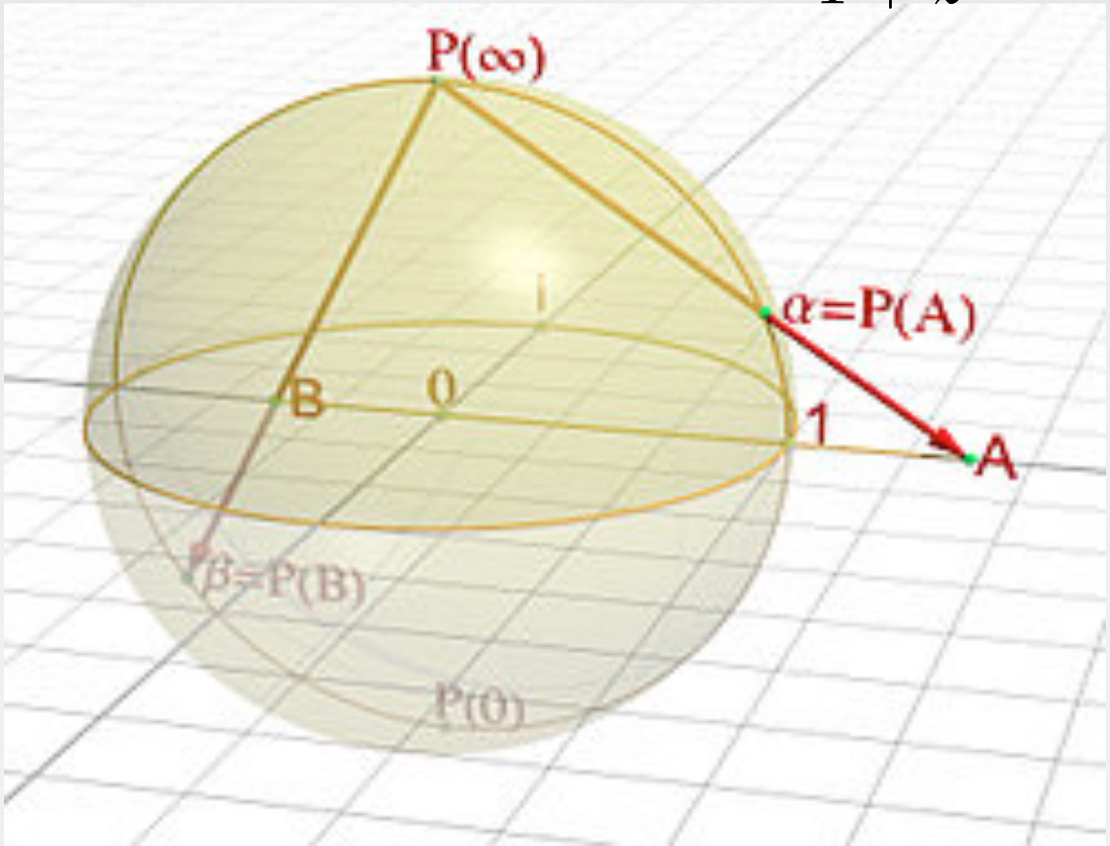
Cylindar: There are 2 solutions (periodic or anti periodic)

Torus: There are 4 solutions: (periodic/anti-periodic): Non-contractible loops.

Category Theory: A spin structure is a property shared between any simplicial complex and

