Strings on the lattice, and AdS/CFT

Valentina Forini

Humboldt University Berlin
Emmy Noether Research Group

based on 1601.04670, 1605.01726
with L. Bianchi, M. S. Bianchi, B. Leder, E.Vescovi + P. Töpfer

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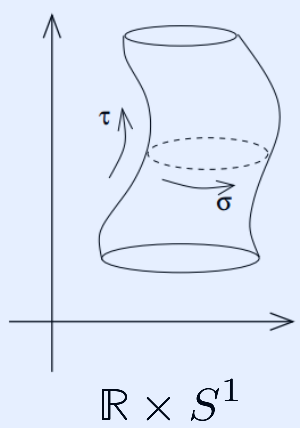
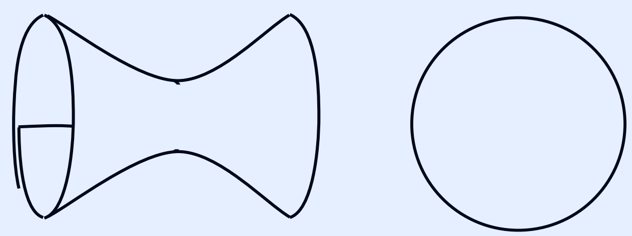
Gauge/string correspondence

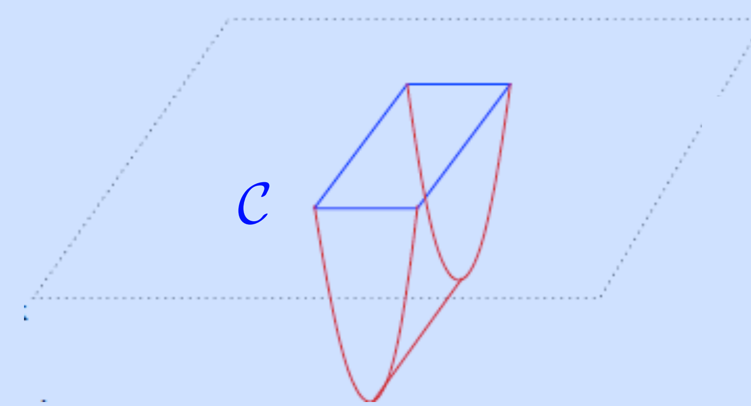




Type IIB strings in $AdS_5 \times S^5$

“Quark-antiquark” potential

$$Z_{\text{string}}\big|_{\mathcal{C}} = \int \mathcal{D}X \, e^{-S_{\text{string}}} \sim e^{-\text{Area}}$$



AdS/CFT

$\mathcal{N} = 4$ super Yang-Mills in 4d

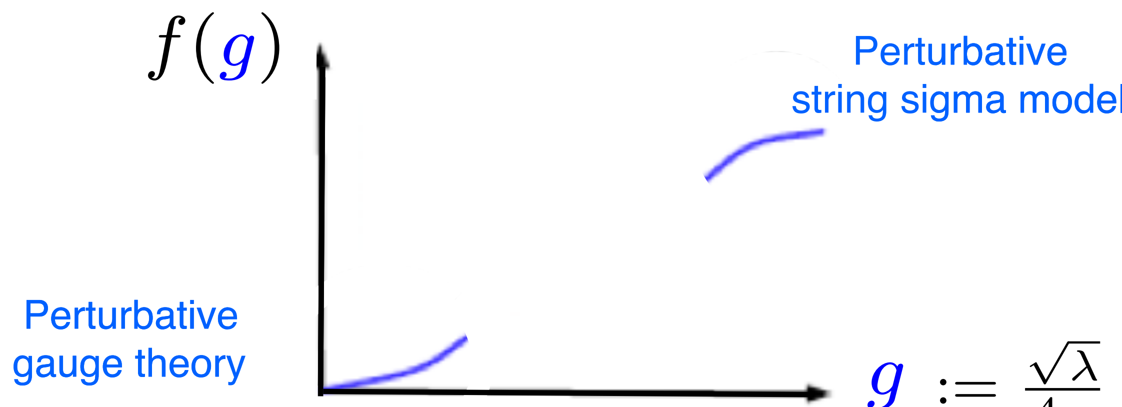
$$\langle W[\mathcal{C}] \rangle = \frac{1}{N} \mathrm{Tr} \mathcal{P} \, e^{\oint (i A_\mu \, \dot{x}^\mu + \Phi_i \, \dot{y}^i) \, ds}$$

Motivation

Main merit: allows studying regimes not accessible via standard analytical tools.

$$f(g) = c\,g + d + \frac{e}{g} + \dots$$

$$f(g) = a\,g^2 + b\,g^4 + \dots$$



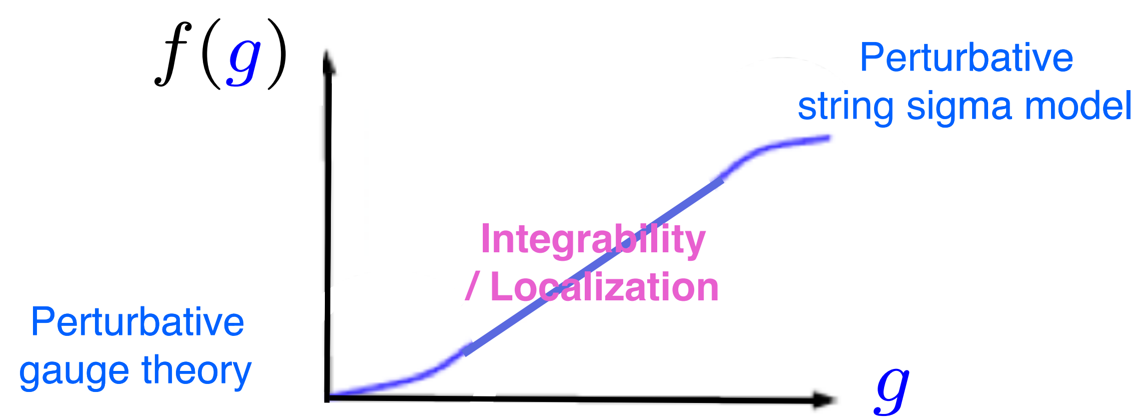
$$g := \frac{\sqrt{\lambda}}{4\pi} = \frac{\sqrt{g_{YM}^2\,N}}{4\pi} \equiv \frac{R^2}{4\pi\alpha'}$$

Motivation

Beautiful progress in obtaining exact results within AdS/CFT

- from integrability (assumed)
- from supersymmetric localization (BPS observable)

$$f(g) = c\,g + d + \frac{e}{g} + \ldots$$



$$f(g) = a\,g^2 + b\,g^4 + \ldots$$

Superstrings in $AdS_5 \times S^5$ with RR fluxes: complicated interacting 2d field theory

$$S_{\rm IIB} = \frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma \Big[\partial_a X \partial^a X + \bar{\theta}\,\Gamma^a (D + F_5)\,\theta\,\partial_a X + \bar{\theta}\theta\bar{\theta}\theta\,\partial_a X \partial^a X + \cdots\Big]$$

