Computing the static potential using non-string-like trial states
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Motivation
For calculating the static potential with a high resolution we have to work with off axis separated quarks.

e.g. matching the lattice QCD potential with the perturbative potential to determine $\Lambda_{\overline{MS}}$ in Fourier space. [F. Karbstein, A. Peters and M. Wagner, JHEP 1409, 114 (2014) [arXiv:1407.7503 [hep-ph]]]
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- The quantity of interest is the Wilson loop, which connects the two quarks like a string.
- To compute the spatial part of the Wilson loop one has to go over stair-like paths through the lattice.
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- Idea: Substitute the spatial part of the Wilson loop by another object to avoid the calculation of stair-like paths.
The Technical Part
- The spatial Wilson line is needed to ensure gauge invariance of the $q\bar{q}$ trial state.

\[ U'(x, y) = G(x) U(x, y) G^\dagger(y) \]

We explore an idea, which has been used in the context of Polyakov loops and the static potential at finite temperature.

\[ \Delta f = \frac{1}{a^2} \left( U^\dagger_1(x-a, y, z) f(x-a, y, z) - 2f(x) + U_1(x, y, z) f(x+a, y, z) \right) + \cdots \]
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- Consider the covariant lattice Laplace operator:

  $$\Delta f = \frac{1}{a^2} \left( U_1^+(x - a, y, z)f(x - a, y, z) - 2f(x) + U_1(x, y, z)f(x + a, y, z) \right)$$

  $$+ \frac{1}{a^2} \left( U_2^+(x, y - a, z)f(x, y - a, z) - 2f(x) + U_2(x, y, z)f(x, y + a, z) \right)$$

  $$+ \frac{1}{a^2} \left( U_3^+(x, y, z - a)f(x, y, z - a) - 2f(x) + U_3(x, y, z)f(x, y, z + a) \right)$$
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- Consider the **covariant lattice Laplace operator**:

  \[
  \Delta f = \frac{1}{a^2} \left( U_1^\dagger(x - a, y, z)f(x - a, y, z) - 2f(x) + U_1(x, y, z)f(x + a, y, z) \right) \\
  + \frac{1}{a^2} \left( U_2^\dagger(x, y - a, z)f(x, y - a, z) - 2f(x) + U_2(x, y, z)f(x, y + a, z) \right) \\
  + \frac{1}{a^2} \left( U_3^\dagger(x, y, z - a)f(x, y, z - a) - 2f(x) + U_3(x, y, z)f(x, y, z + a) \right)
  \]

- Transformation behavior: $\Delta' = G(x)\Delta G^\dagger(x)$
- Writing $f(x)$ as a vector in position space $\Delta$ can be written as a matrix.
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- Consider $f(\mathbf{x})$ is now an eigenvector of the covariant Laplace operator.

\[ \Delta f(\mathbf{x}) = \lambda f(\mathbf{x}) \]
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- Consider $f(x)$ is now an \textit{eigenvector} of the \textit{covariant Laplace operator}.

\[
\Delta f(x) = \lambda f(x) \\
\Delta' f'(x) = \lambda f'(x)
\]

- Apply an gauge transformation on the eigenvector-equation.
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G(x) \Delta G^\dagger(x) f'(x) = \lambda f'(x)
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- We see: $G^\dagger(x) f'(x)$ is again eigenvector to the covariant Laplace operator.
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- Now we know: \( f(x) \) and \( G^\dagger(x)f'(x) \) are members of the same eigenspace.

- In SU(3) the eigenvalues are in general nondegenerate. This means:

\[
f(x)e^{i\phi} = G^\dagger(x)f'(x)
\]

- In SU(2) however, the eigenvalues are always two fold degenerate. This means:

\[
\alpha f_1(x) + \beta f_2(x) = G^\dagger(x)f'(x)
\]

- Where \( f_1 \) and \( f_2 \) are an orthonormal basis of the corresponding eigenspace.
Transformation law for SU(3):

\[ f(x)e^{i\phi} = G^\dagger(x)f'(x) \]

Transformation law for SU(2):

\[ \alpha f_1(x) + \beta f_2(x) = G^\dagger(x)f'(x) \]