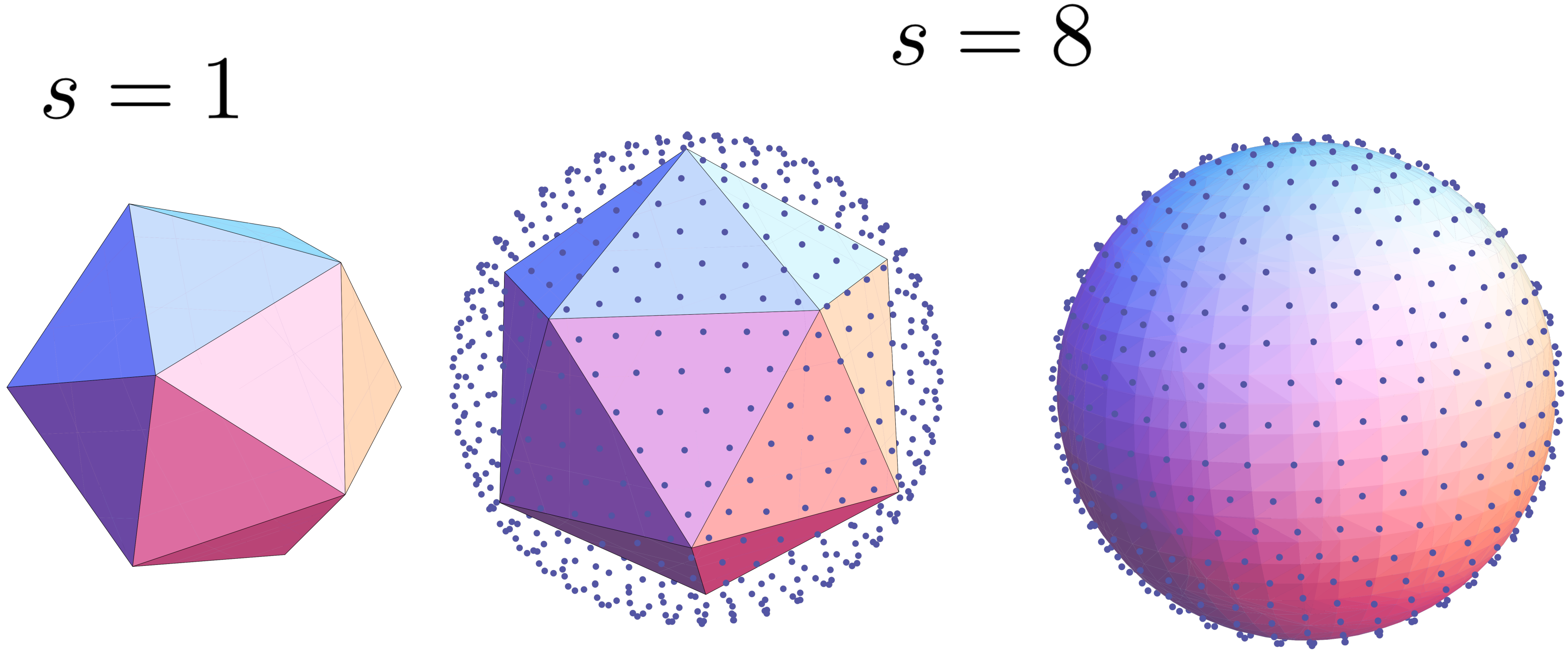
TEST ON THE RIEMANN SPHERE

projection $$\xi = \tan(\theta/2)e^{-i\phi} = \frac{x+iy}{1+z}$$



Conformal Projection + Weyl Rescaling to the Sphere

2D SphereTest Case



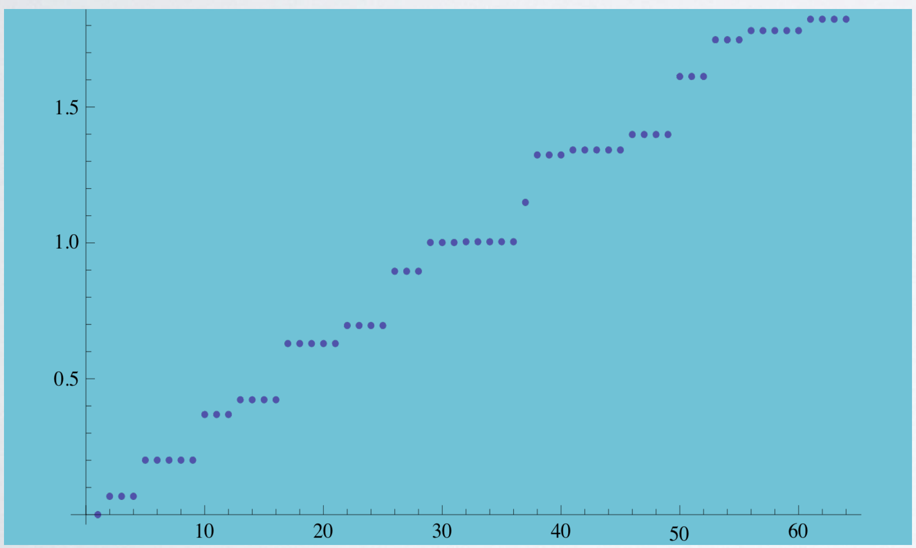
l = 0 (A),1 (T1) , 2 (H) are irreducible 120
Iscosahedral subgroup of O(3)

THE LAPLACIAN ON THE SPHERE

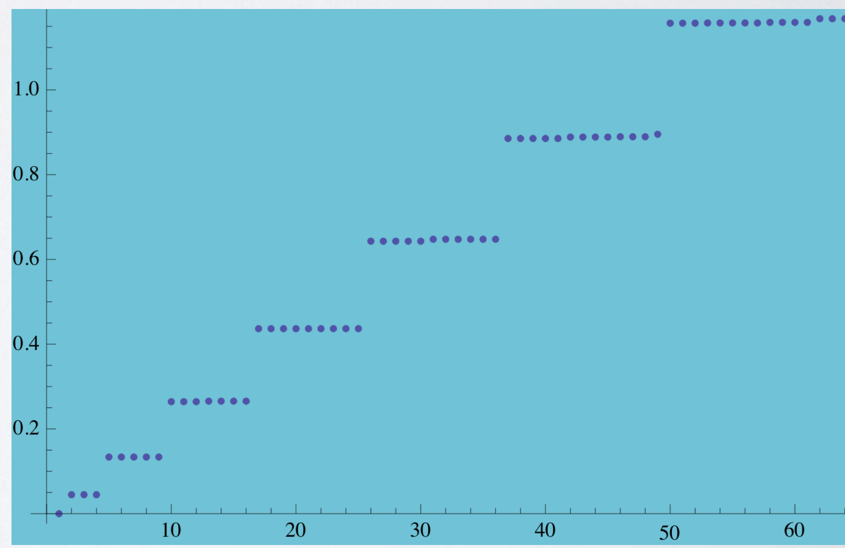
For s = 8 first (l+1)*(l +1) = 64 eigenvalues

BEFORE (K = 1)

AFTER (FEM K's)

