Disconnected and light connected HVP contributions to the muon anomalous magnetic moment

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RBC and UKQCD Collaborations

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Hadronic contributions to $a_\mu$

<table>
<thead>
<tr>
<th>Contribution</th>
<th>Value $\times 10^{10}$</th>
<th>Uncertainty $\times 10^{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED (5 loops)</td>
<td>11 658 471.895</td>
<td>0.008</td>
</tr>
<tr>
<td>EW</td>
<td>15.4</td>
<td>0.1</td>
</tr>
<tr>
<td><strong>HVP LO</strong></td>
<td>692.3</td>
<td>4.2</td>
</tr>
<tr>
<td>HVP NLO</td>
<td>-9.84</td>
<td>0.06</td>
</tr>
<tr>
<td>HVP NNLO</td>
<td>1.24</td>
<td>0.01</td>
</tr>
<tr>
<td><strong>Hadronic light-by-light</strong></td>
<td>10.5</td>
<td>2.6</td>
</tr>
<tr>
<td><strong>Total SM prediction</strong></td>
<td>11 659 181.5</td>
<td>4.9</td>
</tr>
<tr>
<td>BNL E821 result</td>
<td>11 659 209.1</td>
<td>6.3</td>
</tr>
<tr>
<td>Fermilab E989 target</td>
<td></td>
<td>$\approx 1.6$</td>
</tr>
</tbody>
</table>

A reduction of uncertainty for HVP and HLbL is needed. For HLbL only model estimations exist. ⇒ First-principles non-perturbative determination desired.
Classification of hadronic contributions

HVP LO  HVP NLO  HVP NNLO  HLbL

Talk by L. Jin
The lattice approach to HVP LO

Quark-connected piece with > 90% of the contribution with by far dominant part from up and down quark loops (Below focus on light contribution only); Talk by M. Spraggs for strange-quark analysis

Quark-disconnected piece with \( \approx 1.5\% \) of the contribution (1/5 suppression already through charge factors); Phys.Rev.Lett. 116 (2016) 232002

QED and isospin-breaking corrections, estimated at the few-per-cent level; Talks by V. Gülpers and J. Harrison
HVP quark-connected contribution

Biggest challenge to direct calculation at physical point is to control statistics and potentially large finite-volume errors (Estimated at $O(10\%)$ Aubin et al. 2015)

Statistics: for strange and charm solved issue, for up and down quarks existing methodology (such as HPQCD moments approach) less effective

Finite-volume errors are exponentially suppressed in the simulation volume but seem to be sizeable in QCD boxes with $m_\pi L = 4$
Starting from

\[ \sum_x e^{iqx} \langle J_\mu(x) J_\nu(0) \rangle = (\delta_{\mu\nu} q^2 - q_\mu q_\nu) \Pi(q^2) \]  

(1)

with vector current \( J_\mu(x) = i \sum_f Q_f \bar{\Psi}_f(x) \gamma_\mu \Psi_f(x) \) and using the subtraction prescription of Bernecker-Meyer 2011

\[ \Pi(q^2) - \Pi(q^2 = 0) = \sum_t \left( \frac{\cos(qt) - 1}{q^2} + \frac{1}{2} t^2 \right) C(t) \]  

(2)

with \( C(t) = \frac{1}{3} \sum_{\vec{x}} \sum_{j=0,1,2} \langle J_j(\vec{x}, t) J_j(0) \rangle \) we may write

\[ a_\mu^{\text{HVP}} = \sum_{t=0}^{\infty} w_t C(t) , \]  

(3)

where \( w_t \) captures the QED part of the diagram.
Integrand $w_T C(T)$ for the light-quark connected contribution:

$m_\pi = 140$ MeV, $a = 0.11$ fm (RBC/UKQCD $48^3$ ensemble)

Statistical noise from long-distance region
Approaches to the long-distance noise problem:

- Only use short-distance lattice data ($\lesssim 0.5$ fm–$1.5$ fm), beyond that multi-exponentials from fit

- RBC in progress: improved stochastic estimator
Low-mode saturation for physical pion mass (here 2000 modes):

\[ a_{\mu} \times 10^{10} \]

\[ T \]

Conf = 82

- Full temporal integrand
- Sloppy temporal integrand
- No new physics
- Low-mode only integrand
- NLO FV ChPT temporal integrand
From Aubin et al. 2015 (arXiv:1512.07555v2)

MILC lattice data with $m_\pi L = 4.2$, $m_\pi \approx 220$ MeV; Plot difference of $\Pi(q^2)$ from different irreps of 90-degree rotation symmetry of spatial components versus NLO FV ChPT prediction (red dots)

While the absolute value of $a_\mu$ is poorly described by the two-pion contribution, the volume dependence may be described sufficiently well to use ChPT to control FV errors at the 1% level; this needs further scrutiny