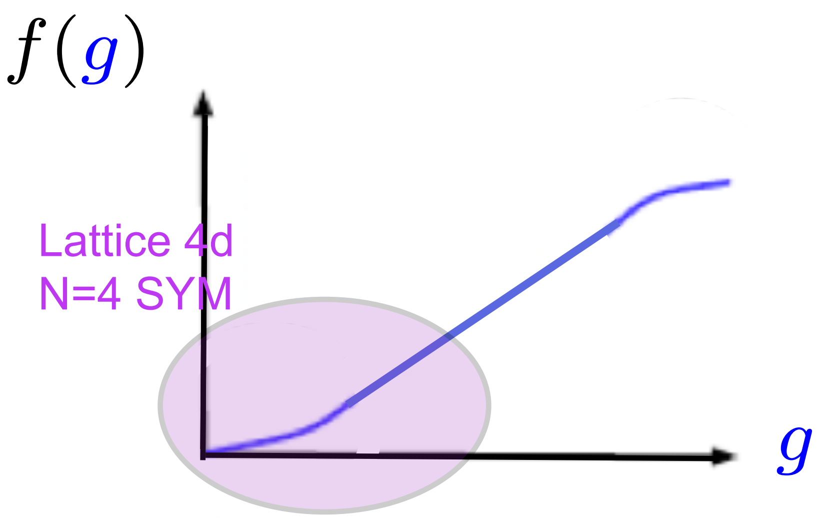
under control perturbatively (and with some caveats).

Need of genuine 2d QFT to cover the finite-coupling region.

Motivation

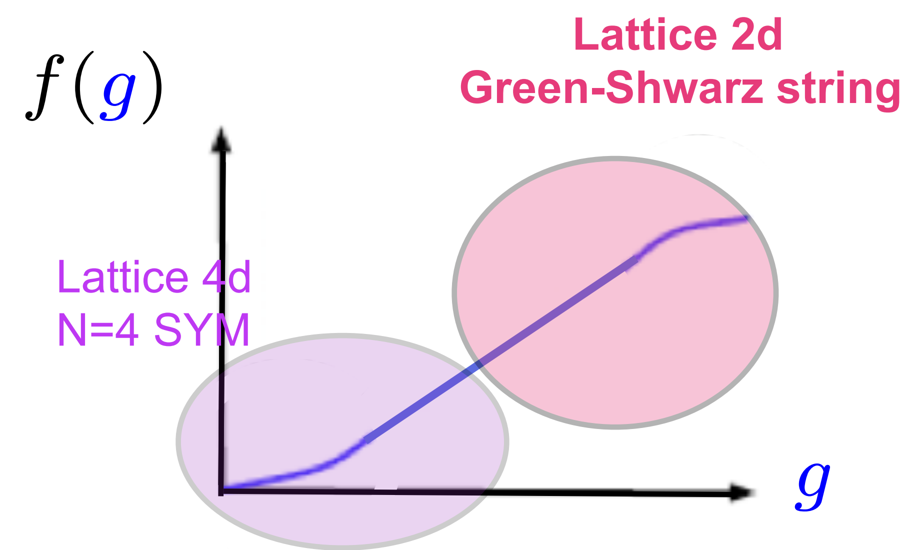
Lattice techniques in AdS/CFT:
(next talk! Berkowitz)
exciting program on the 4d susy CFT side,
subtleties with supersymmetry.

Talks by Shaich, Giedt, Anosh



Motivation

Lattice for superstring world-sheet
in $AdS_5 \times S^5$



[previous study: Roiban McKeown 2013]

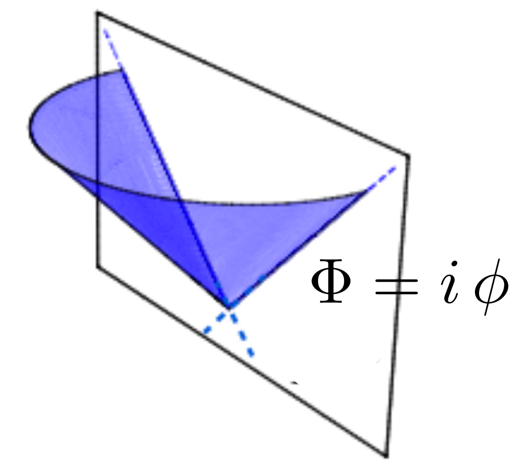
Features:

- 2d: computationally cheap
- no supersymmetry (only as flavour symmetry, Green-Schwarz)
- all gauge symmetries are fixed (no formulation à la Wilson), only scalar fields (some of which anti-commuting)

Non-trivial 2d qft with strong coupling analytically known, finite-coupling (numerical) prediction.

The cusp anomaly of $\mathcal{N} = 4$ SYM from string theory

Completely solved via integrability. [Beisert Eden Staudacher 2006]



Expectation value of a light-like cusped Wilson loop

$$\langle W[C_{\text{cusp}}]\rangle \sim e^{-f(g)\,\phi\,\ln\frac{L_{\text{IR}}}{\epsilon_{\text{UV}}}}$$

AdS/CFT

$$Z_{\text{cusp}} = \int [D\delta X][D\delta\theta]\, e^{-S_{\text{IIB}}(X_{\text{cusp}}+\delta X,\delta\theta)} = e^{-\Gamma_{\text{eff}}} \equiv e^{-f(g)\,V_2}$$

String partition function with "cusp" boundary conditions, evaluated perturbatively

$$f(g)|_{g\to 0} = 8g^2\left[1 - \frac{\pi^2}{3}g^2 + \frac{11\,\pi^4}{45}g^4 - \left(\frac{73}{315} + 8\,\zeta_3\right)g^6 + \ldots\right]$$ [Bern et al. 2006]

$$f(g)|_{g\to\infty} = 4g\left[1 - \frac{3\ln 2}{4\pi}\frac{1}{g} - \frac{K}{16\pi^2}\frac{1}{g^2} + \ldots\right]$$ [Giombi et al. 2009]

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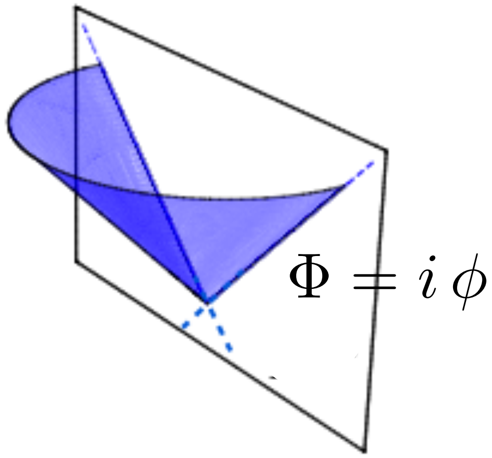

$$\langle W[C_{\rm cusp}]\rangle \sim e^{-f(g)\,\phi\,\ln\frac{L_{\rm IR}}{\epsilon_{\rm UV}}}$$