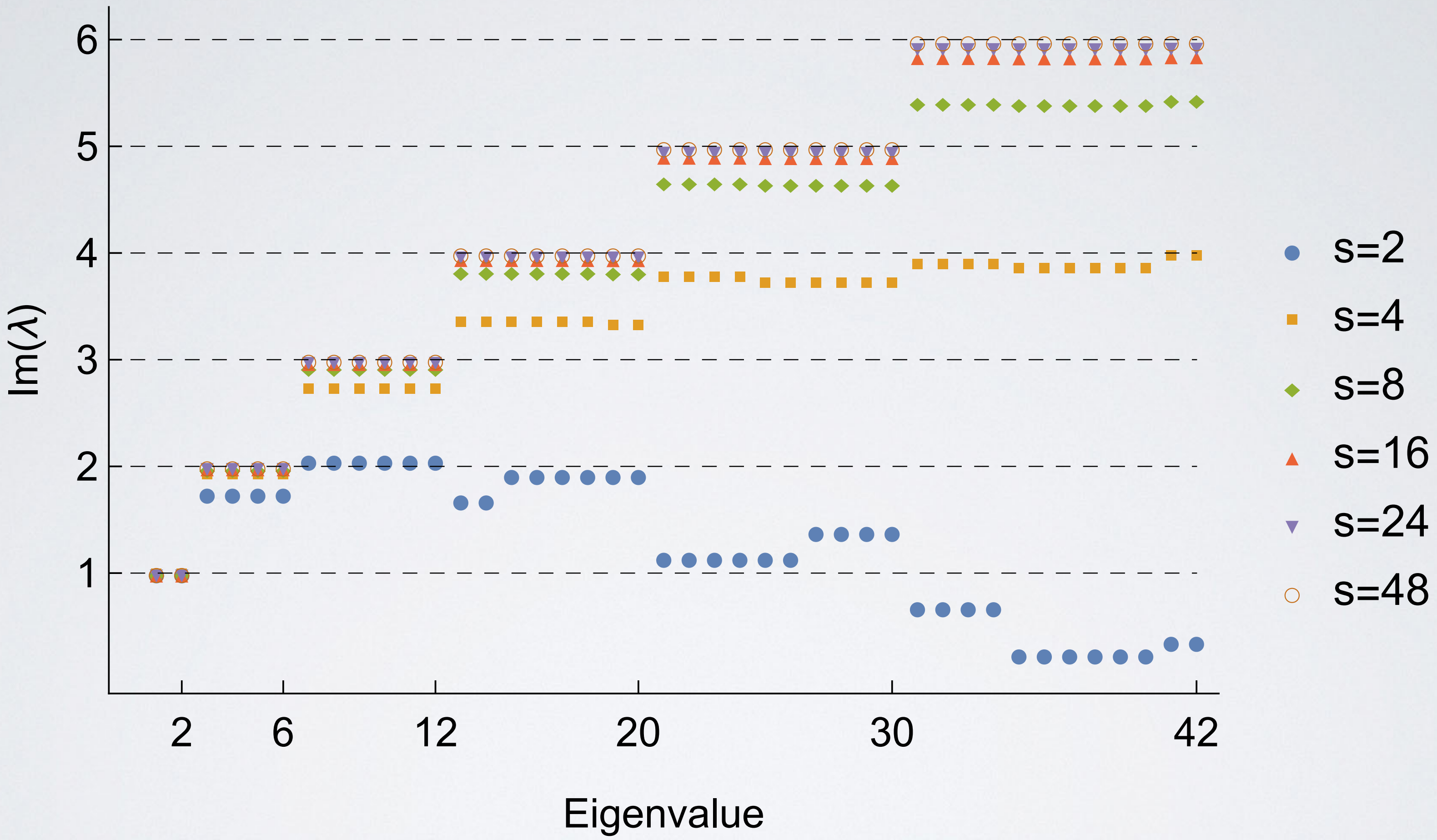


l, m



l, m

DIRAC DISPERSION RELATION



$$m + i\lambda = m \pm i\,(j + 1/2)$$

No Zero Mode in Chiral limit

$$j = 1/2, 3/2, 5/2, \cdots \quad \text{Digeneracy} \quad 2j + 1$$

ISING: FREE MAJORANA FERMIONS

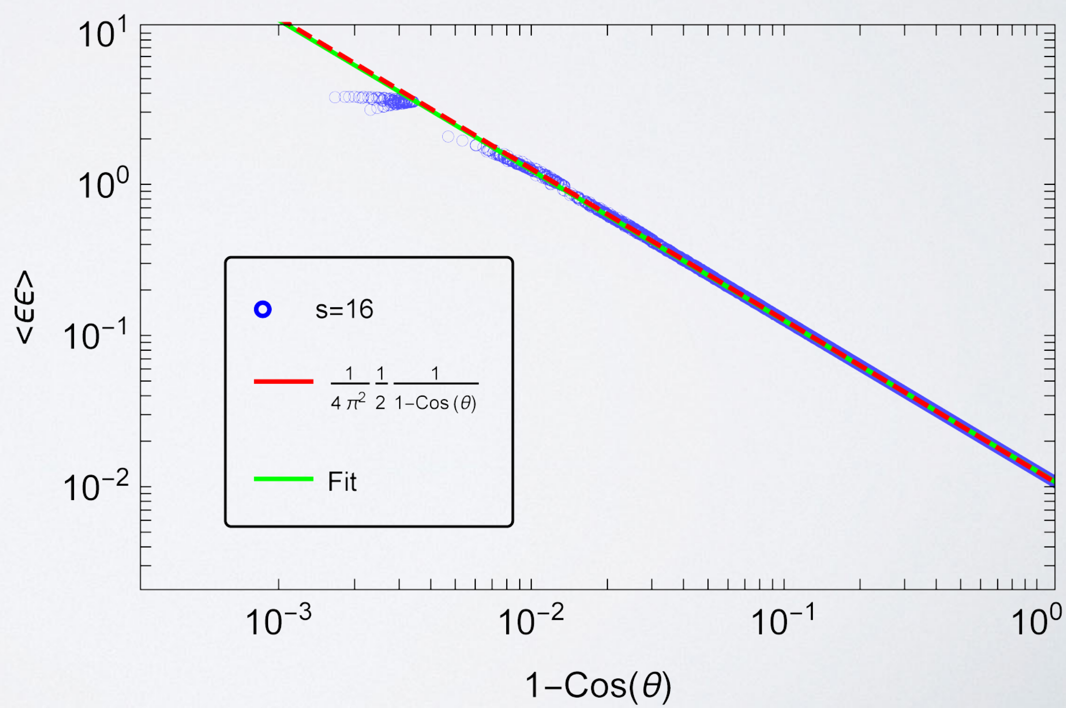
c =1/2 Minimal Model OPE: $\sigma \times \sigma = \mathbf{1} + \epsilon \quad , \quad \epsilon \times \sigma = \epsilon \quad , \quad \epsilon \times \epsilon = \mathbf{1}$

Even Ope $\epsilon(z) = i\bar{\psi}(z)\psi(z)$ Odd operator is twist $\sigma(z)$

$$S_{Dirac} = \int d^2x [\psi \partial_{\bar{z}} \psi + \bar{\psi} \partial_z \bar{\psi}]$$

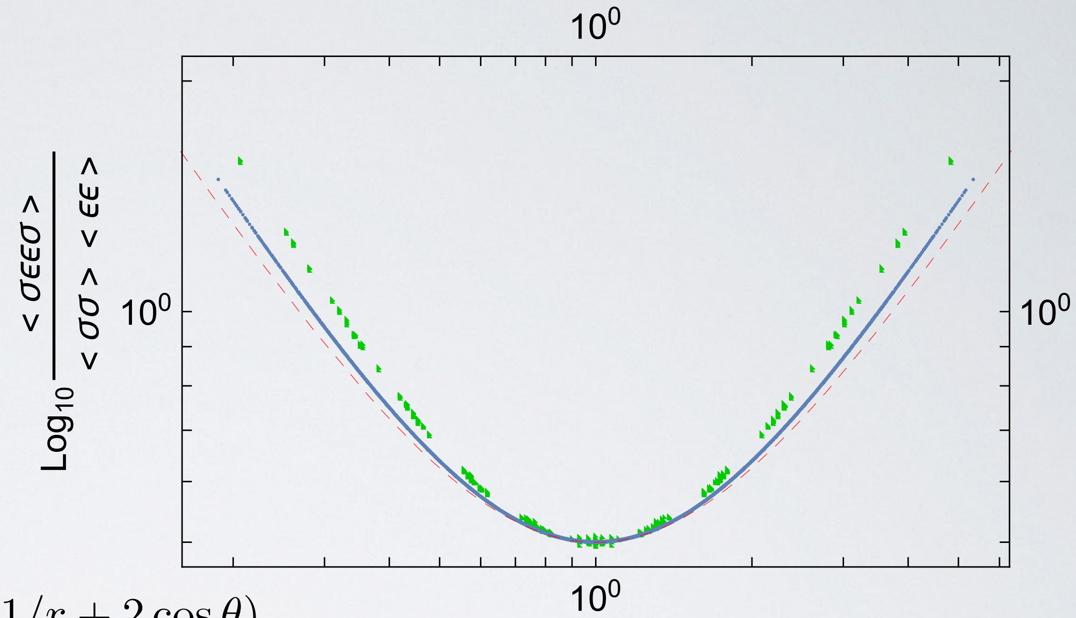
$$\langle \psi(z_1)\bar{\psi}(z_1)\bar{\psi}(z_1)\psi(z_2) \rangle = \left[\frac{1}{\partial}\right]_{z_1,z_2} \left[\frac{1}{\bar{\partial}}\right]_{z_1,z_2} = \frac{1}{4\pi^2} \frac{1}{|z_1 - z_2|^2}$$

