Phase diagram of the O(3) model from dual lattice simulations

Falk Bruckmann
(University of Regensburg)

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Tin Sulejmanpašić [1408.2229]
plus Thomas Kloiber, Christof Gattringer [1507.04253, 1607.02457]

The $O(3)$ model ...

in 1+1 dimensions, action:

$$S = \frac{1}{2g_0^2} \int d^2x \, (\partial_\nu n_a)^2 \,, \quad n_a^2 = 1 \,, \quad a = 1, 2, 3$$

many nonperturbative effects like QCD:

- asymptotic freedom
- dynamical mass generation (g_0 dimensionless)
  low energy excitations: particle triplet
- topology, instantons, Θ-term & renormalons
- lattice discretisation = Heisenberg model
- O(3) symmetry ⇒ Noether current

. . . at nonzero chemical potential

μ in (n_1, n_2) plane:

$$S = \frac{1}{2g_0^2} \int d^2x \left[(\partial_\nu n_a)^2 + \underbrace{2i\mu\,(n_1\partial_0 n_2 - n_2\partial_0 n_1)}_{\text{complex}} \underbrace{-\mu^2(n_1^2 + n_2^2)}_{+\mu^2(n_3^2 - 1)} \right]$$

- $\mathcal{O}(\mu)$ imaginary

  complex action/sign problem!

  dual variables — cf. Ising, *XY* model

- $\mathcal{O}(\mu^2)$ in bosonic theories, from second derivative $(\partial_\nu - i\delta_{\nu,0}\mu T_3)^2$

  suppression of perpendicular component $\Rightarrow$ planar, i.e. $O(2)$

  vortices, Berezinskii-Kosterlitz-Thouless transition?

  phase diagram — mostly fixed lattice spacing

Dualisation in a nutshell

- polar coordinates in *n*-space:

$$S = -J \sum_{x,\nu} \Big(\cos\vartheta(x)\cos\vartheta(x+\hat{\nu}) + \sin\vartheta(x)\sin\vartheta(x+\hat{\nu}) \times \tfrac{1}{2}\big\{ e^{i(\phi(x+\hat{\nu})-\phi(x))} e^{\mu\,\delta_{\nu,0}} + \underbrace{e^{-i(\phi(x+\hat{\nu})-\phi(x))} e^{-\mu\,\delta_{\nu,0}}}_{\text{not c.c.}} \big\} \Big)$$

Dualisation in a nutshell

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idea:

- for each term S_A in the action, expand e^{S_A} with a dual variable k_A
  here bond and internal indices: $k_\nu^{(.)}(x)$
- then integrate out the original fields, e.g.

$$\forall x \quad \int d\phi(x)\, e^{-i\phi(x)\left\{\sum_\nu [k_\nu^{(1)} - k_\nu^{(2)}](x) - (x \to x-\hat{\nu})\right\}} \sim \delta(\nabla_\nu \underbrace{[k_\nu^{(1)} - k_\nu^{(2)}]}_{m_\nu(x) \in \mathbb{Z}})$$

$\Rightarrow$ the dual variable $m_\nu(x)$ is divergence-free: symmetry manifest

$\Rightarrow$ closed m-loops: vacuum bubbles and winding loops

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$$\times\frac{1}{2}\Big\{\underbrace{e^{i(\phi(x+\hat\nu)-\phi(x))}e^{\mu\,\delta_{\nu,0}}}_{\text{expand } e^{\cdots} \text{ with } k_\nu^{(1)}(x)}+\underbrace{e^{-i(\phi(x+\hat\nu)-\phi(x))}e^{-\mu\,\delta_{\nu,0}}}_{\text{expand } e^{\cdots} \text{ with } k_\nu^{(2)}(x)}\Big\}\Big)$$

chemical potential:

- enters in a similar way:

$$\prod_x e^{\mu\,\delta_{\nu,0}\sum_\nu[k_\nu^{(1)}-k_\nu^{(2)}](x)} \sim e^{\mu\sum_x m_0(x)} = e^{\frac{\mu}{T}\sum_{x_1} m_0(x)}$$

$\Rightarrow \mu$ couples to the conserved charge of m (def. grandcanonical)

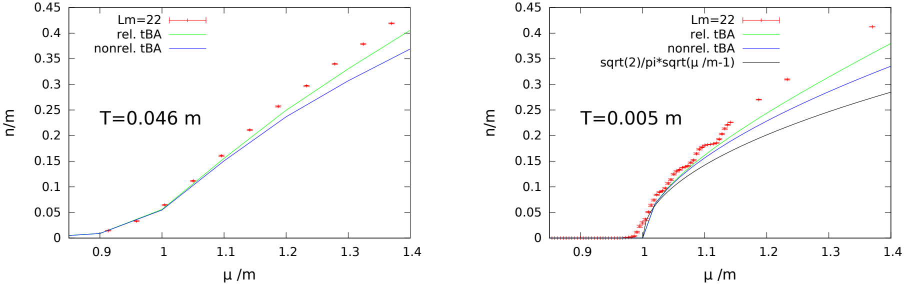
$\Rightarrow$ an integer for every configuration
= net m flux through every time-slices
= total winding number of particle worldlines

- sign problem solved (weight $e^{\mu\text{ real}} > 0$)

Thermodynamics

density as a function of μ rises near dyn. mass m:

checks of worm alg.