AdS/CFT

$$Z_{\rm cusp} = \int [D\delta X][D\delta\theta]\, e^{-S_{\rm IIB}(X_{\rm cusp}+\delta X,\delta\theta)} = e^{-\Gamma_{\rm eff}} \equiv e^{-f(g)\,V_2}$$

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A lattice approach prefers expectation values

$$\langle S_{\rm cusp}\rangle = \frac{\int [D\delta X][D\delta\Psi]\, S_{\rm cusp}\, e^{-S_{\rm cusp}}}{\int [D\delta X][D\delta\Psi]\, e^{-S_{\rm cusp}}} = -g\,\frac{d\ln Z_{\rm cusp}}{dg} \equiv g\,\frac{V_2}{8}\,f'(g)$$

Linearization

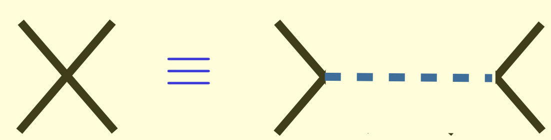
The relevant (gauge-fixed) action has quartic fermionic interactions

$$S_{\text{cusp}} = g \int dt ds \mathcal{L}_{\text{cusp}}$$

$$\begin{aligned}\mathcal{L}_{\text{cusp}} = {} & |\partial_t x + \tfrac{1}{2}x|^2 + \tfrac{1}{z^4}|\partial_s x - \tfrac{1}{2}x|^2 + \left(\partial_t z^M + \tfrac{1}{2}z^M + \tfrac{i}{z^2} z_N \eta_i \left(\rho^{MN}\right)^i{}_j \eta^j\right)^2 + \tfrac{1}{z^4}\left(\partial_s z^M - \tfrac{1}{2}z^M\right)^2 \\ & + i\left(\theta^i \partial_t \theta_i + \eta^i \partial_t \eta_i + \theta_i \partial_t \theta^i + \eta_i \partial_t \eta^i\right) - \tfrac{1}{z^2}\left(\eta^i \eta_i\right)^2 \\ & + 2i\Big[\tfrac{1}{z^3} z^M \eta^i \left(\rho^M\right)_{ij}\left(\partial_s \theta^j - \tfrac{1}{2}\theta^j - \tfrac{i}{z}\eta^j\left(\partial_s x - \tfrac{1}{2}x\right)\right) + \tfrac{1}{z^3} z^M \eta_i (\rho_M^\dagger)^{ij}\left(\partial_s \theta_j - \tfrac{1}{2}\theta_j + \tfrac{i}{z}\eta_j\left(\partial_s x - \tfrac{1}{2}x\right)^*\right)\Big]\end{aligned}$$

To formally integrate out Graßmann-odd fields, $P[\Phi_i] = \dfrac{e^{-S_E[\Phi_i]} \det \mathcal{O}_F}{Z}$

linearize



Introduce **auxiliary** fields
(7 complex bosons)

and enforce definite positive weight

$$\det O_F \quad \longrightarrow \quad \sqrt{\det(\mathcal{O}_F\, \mathcal{O}_F^\dagger)} = \int D\zeta\, D\bar{\zeta}\ e^{-\int d^2\xi\, \bar{\zeta}(\mathcal{O}_F \mathcal{O}_F^\dagger)^{-\frac{1}{4}}\zeta}$$

Green-Schwarz string in the null cusp background

After linearization the Lagrangian reads ($m \sim P_+$)

$$\mathcal{L}_{\rm cusp} = \left|\partial_t x + \frac{m}{2} x\right|^2 + \frac{1}{z^4}\left|\partial_s x - \frac{m}{2} x\right|^2 + (\partial_t z^M + \frac{m}{2} z^M)^2 + \frac{1}{z^4}(\partial_s z^M - \frac{m}{2} z^M)^2$$
$$+ \frac{1}{2}\phi^2 + \frac{1}{2}(\phi_M)^2 + \psi^T O_F \psi \ ,$$

- 8 bosonic coordinates: x, x^*, z^M $(M = 1, \cdots, 6)$, $z = \sqrt{z_M z^M}$;
- 7 auxiliary fields ϕ, ϕ^M $(M = 1, \cdots, 6)$);
- 8 fermionic variables, $\psi \equiv (\theta^i, \theta_i, \eta^i, \eta_i)$, and $\theta^i = (\theta_i)^\dagger$, $\eta^i = (\eta_i)^\dagger$, $i = 1, 2, 3, 4$ transforming in the fundamental of $SU(4)$

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$$O_F = \begin{pmatrix} 0 & i\partial_t & -\mathrm{i}\rho^M \left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} & 0 \\ \mathrm{i}\partial_t & 0 & 0 & -\mathrm{i}\rho^\dagger_M \left(\partial_s + \frac{m}{2}\right)\frac{z^M}{z^3} \\ \mathrm{i}\frac{z^M}{z^3}\rho^M\left(\partial_s - \frac{m}{2}\right) & 0 & 2\frac{z^M}{z^4}\rho^M\left(\partial_s x - m\frac{x}{2}\right) & i\partial_t - A^T \\ 0 & \mathrm{i}\frac{z^M}{z^3}\rho^\dagger_M\left(\partial_s - \frac{m}{2}\right) & \mathrm{i}\partial_t + A & -2\frac{z^M}{z^4}\rho^\dagger_M\left(\partial_s x^* - m\frac{x}{2}^*\right) \end{pmatrix}$$

$$A = \frac{1}{\sqrt{2}z^2}\phi_M \rho^{MN} z_N - \frac{1}{\sqrt{2}z}\phi + \mathrm{i}\,\frac{z_N}{z^2}\rho^{MN}\,\partial_t z^M$$

and ρ^M are off-diagonal blocks of SO(6) Dirac matrices $\gamma^M \equiv \begin{pmatrix} 0 & \rho^\dagger_M \\ \rho^M & 0 \end{pmatrix}$.

Manifest global symmetry is $SO(6) \times SO(2)$.