$$\langle \epsilon(z_1)\epsilon(z_2) \rangle$$



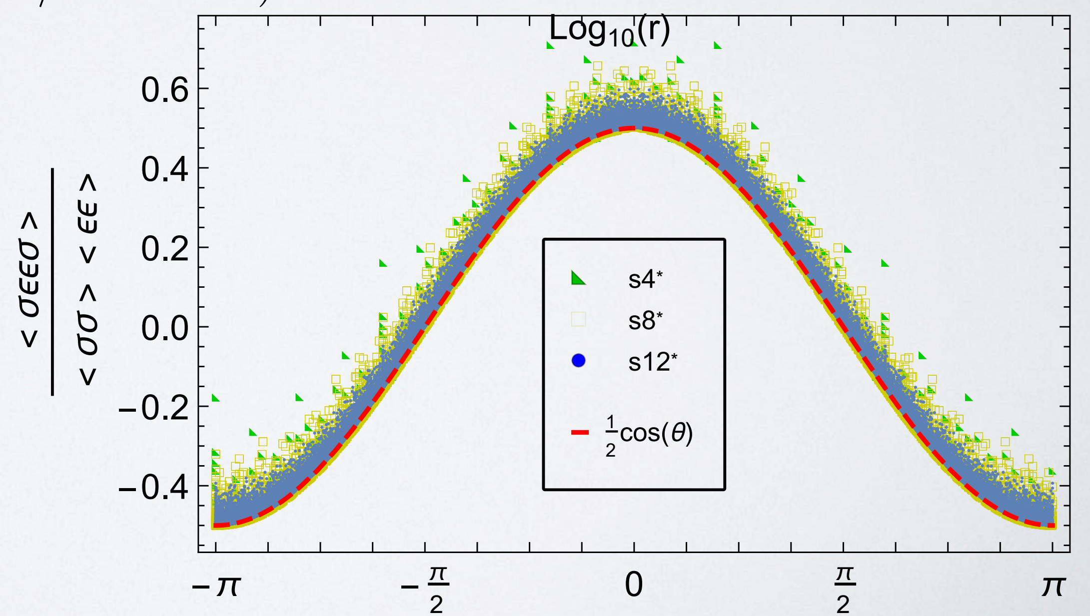
ADD TWISTS AT N/S POLES

Subtract Cos term



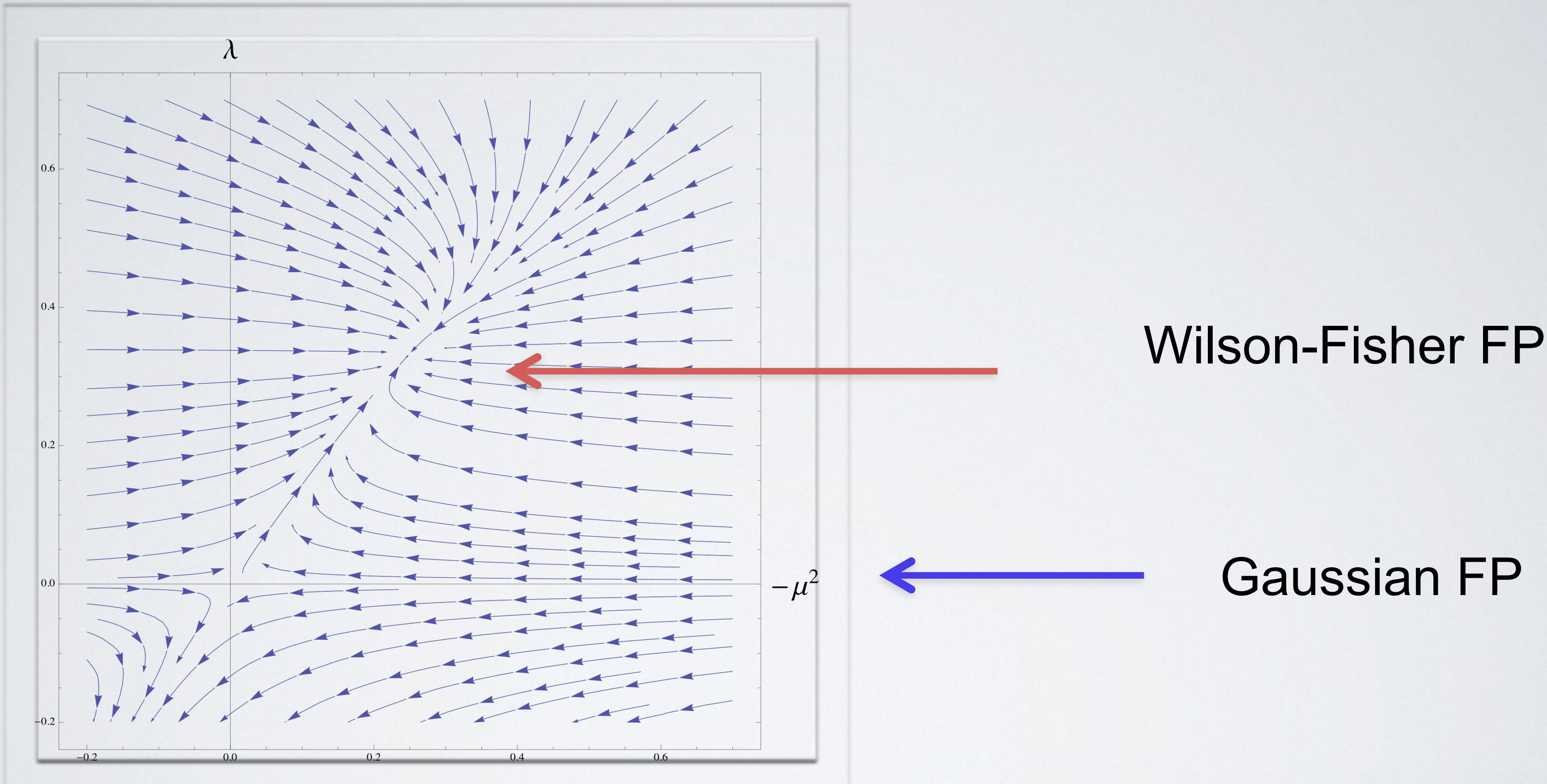
$$\frac{\langle\sigma(0)\epsilon(z_2)\epsilon(z_3)\sigma(\infty)\rangle}{\langle\epsilon(z_2)\epsilon(z_3)\rangle}$$

$$=\frac{1}{4}|\sqrt{z_1/z_2}+\sqrt{z_2/z_1}|^2=\frac{1}{4}(r+1/r+2\cos\theta)$$



Substract r term

Replace Ising Model by phi 4th



$$\mathcal{L}(x) = -\frac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - \mu^2/2\lambda)^2$$