- crossover at $T > 0$
  volume doubled: no change

plots

- thermodynamic Bethe ansätze — Lieb, Liniger 63; Yang[2] 69
  = shortrange repulsive bosons in 1d at low densities — no antibosons
- universal square root $\sqrt{\frac{\mu}{m} - 1}$ at very low densities and $T \to 0$
  = fermions in 1d (infinite repulsion or IR limit)

Quantum phase transition

- phase transition at $T = 0$

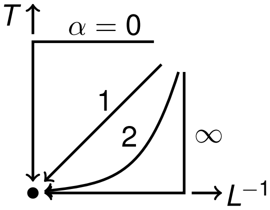
  as crossovers get sharper

  universal square root $\Rightarrow$ second order

- scaling: $T \to 0$ and $L \to \infty$ ($N_t, N_s \to \infty$) how?

  let's keep

  $$TL^\alpha \text{ constant}$$



- consecutive limits $\alpha = 0$ or ∞ ($L \to \infty$ or $T \to 0$ first) impractical

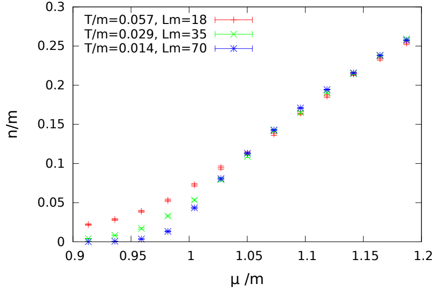
- simultaneous limits $\alpha = 1, 2$:

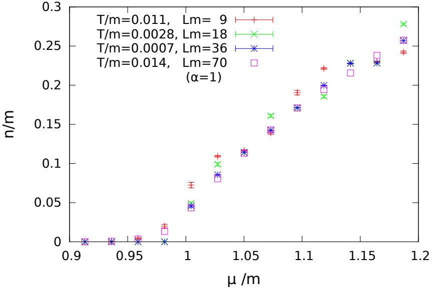
$\alpha = 1$: $L = 1/T$

(square lattices)

$\alpha = 2$: $L^2 = \#/T$

(time-elongated lattices)





$\Rightarrow$ insensitive to how $T \to 0$ and $L \to \infty$: bulk quantity

Spin stiffness σ (or superfluid density)

- twist spatial boundary conditions: $\phi(x_0, x_1 + L) = \phi(x_0, x_1) + \varphi$

  define σ as the corresponding susceptibility of the free energy:

$$\sigma := L\,\partial_\varphi^2 F\big|_{\varphi=0} \qquad \big(\partial_\varphi F\big|_{\varphi=0} = 0\big)$$

- connected to spatial order/correlation length ξ_{spat}

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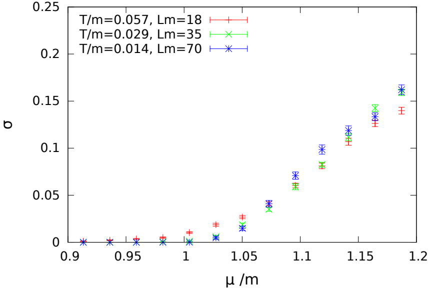
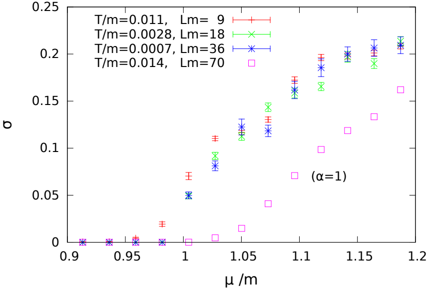
- connected to spatial order/correlation length ξ_{spat}
- results:

$\alpha = 1$: $L = 1/T$



L too large: correlation lost

$\alpha = 2$: $L^2 = \#/T$



(stiffness = density, understood)

Dynamical critical exponent

in Quantum phase transitions:

- correlation length in Euclidean time as well, scales like:

$$\xi_{\text{temp}} \sim (\xi_{\text{spat}})^z \qquad z \ldots \text{dynamical crit. exponent}$$

- if Euclidean invariance $\Rightarrow z = 1$

  μ breaks it: particle worldlines winding in Euclidean time

- note the similarity to our scaling

$$\beta \sim L^\alpha$$

- our results indicate $z \simeq 2$

  in agreement with 1d fermions

Summary

- dual variables solve the $O(3)$ sign problem
- μ couples to the winding number of particle wordlines
- $T > 0$: crossover of the density as a function of μ