Discretization

Suppress fermion doublers with the Wilson-like discretization

$$K_F{}^W = \begin{pmatrix} W_+ & -\mathring{p}_0 \mathbb{1} & (\mathring{p}_1 - i\frac{m}{2})\rho^M u_M & 0 \\ -\mathring{p}_0 \mathbb{1} & -W_+^\dagger & 0 & (\mathring{p}_1 - i\frac{m}{2})\rho_M^\dagger u^M \\ -(\mathring{p}_1 + i\frac{m}{2})\rho^M u_M & 0 & W_- & -\mathring{p}_0 \mathbb{1} \\ 0 & -(\mathring{p}_1 + i\,\frac{m}{2})\rho_M^\dagger u^M & -\mathring{p}_0 \mathbb{1} & -W_-^\dagger \end{pmatrix}$$

where $W_\pm = \frac{r}{2}\left(\hat{p}_0^2 \pm i\,\hat{p}_1^2\right)\rho^M u_M$, $|r| = 1$, and $\hat{p}_\mu \equiv \frac{2}{a}\sin\frac{p_\mu a}{2}$. It leads to

$$\Gamma_{\rm LAT}^{(1)} = \frac{V_2}{2\,a^2}\int_{-\pi}^{+\pi}\frac{d^2p}{(2\pi)^2}\ln\Big[\frac{4^8(\sin^2\frac{p_0}{2}+\sin^2\frac{p_1}{2})^5(\sin^2\frac{p_0}{2}+\sin^2\frac{p_1}{2}+\frac{M^2}{8})^2(\sin^2\frac{p_0}{2}+\sin^2\frac{p_1}{2}+\frac{M^2}{4})}{\left(\sin^2 p_0+\sin^2 p_1+\frac{M^2}{4}+4\sin^4\frac{p_0}{2}+4\sin^4\frac{p_1}{2}\right)^8}\Big]$$

$$\xrightarrow{a\to 0}\ \boxed{-\frac{3\ln 2}{8\pi}}\,V_2\,m^2\,,$$ cusp anomaly at strong coupling $(\,|r| = 1, M = m\,a.)$

- Does not induce (additional) complex phases:

$$\left(O_F{}^W\right)^\dagger = \Gamma^5\,O_F{}^W\,\Gamma^5 \quad \text{and} \quad \left(O_F{}^W\right)^T = -O_F{}^W$$

  with $\Gamma_5^\dagger\Gamma_5 = \mathbb{1}$, $\Gamma_5^\dagger = -\Gamma_5$ ensures $\det O_F{}^W$ to be real and non-negative.
- It **preserves** the **SO(6)** global symmetry, breaks the SO(2).

The simulation: parameter space

- In the continuum model there are two parameters, $g = \frac{\sqrt{\lambda}}{4\pi}$ and $m \sim P_+$. In perturbation theory divergences cancel, dimensionless quantities are pure functions of the (bare) coupling

$$F = F(g)$$

- Our discretization cancels (1-loop) divergences, and reproduces the 1-loop cusp anomaly. Assume it is true nonperturbatively, for lattice regularization.

  Only additional scale: lattice spacing a.

  Three dimensionless (input) parameters:

$$g\,, \qquad N \equiv \frac{L}{a}\,, \qquad M \equiv m\,a$$

  Therefore

$$F_{\rm LAT} = F_{\rm LAT}(g, N, M)$$

Line of constant physics

In the continuum, "effective" masses undergo a *finite* renormalization

$$m_x^2(g) = \frac{m^2}{2}\left(1 - \frac{1}{8\,g} + \mathcal{O}(g^{-2})\right) \qquad (\star)$$

The dimensionless physical quantity to keep constant when $a \to 0$ is

$$L^2\, m_x^2 = \text{const}\,, \qquad \text{leading to} \qquad \boxed{(L\,m)^2 \equiv (NM)^2 = \text{const}}\,,$$

if $(\star)$ is still true on the lattice and g is not (infinitely) renormalized.

Continuum limit $a \to 0$

We assume that, on the lattice, no further scale but a is present.

A generic observable

$$F_{\rm LAT} = F_{\rm LAT}(g, N, M) = F(g) + \mathcal{O}\Big(\frac{1}{N}\Big) + \mathcal{O}\left(e^{-MN}\right)$$

where

$$g = \frac{\sqrt{\lambda}}{4\pi}\,, \qquad N = \frac{L}{a}\,, \qquad M = a\,m\,.$$

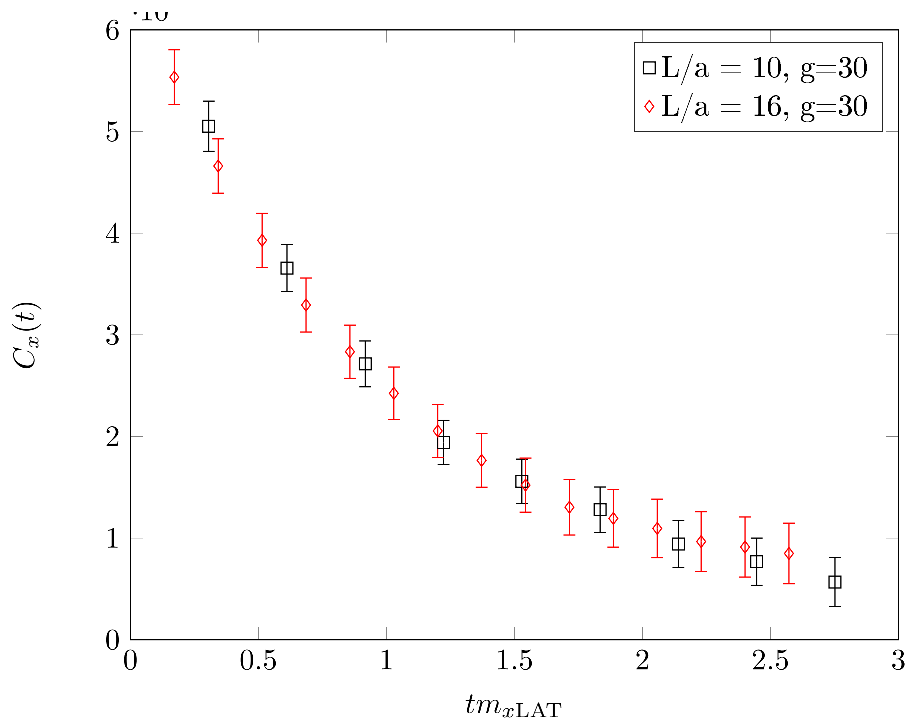
Recipe:

- fix g
- fix MN, large enough so to to keep small finite volume effects
- evaluate $F_{\rm LAT}$ for $N = 6, 8, 10, 12, 16, \cdots$
- obtain $F(g)$ extrapolating to $N \to \infty$.