NOW BINDER CUMULANT CONVERGES

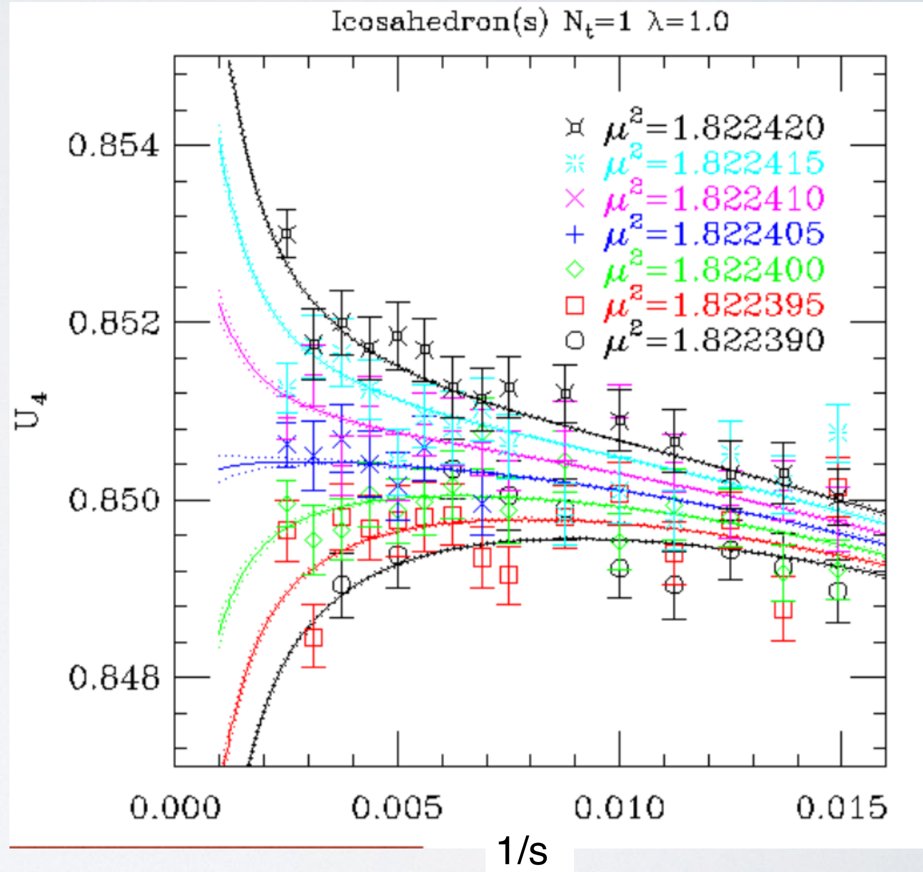
$$U_{2n}(\mu^2, \lambda, s) = U_{2n,cr} + a_{2n}(\lambda)[\mu^2 - \mu^2_{cr}]s^{1/\nu} + b_{2n}(\lambda)s^{-\omega}$$

FIT: $U_4 = 0.85081(10)$

EXACT: $U_4^{exact} = 0.851021(5)$

HIGHER MOMENT 2n = 4,6,8,10,12

$U_6 = 0.77280(13)$



$$U_4 = \frac{3}{2}[1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle \langle M^2 \rangle}]$$

$$\mu^2_{cr} = 1.82240070(34)$$

Simultaneous fit for s up 800: E.G. 6,400,002 Sites on Sphere

$$dof = 1701 \quad , \quad \chi^2/dof = 1.026$$

EXACT SOLUTION TO CFT

Exact Two point function

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \rightarrow \frac{1}{|1 - cos\theta_{12}|^{\Delta}}$$

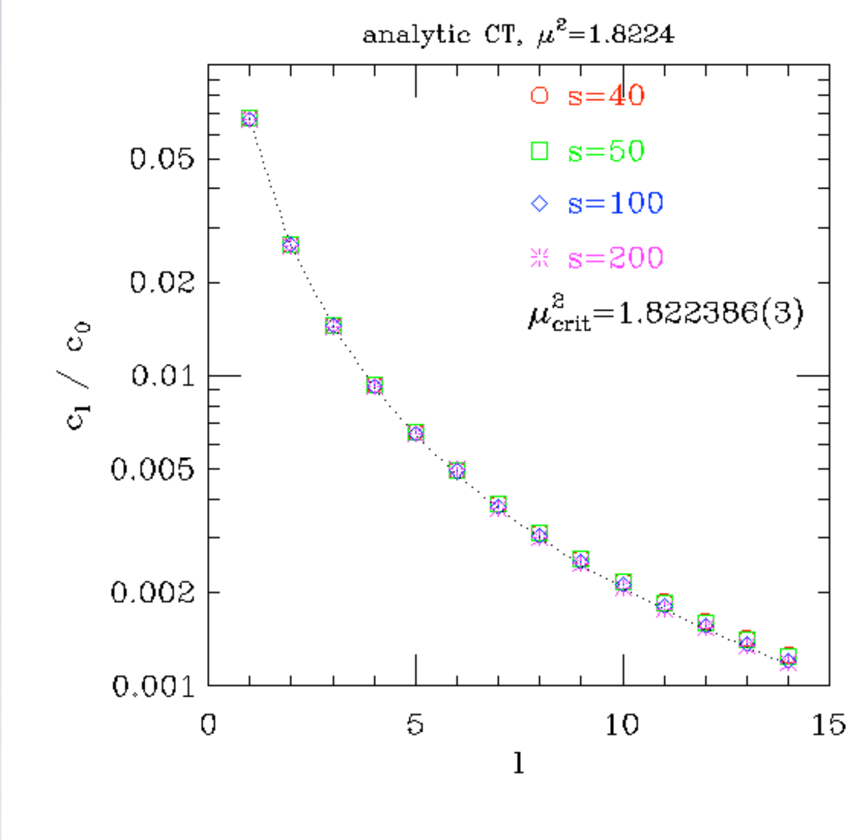
$$\Delta = \eta/2 = 1/8 \qquad x^2 + y^2 + z^2 = 1$$