  $\uparrow$ agree with thermodyn. Bethe ansätze

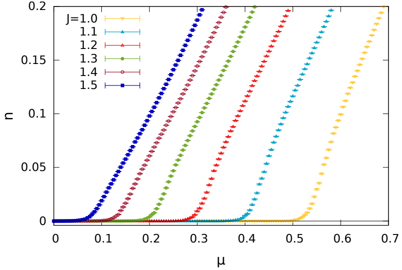
  $\downarrow$ in particular the universal square root
- $T = 0$: quantum phase transition of second order
- spin stiffness sensitive to how $T \to 0$ and $L \to \infty$
- dynamical critical exponent: $z \simeq 2$

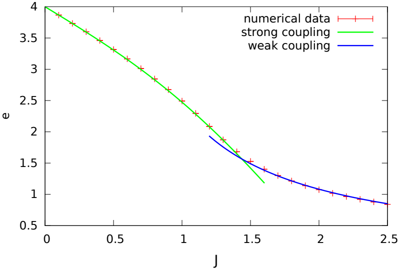
Back-up: some numerical checks

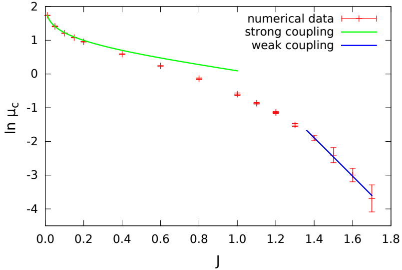
- of our worm algorithm:

energy density (plaquette)

mass threshold transition

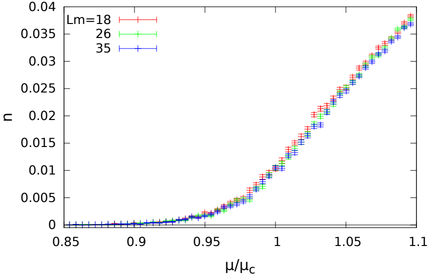




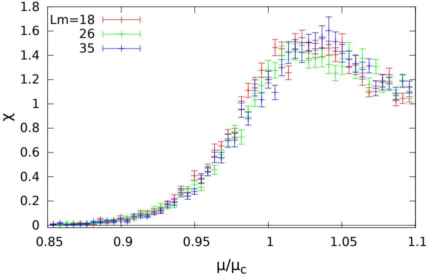


- at $T = 0.023m$, doubling L:

density



its susceptibility



Back-up: full dual partition function, extended to O(4)

$$Z = \sum_{\substack{m\in\mathbb{Z}\\ \bar{m},k^{(3)},k^{(4)}\in\mathbb{N}_0}} \prod_{x,\nu} \frac{(J/2)^{|m_\nu|+2\bar{m}_\nu} J^{k_\nu^{(3)}+k_\nu^{(4)}}}{(|m_\nu^{(j)}|+\bar{m}_\nu)!\,\bar{m}_\nu!k_\nu^{(3)}!k_\nu^{(4)}!} \qquad \text{(expansions of exp.s)}$$

$$\times \prod_x \frac{\Gamma(1+\frac{a}{2})\Gamma(\frac{1}{2}+\frac{b^{(3)}}{2})\Gamma(\frac{1}{2}+\frac{b^{(4)}}{2})}{\Gamma(1+\frac{a}{2}+\frac{1}{2}+\frac{b^{(3)}}{2}+\frac{1}{2}+\frac{b^{(4)}}{2})} \qquad \text{(integrated out } \vartheta\text{)}$$

$$\cdot\,\delta\big(\nabla_\nu m_\nu\big)\, e^{\mu\sum_{x_1} m_0} \qquad \text{(symmetry and chem. potential)}$$

$$\cdot\,\delta_{\text{even}}\big(b^{(3)}\big)\,\delta_{\text{even}}\big(b^{(4)}\big) \qquad \text{(more constraints)}$$

where $b^{(j)}(x) = \sum_\nu \left[k_\nu^{(j)}(x) + k_\nu^{(j)}(x+\hat{\nu})\right] \geq 0$, $a(x)$ similar

- indeed positive weights
- similar for all $O(N)$ and $CP(N-1)$ models
- there must be a conserved current in e.g. the (3,4)-components not manifest, since those not expanded in polar decomposition

Back-up: def. spin stiffness

- $O(2)$ angle periodic up to a twist angle:

$$\phi(x_0, x_1 + L) = \phi(x_0, x_1) + \varphi$$

- free energy increases if system correlated/ordered:

$$\sigma := L \left.\frac{\partial^2 F}{\partial \varphi^2}\right|_{\varphi=0} = -L\,T\,\frac{1}{Z} \left.\frac{\partial^2 Z}{\partial \varphi^2}\right|_{\varphi=0} \qquad \left(\left.\frac{\partial Z}{\partial \varphi}\right|_{\varphi=0} = 0 \right)$$

viewed as imaginary chemical potential φ/L in spat. direction:

$$\sigma = \left.\frac{\partial^2 F/L}{\partial(\varphi/L)^2}\right|_{\varphi=0} \quad \ldots \text{ a density } \checkmark$$

- related to spatial winding numbers of the dual variable m:

$$\sigma = L\,T\,\langle w_{\text{spat}}^2 \rangle \qquad (\langle w_{\text{spat}} \rangle = 0)$$