Measure I: mass of x boson

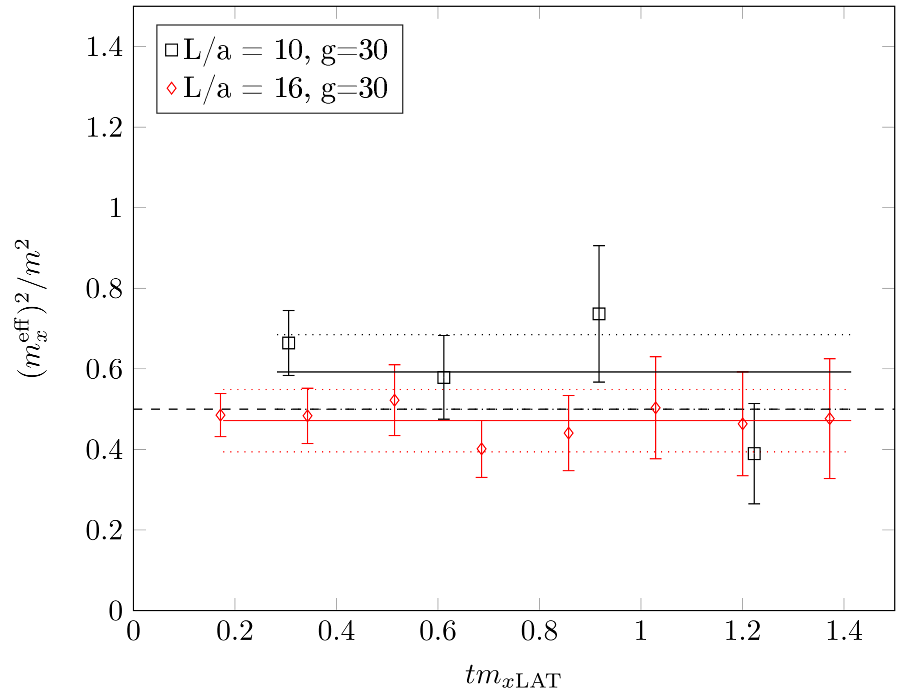
From the correlator of the x fields

$$C_x(t) = \sum_{s_1,\, s_2} \langle x(t,\, s_1) x^*(0,\, s_2) \rangle$$
$$= c_0\, e^{-t\, m_{x\mathrm{LAT}}} + \ldots$$



extract the x-mass

$$m_{x\,\mathrm{LAT}} = \lim_{T,\, t \to \infty} m^{\mathrm{eff}_x}$$
$$\equiv \lim_{T,\, t \to \infty,\, a} \frac{1}{a} \log \frac{C_x(t;\, 0)}{C_x(t + a;\, 0)}$$



No infinite renormalization occurring, no need of tuning m to adjust for it.
This corroborates our choice of line of constant physics.

Measure II: the cusp action

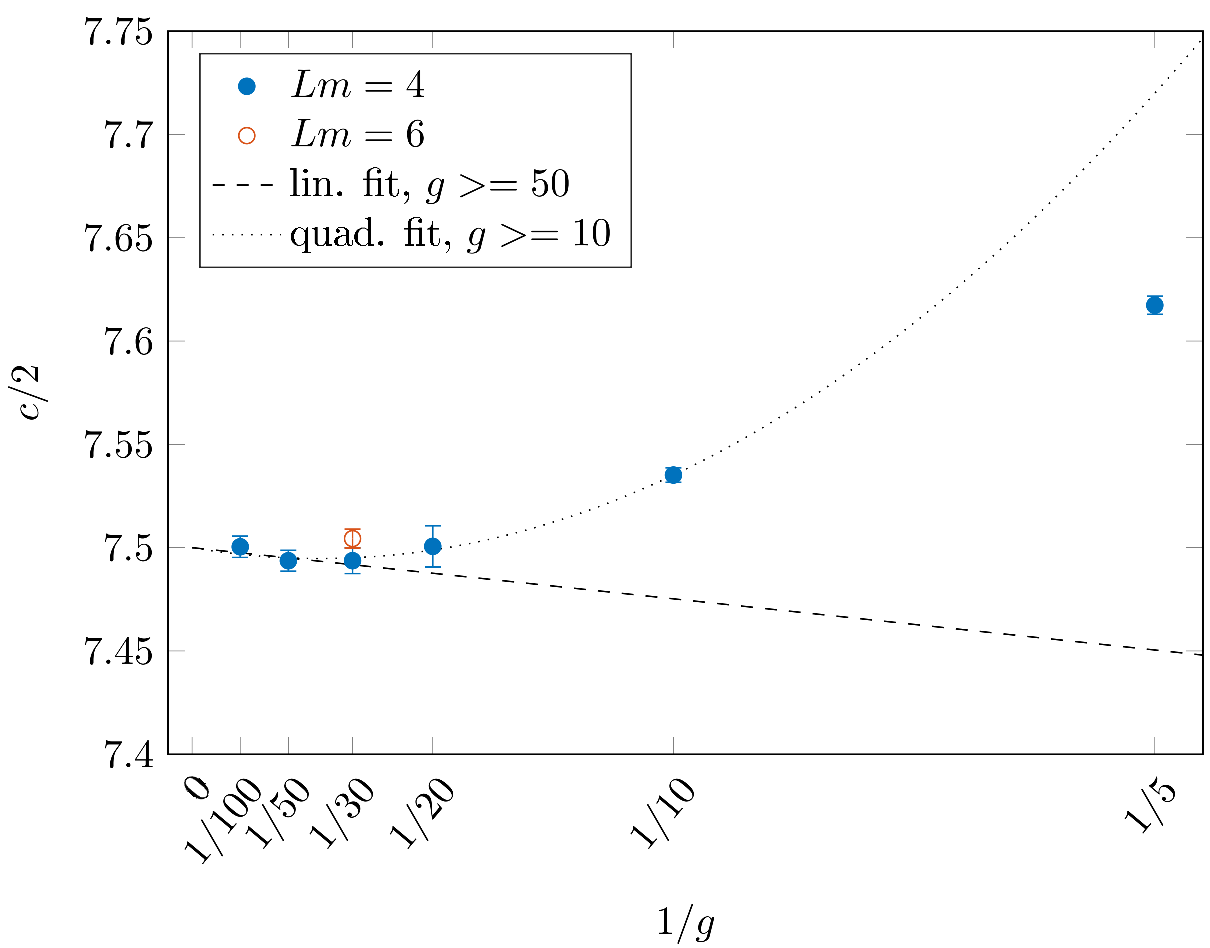
In measuring $\langle S_{\rm cusp}\rangle \equiv g\,\frac{V_2\,m^2}{8}\,f'(g)$ quadratic divergences appear.

At large g,

$$\langle S_{\rm LAT}\rangle \equiv g\,\frac{N^2\,M^2}{4}\,4 + \frac{c}{2}(2N^2)$$

where $c = \mathsf{n}_{\rm bos}$. This is because $S = -\frac{\partial \ln Z}{\partial \ln g}$ and $Z \sim \Pi_{\mathsf{n}_{\rm bos}} (\det g\,\mathcal{O})^{-\frac{1}{2}}$.

Here, $\mathsf{n}_{bos}{}^{\rm phys} + \mathsf{n}_{\rm bos}{}^{aux} = 8 + 7 = 15$.