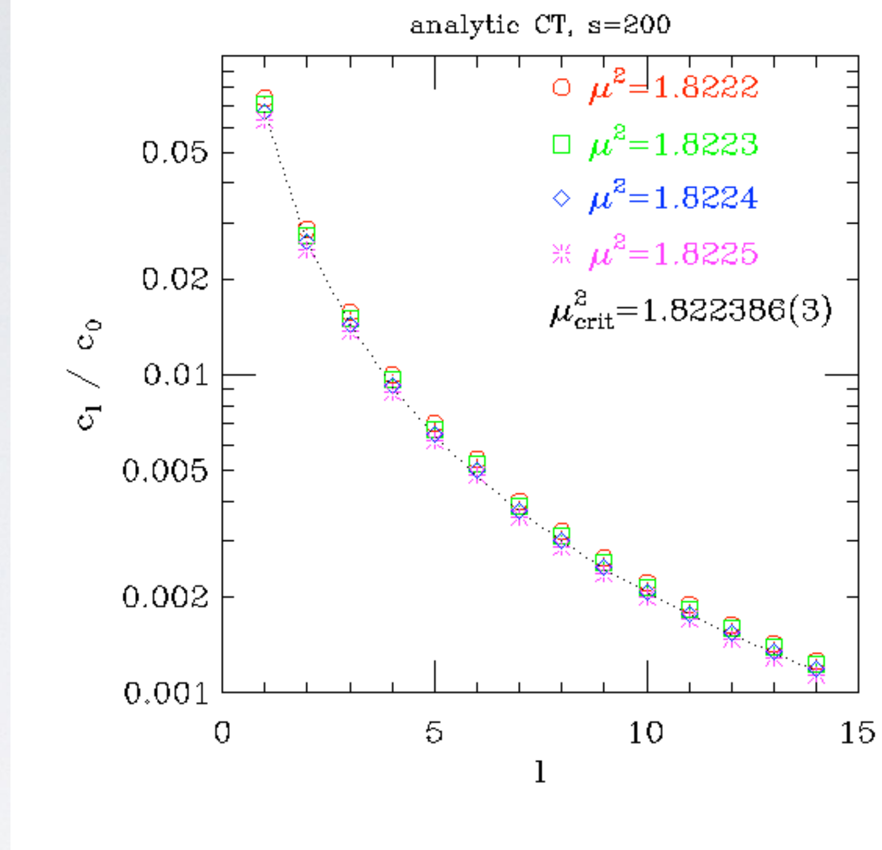
4 pt function $(x_1, x_2, x_3, x_4) = (0, z, 1, \infty)$

$$g(0, z, 1, \infty) = \frac{1}{2|z|^{1/4}|1 - z|^{1/4}}[|1 + \sqrt{1 - z}| + |1 - \sqrt{1 - z}|]$$

Critical Binder Cumulant $U_B^* = 1 - \frac{\langle M^4\rangle}{3\langle M^2\rangle\langle M^2\rangle} = 0.567336$

Dual to Free Fermion $u = \frac{r_{12}^2 r_{34}^2}{r_{13}^2 x_{24}^2}$, $v = \frac{r_{14}^2 r_{32}^2}{r_{13}^2 r_{24}^2}$ where $r_{ij}^2 = (\vec{r}_i - \vec{r}_j)^2 = 2(1 - \cos\theta_{ij})$





$$\int_{-1}^{1} dz \left(\frac{2}{1-z}\right)^{1/8} P_l(z)$$

$$\Delta_\sigma = \eta/2 = 1/8 \simeq 0.128$$

$$\Longrightarrow \frac{16}{7}, \frac{16}{35}, \frac{48}{161}, \frac{816}{3565}, \frac{12240}{64883}, \frac{493680}{3049501}, \frac{33456}{234577}, \frac{55760}{435643}, \frac{3602096}{30930653}, \frac{20129360}{187963199}, \frac{541373840}{5450932771} \cdots$$

Very fast cluster algorithm:
Brower,Tamayo 'Embedded Dynamics for phi 4th Theory" PRL 1989. Wolff single cluster + plus Improved Estimators etc

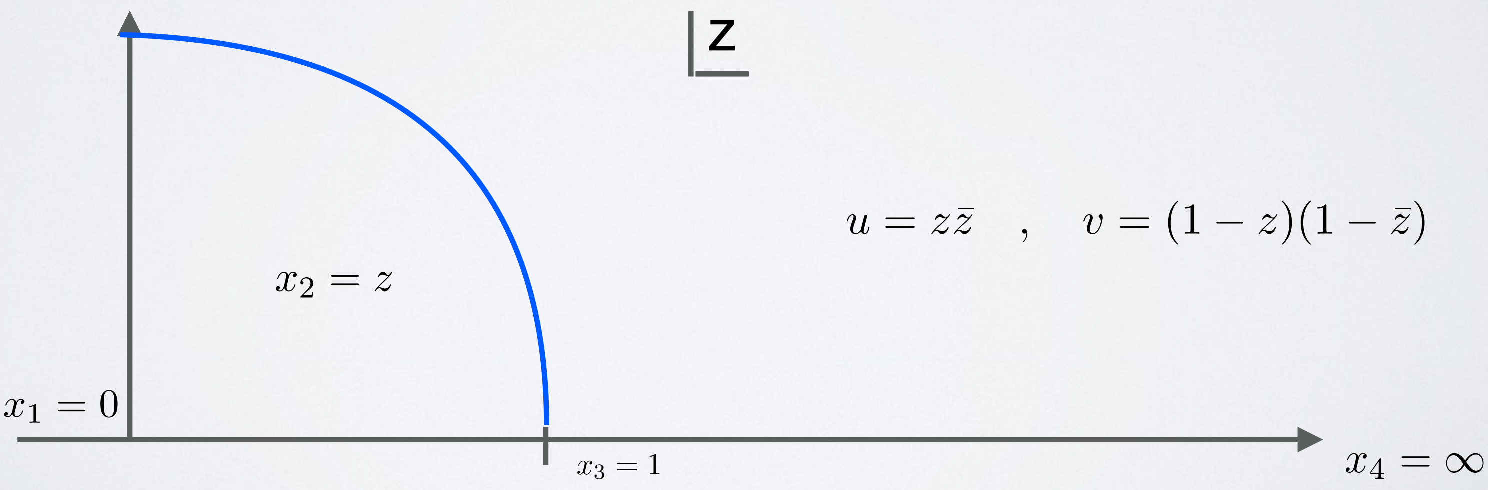
EXACT FOUR POINT FUNCTION

OPE Expansion: $\phi \times \phi = \mathbf{1} + \phi^2$ or $\sigma \times \sigma = \mathbf{1} + \epsilon$

$$g(u,v) = \frac{\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle}{\langle \phi_1 \phi_3 \rangle \, \langle \phi_2 \phi_4 \rangle}$$

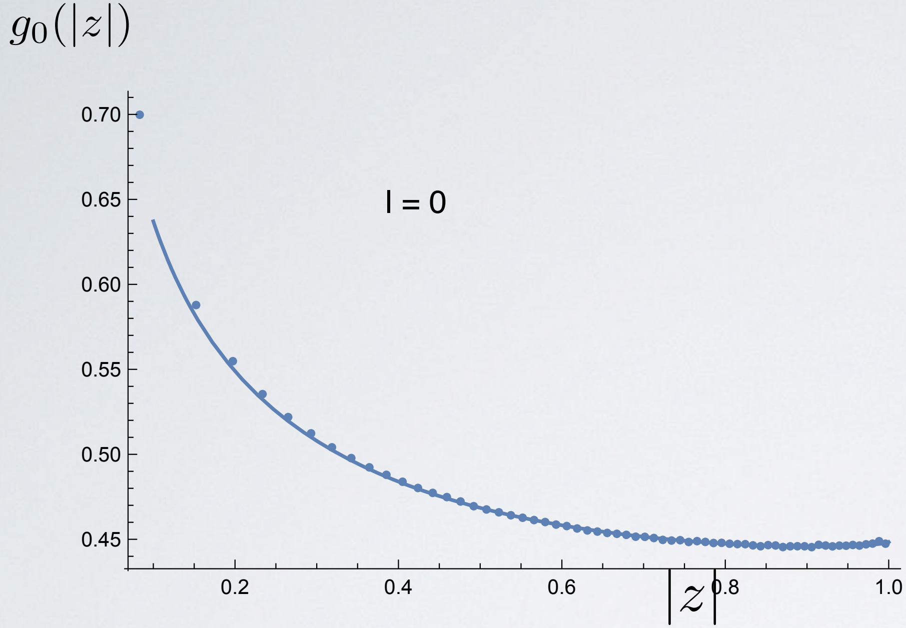
$$= \frac{1}{\sqrt{2}|z|^{1/4}|1-z|^{1/4}} \left[|1 + \sqrt{1-z}| + |1 - \sqrt{1-z}| \right]$$