Wilson Line:

\[ U'(x, y) = G(x)U(x, y)G^\dagger(y) \]

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SU(3) - Case:
\[ f'(x)f'^\dagger(y) = G(x)f(x)f^\dagger(y)G^\dagger(y) \]

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SU(3) - Case:

\[ f'(x)f'^\dagger(y) = G(x)f(x)f^\dagger(y)G^\dagger(y) \]

SU(2) - Case:

\[ \sum_{i=1}^{2} f'_i(x)f'^\dagger_i(y) = G(x)\left( \sum_{i=1}^{2} f_i(x)f^\dagger_i(y) \right)G^\dagger(y) \]

- Now it is easy to create an object with the needed transformation behavior.

- Where \( f_1 \) and \( f_2 \) are an orthonormal basis of the corresponding eigenspace.
- We found objects with the required transformation behavior given by
  \[ f(x)f^\dagger(y) \text{ for SU}(3) \text{ and } \sum_{i=1}^{2} f_i(x)f_i^\dagger(y) \text{ for SU}(2). \]
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- With these new objects it is not necessary to distinguish a certain path between \( x \) and \( y \).
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Advantages:
The computation of stair-like paths is not longer needed.
- We found objects with the required transformation behavior given by
  \[ f(x)f^\dagger(y) \quad \text{for SU}(3) \quad \text{and} \quad \sum_{i=1}^{2} f_i(x)f_i^\dagger(y) \quad \text{for SU}(2). \]

- With these new objects it is not necessary to distinguish a certain path between \( x \) and \( y \).

**Advantages:**
The computation of stair-like paths is not longer needed.

**Price to pay:**
One has to compute the eigenvectors of \( \Delta \) first.
Figure: Runtime of the eigenvector calculation and the remaining computations using the new method.
Results
Figure: Effective mass in units of the lattice spacing for the ordinary Wilson loop - using 100 basically independent SU(2) gaugelink configurations with $\beta = 2.5 \ (\approx 0.089 fm)$ on a 24x24 Lattice.
Figure: Effective mass in units of the lattice spacing for the new method - using 100 basically independent SU(2) gaugelink configurations with $\beta = 2.5 \ (\approx 0.089 \text{fm})$ on a 24x24 Lattice.
Figure: Comparison of the effective masses from the ordinary Wilson loop and the new method - using 100 basically independent SU(2) gaugelink configurations with $\beta = 2.5$ ($\approx 0.089\text{fm}$) on a 24x24 Lattice.
Figure: Potential for the static $q\bar{q}$ pair in units of the lattice spacing - using the ordinary Wilson loop on 100 basically independent SU(2) gaugelink configurations with $\beta = 2.5$ ($\approx 0.089 fm$) on a 24×24 Lattice
Figure: Potential for the static $q\bar{q}$ pair in units of the lattice spacing - using the new method on 100 basically independent SU(2) gaugelink configurations with $\beta = 2.5$ ($\approx 0.089\text{fm}$) on a 24x24 Lattice
Figure: Off-axis potential for the static $q\bar{q}$ pair in units of the lattice spacing - using the new method on 100 basically independent SU(2) gaugelink configurations with $\beta = 2.5$ ($\approx 0.089\text{fm}$) on a 24x24 Lattice
Figure: On-axis potential for the static $q\bar{q}$ pair in units of the lattice spacing - using the new method on 60 basically independent SU(3) gaugelink configurations with $\beta = 3.9$ on a 48x24 Lattice
Summary

- In the ordinary approach the calculation of the static $q\bar{q}$-potential, for off-axis separations, requires the computation of time consuming stair-like paths.

- These stair-like paths come from the Wilson loop, an object that ensures gauge invariance of the used $q\bar{q}$ trial state.

- By using the eigenvectors of the covariant Laplace operator we were able to substitute the spatial part of the Wilson loop by a new object.

- Advantages: Fast computation times for off axis calculations and nearly similar quality of the results (error bars).

- Possible application: The potential with fine resolution can be used for better modeling and comparison with perturbative theories ($\Lambda_{\overline{MS}}$-determination, $b\bar{b}$-spectrum).