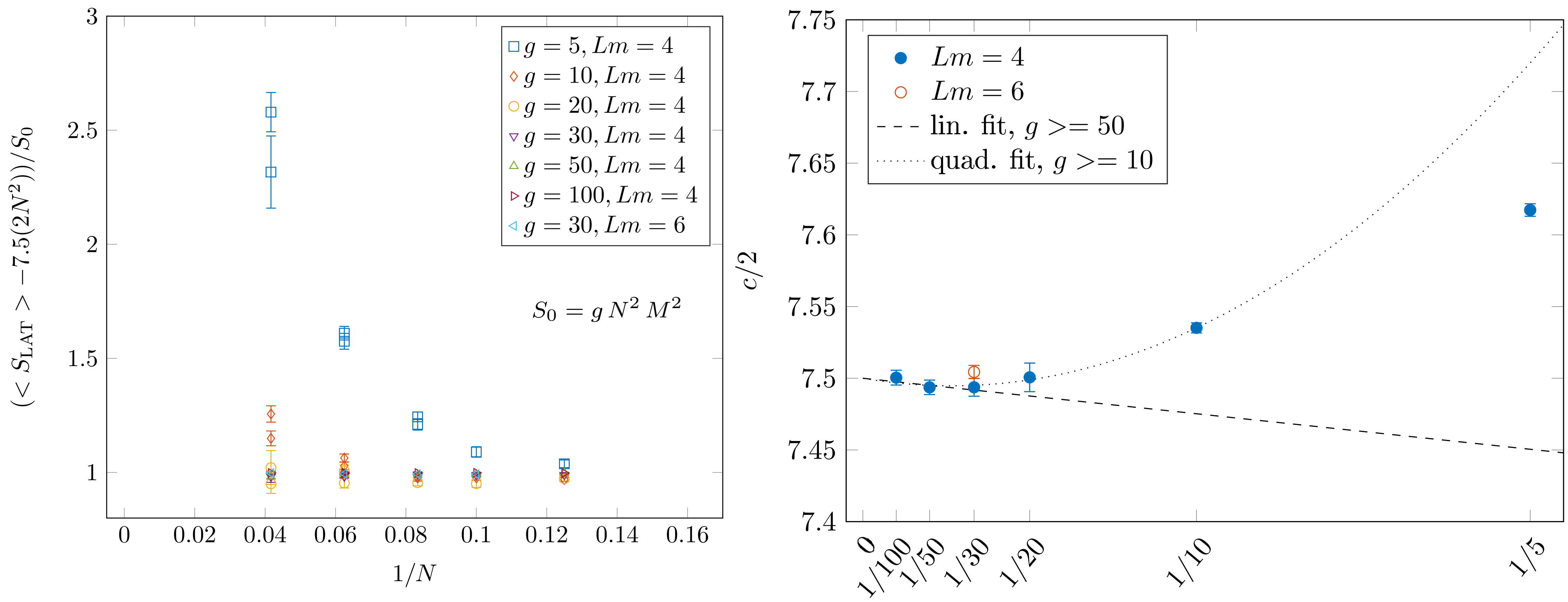


Measure II: the cusp action

In measuring $\langle S_{\rm cusp}\rangle \equiv g\,\frac{V_2\,m^2}{8}\,f'(g)$ quadratic divergences appear.

They appear also at finite g,

$$\langle S_{\rm LAT}\rangle \equiv g\,\frac{N^2\,M^2}{4}\,f'(g)_{\rm LAT} + \frac{c(g)}{2}(2N^2)$$



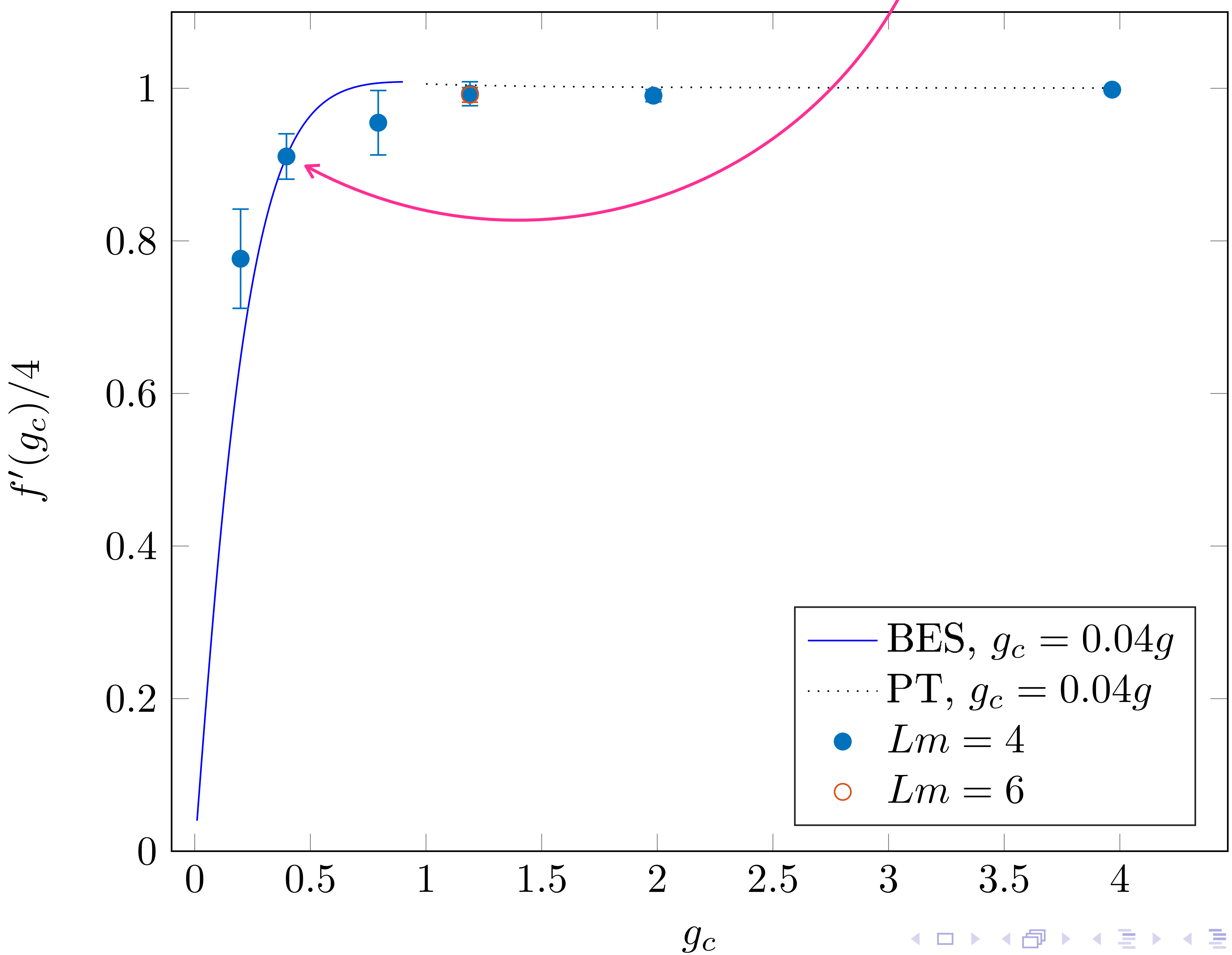
In continuum perturbation theory dim. reg. set them to zero.