Crossing Sym: $|1 + \sqrt{1-z}| + |1 - \sqrt{1-z}| = \sqrt{2 + 2\sqrt{(1-z)(1-\bar{z})} + 2\sqrt{z\bar{z}}}$



$$u = z\bar{z} \quad , \quad v = (1-z)(1-\bar{z})$$

2 TO 2 SCATTERING DATA

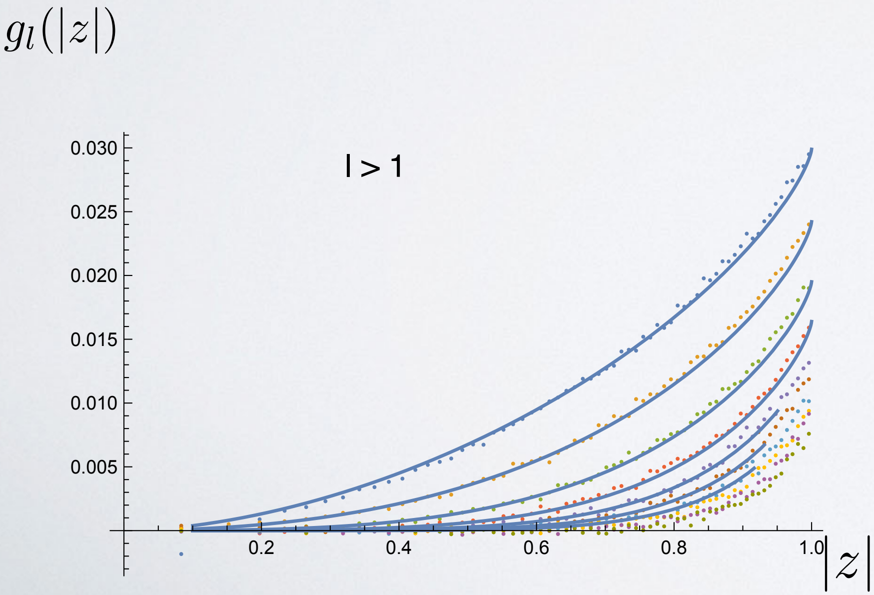


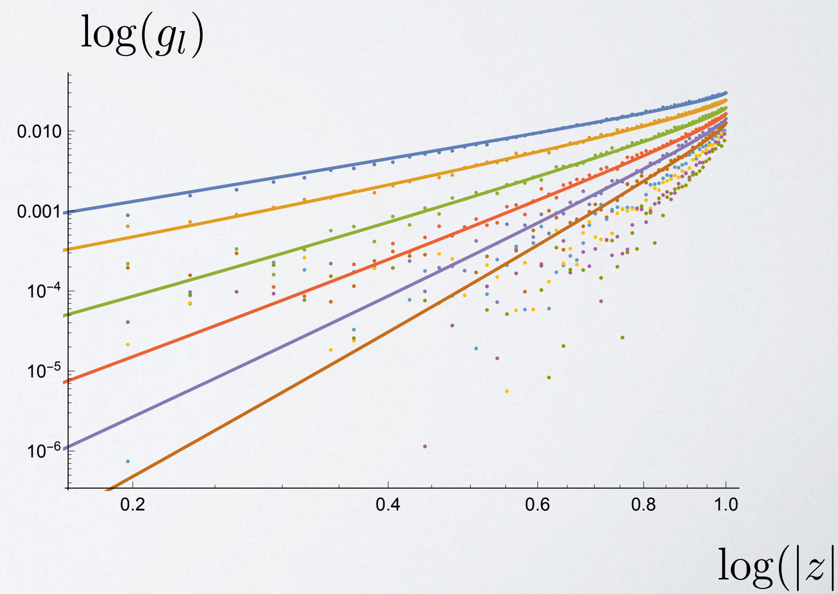
ZERO PARAMETER FIT

s= 10 Run for 1/2 hour

$$g(u,v) = \sum_l g_l(|z|)\cos(l\theta)$$

$$z = |z|e^{il\theta}$$





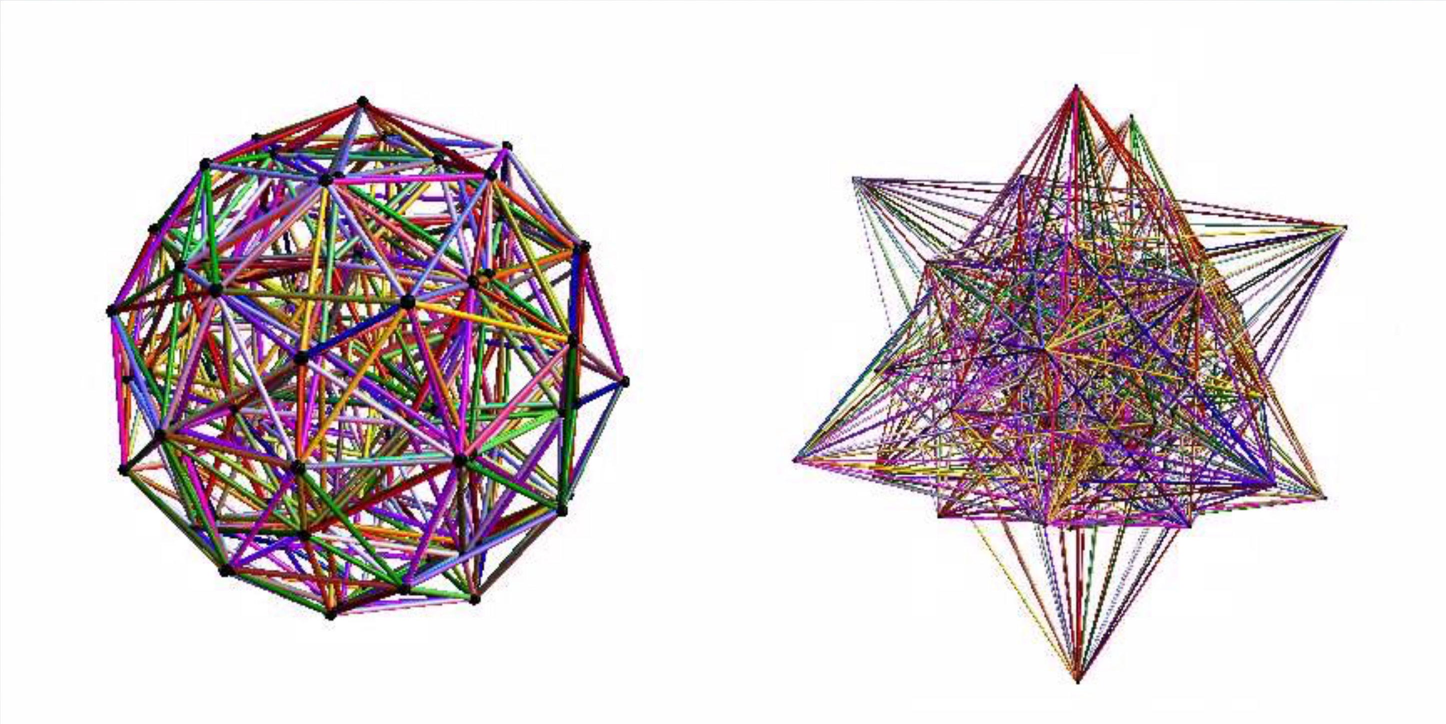
CONCLUSIONS: QFE PLANS & DREAMS

- COMPUTATION:
  - 2+1 Radial Phi 4th/3D Ising CFT (with cluster algorithm)
  - Extend Peter Boyle's GRID to HMC on Simplicial Spheres (Interesting 3D Problem for Dirac/Scalar Theories.)
  - 3 Sphere starting with 600 cell: 4 Sphere ?
- THEORY:
  - Prove QFE for all super renormalizable theories
  - Classify all CT that break diffeomorphism invariance.
  - Renormalization of 4d non-Abelian FT
  - Clarity DEC for Quantum FT

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