Here, expected mixing of the Lagrangian with lower dimension operator

$$\mathcal{O}(\phi(s))_r = \sum_{\alpha:[O_\alpha]\le D} Z_\alpha\,\mathcal{O}_\alpha(\phi(x))\,, \qquad Z_\alpha \sim \Lambda^{(D-[\mathcal{O}_\alpha])} \sim a^{-(D-[\mathcal{O}_\alpha])}$$

Cusp on the lattice vs cusp in the continuum

Subtract divergences, assume $g = \alpha\, g_c$, from $f'(g) = f'(g_c)_c$ get $g_c = 0.04g$.



The phase

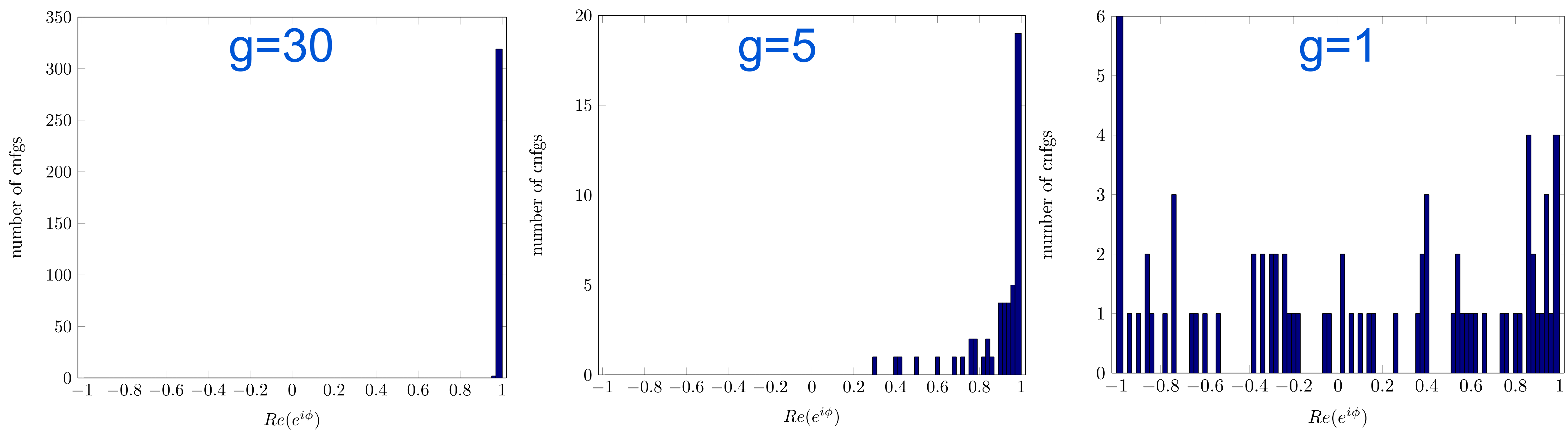
After linearization $\mathcal{L}_F = \psi^T \, \mathcal{O}_F \, \psi$, integrating fermions leads to a complex Pfaffian $\mathrm{Pf}\, O_F = |(\det O_F)^{\frac{1}{2}}|\, e^{i\theta}$.

The phase is encoded in the linearization: we deal with a fermion hermitian bilinear $b \sim \eta^2$ whose corresponding quartic interaction

$$e^{-\mathcal{L}_4^{\mathrm{ferm}}} = e^{-\frac{b^2}{4\,a}} = \int dx\, e^{-a\,x^2 + i\,b\,x}$$

comes in the exponential as a “repulsive” potential.

In the interesting ($g = 1$) region the real part of the phase has a flat distribution,



and reweighting $\langle \mathcal{O} \rangle_{\mathrm{reweight}} = \dfrac{\langle \mathcal{O}\, e^{i\theta} \rangle_{\theta=0}}{\langle e^{i\theta} \rangle_{\theta=0}}$ breaks down.

Alternative linearization

Exploit the Graßmann nature of fermions

$$\begin{aligned}\mathcal{L}_{F4} &= -\frac{1}{z^2}(\eta^2)^2+\frac{1}{z^2}(i\,\eta_i(\rho^{MN})^i{}_j n^N \eta^j)^2 \\ &= -\frac{1}{z^2}(\eta^2)^2 \mp 2(\eta^2)^2 \mp \Sigma_\pm{}_i^j \Sigma_\pm{}_j^i\end{aligned}$$

where $\Sigma_\pm{}_i^j = \Sigma_i^j \pm \tilde{\Sigma}_i^j$ and

$$\Sigma_i{}^j = \eta_i \eta^j \qquad \tilde{\Sigma}_j{}^i = (\rho^N)^{ik} n_N (\rho^L)_{jl} n_L \eta_k \eta^l$$

Choose the good sign ($-$). This ensure a Pfaffian real, but not definite positive ($\mathrm{Pf}\, O_F = \pm(\det O_F)^{\frac{1}{2}}$).

Simulations for the new Yukawa terms (now $1+16$ real auxiliary fields)

$$\mathcal{L}_{F4} \longrightarrow \frac{12}{z}\eta^2\phi + 6\phi^2 + \frac{2}{z}\Sigma_\pm{}_j^i \phi_i^j + \phi_j^i \phi_i^j$$

are ongoing, reweighting seems problematic.

However, similar Lagrangean rearrangement do eliminate the sign problem in other models with quartic fermionic interactions.

[Catterall 2015, Wipf et al. unpublished]

Conclusions

Solving a non-trivial 4d QFT is hard $\longrightarrow$ reduce the problem via AdS/CFT:

solve a non-trivial 2d QFT.

Lattice simulation of gauge-fixed Green-Schwarz string, Wilson-like fermion discretizations, standard methods (Rational Hybrid Monte Carlo).

- Results seems not to be sensitive to the discretization adopted
- Observables measured are in good agreement with expectation at large g
- At small g, complex phase and sign problem.
  Qualitative agreement with nonperturbative expectation from AdS/CFT

Comparison assumes trivial relation between g and g_c - if not, no predictivity.

Then continuum prediction is the point where to study the theory, lattice bare coupling tuned accordingly and used for fully predictive measurements of e.g. masses.

Outlook

- Simulations with phase-free linearization
- Further observables, different backgrounds (e.g. AdS$_4$/CFT$_3$)
- Correlators of string vertex operators (gauge theory 3-point functions)
- ...
- ...

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- Correlators of string vertex operators (gauge theory 3-point functions)
- $\ldots$
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Thanks for your attention.

Integrability in Gauge and String Theory 2016

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Campus Adlershof

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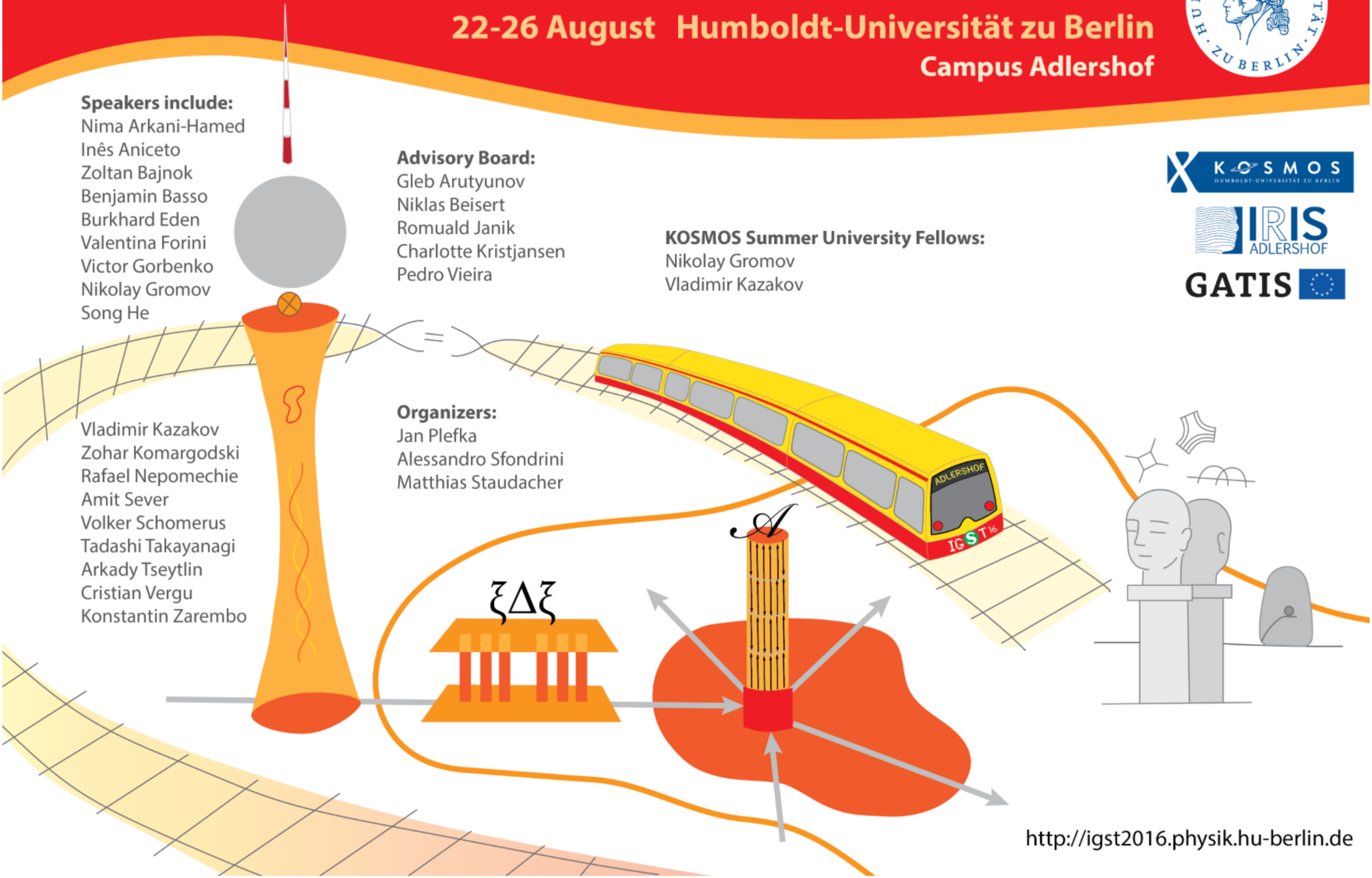

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Parameters of the simulations

g	$T/a \times L/a$	Lm	am	τ_{int}^{S}	$\tau_{\text{int}}^{m_x}$	statistics [MDU]
5	16×8	4	0.50000	0.8	2.2	900
	20×10	4	0.40000	0.9	2.6	900
	24×12	4	0.33333	0.7	4.6	900,1000
	32×16	4	0.25000	0.7	4.4	850,1000
	48×24	4	0.16667	1.1	3.0	92,265
10	16×8	4	0.50000	0.9	2.1	1000
	20×10	4	0.40000	0.9	2.1	1000
	24×12	4	0.33333	1.0	2.5	1000,1000
	32×16	4	0.25000	1.0	2.7	900,1000
	48×24	4	0.16667	1.1	3.9	594,564
20	16×8	4	0.50000	5.4	1.9	1000
	20×10	4	0.40000	9.9	1.8	1000
	24×12	4	0.33333	4.4	2.0	850
	32×16	4	0.25000	7.4	2.3	850,1000
	48×24	4	0.16667	8.4	3.6	264,580
30	20×10	6	0.60000	1.3	2.9	950
	24×12	6	0.50000	1.3	2.4	950
	32×16	6	0.37500	1.7	2.3	975
	48×24	6	0.25000	1.5	2.3	533,652
	16×8	4	0.50000	1.4	1.9	1000
	20×10	4	0.40000	1.2	2.7	950
	24×12	4	0.33333	1.2	2.1	900
	32×16	4	0.25000	1.3	1.8	900,1000
	48×24	4	0.16667	1.3	4.3	150
50	16×8	4	0.50000	1.1	1.8	1000
	20×10	4	0.40000	1.2	1.8	1000
	24×12	4	0.33333	0.8	2.0	1000
	32×16	4	0.25000	1.3	2.0	900,1000
	48×24	4	0.16667	1.2	2.3	412
100	16×8	4	0.50000	1.4	2.7	1000
	20×10	4	0.40000	1.4	4.2	1000
	24×12	4	0.33333	1.3	1.8	1000
	32×16	4	0.25000	1.3	2.0	950,1000
	48×24	4	0.16667	1.4	2.4	541

Table 1: Parameters of the simulations: the coupling g, the temporal (T) and spatial (L) extent of the lattice in units of the lattice spacing a, the line of constant physics fixed by Lm and the mass parameter $M = am$. The size of the statistics after thermalization is given in the last column in terms of Molecular Dynamic Units (MDU), which equals an HMC trajectory of length one. In the case of multiple replica the statistics for each replica is given separately. The auto-correlation times τ of our main observables m_x and S are also given in the same units.