600 CELL ON S3

HTTPS://EN.WIKIPEDIA.ORG/WIKI/600-CELL



16 vertices of the form:[3] (±½, ±½, ±½, ±½),

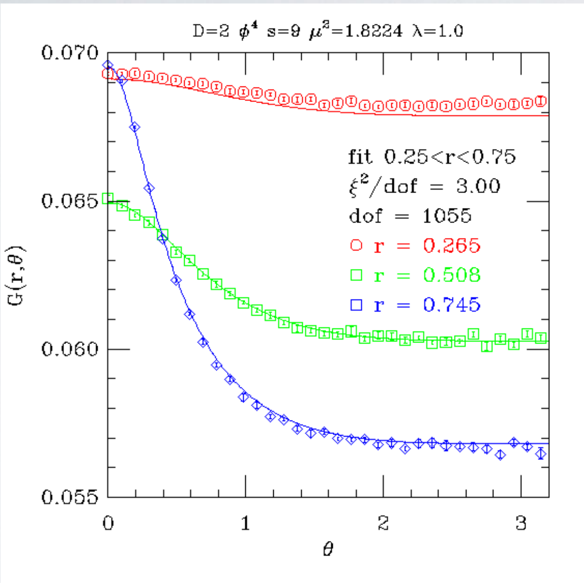
8 vertices obtained from (0, 0, 0, ±1) by permuting coordinates.

96 vertices are obtained by taking even permutations of ½ (±ϕ, ±1, ±1/ϕ, 0).

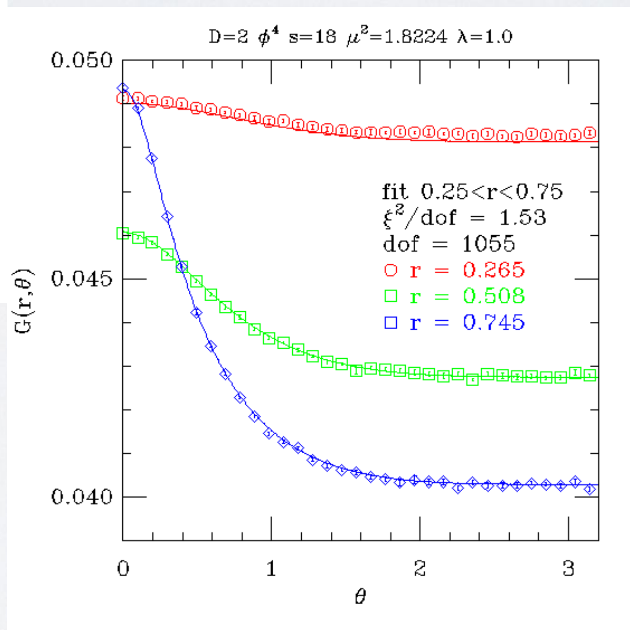
https://en.wikipedia.org/wiki/List_of_regular_polytopes_and_compounds#Five-dimensional_regular_polytopes_and_higher

4PT CONVERGENCE

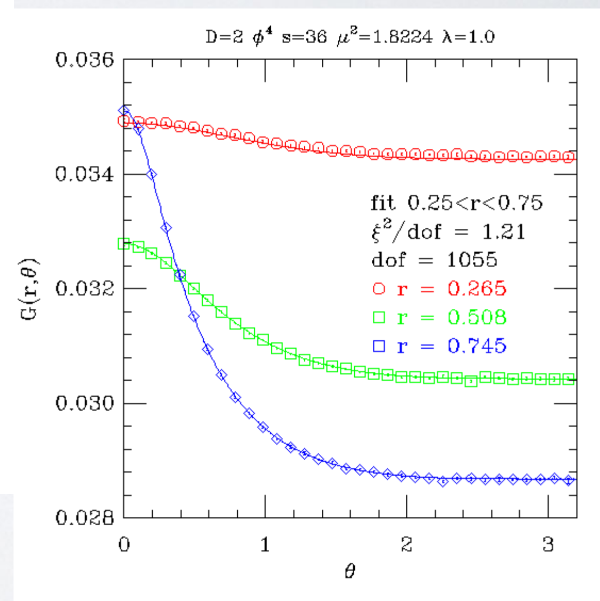
$s = 8 \quad \xi^2/dof = 3.00$



$s = 18 \quad \xi^2/dof = 1.53$



$s = 36 \quad \xi^2/dof = 1.21$



WARNINGS

- Regge Calculus and FEM use piece wise linear interpolations.
    - g(x) —> Curvature are singular delta functions
    - phi(x) —> Laplacian has singular delta functions
    - Solutions: Discrete Exterior Calculus and "smoothed RC"
- Fermions see Manifold Geometry
- UV divergences are not uniform: Need Quantum Finite

K-SIMPLICIES