Aubin et al. find an $O(10\%)$ finite-volume error for $m_\pi L = 4.2$ based on the $A_1 - A_1^{44}$ difference (right-hand plot)
Compare difference of integrand of $48 \times 48 \times 96 \times 48$ (spatial) and $48 \times 48 \times 48 \times 96$ (temporal) geometries with NLO FV ChPT $(A_1 - A_{14}^{44})$:

$m_\pi = 140$ MeV, $p^2 = m_\pi^2 / (4\pi f_\pi)^2 \approx 0.7\%$
Combined lattice and dispersive analysis

We can use the dispersion relation to overlay experimental $e^+e^-$ scattering data (Bernecker, Meyer 2011). Below the experimental result is taken from Jegerlehner 2016:

![Graph showing lattice and dispersive analysis results](image)

The lattice data here includes finite-volume corrections based on NLO FV ChPT. Continuum limit, charm contribution, and QED/IB correction missing. Will study different individual datasets: BaBar, KLOE.
Combined lattice and dispersive analysis

We can use the dispersion relation to overlay experimental $e^+e^-$ scattering data (Bernecker, Meyer 2011). Below the experimental result is taken from Jegerlehner 2016:

![Graph showing data points and error bars for different datasets.

The lattice data here includes finite-volume corrections based on NLO FV ChPT. Continuum limit, charm contribution, and QED/IB correction missing. Will study different individual datasets: BaBar, KLOE.

- BaBar 09: $376.7 \pm 2.0 \pm 1.9$
- KLOE 08: $368.1 \pm 0.4 \pm 2.3 \pm 2.0$
- KLOE 10: $365.3 \pm 0.9 \pm 2.3 \pm 2.2$
- KLOE 12: $366.7 \pm 1.2 \pm 2.4 \pm 0.8$
- BESIII: $368.2 \pm 2.5 \pm 3.3$
The lattice data is precise at shorter distances and the experimental data is precise at longer distances. We can do a combined analysis with lattice and experimental data:

\[ a_\mu = \sum_{t=0}^{t_{\text{lat}/\text{ex}}} w_t C_{\text{lattice}}(t) + \sum_{t=t_{\text{lat}/\text{ex}}+1}^{\infty} w_t C_{\text{exp}}(t) \]

As expected a nice plateau region as a function of \( t_{\text{lat}/\text{ex}} \) is visible.

If we took \( t_{\text{lat}/\text{ex}} = 15a \approx 1.7 \text{ fm} \), we currently have a statistical error of 0.7%

\[ a_\mu^{\text{HVP},u,d,s} = 693(5) \times 10^{-10} . \]

Continuum limit, charm contribution, and QED/IB correction missing.
HVP quark-disconnected contribution

First results at physical pion mass with a statistical signal


Statistics is clearly the bottleneck

New stochastic estimator allowed us to get result

\[ a_\mu^{\text{HVP (LO) DISC}} = -9.6(3.3)_{\text{stat}}(2.3)_{\text{sys}} \times 10^{-10} \quad (4) \]

from 20 configurations at physical pion mass and 45 propagators/configuration.
Our setup:

\[ C(t) = \frac{1}{3V} \sum_{j=0,1,2} \sum_{t'} \langle V_j(t + t') V_j(t') \rangle_{SU(3)} \]  

where \( V \) stands for the four-dimensional lattice volume, \( V_\mu = (1/3)(V^u_\mu - V^s_\mu) \), and

\[ V^f_\mu(t) = \sum_{\vec{x}} \text{Im} \, \text{Tr}[D^{-1}_{\vec{x},t;\vec{x},t}(m_f) \gamma_\mu]. \]  

We separate 2000 low modes (up to around \( m_s \)) from light quark propagator as \( D^{-1} = \sum_n \nu^n(w^n)^\dagger + D^{-1}_{\text{high}} \) and estimate the high mode stochastically and the low modes as a full volume average Foley 2005.

We use a sparse grid for the high modes similar to Li 2010 which has support only for points \( x_\mu \) with \((x_\mu - x_\mu^{(0)}) \mod N = 0\); here we additionally use a random grid offset \( x_\mu^{(0)} \) per sample allowing us to stochastically project to momenta.
Combination of both ideas is crucial for noise reduction at physical pion mass!

Fluctuation of $\mathcal{V}_\mu (\sigma)$:

Since $C(t)$ is the autocorrelator of $\mathcal{V}_\mu$, we can create a stochastic estimator whose noise is potentially reduced linearly in the number of random samples, hence the normalization in the lower panel.
Result for partial sum $L_T = \sum_{t=0}^{T} w_t C(t)$:

For $t \geq 15 \ C(t)$ is consistent with zero but the stochastic noise is $t$-independent and $w_t \propto t^4$ such that it is difficult to identify a plateau region based only on this plot.
Resulting correlators and fit of \( C(t) + C_s(t) \) to \( c_\rho e^{-E_\rho t} + c_\phi e^{-E_\phi t} \) in the region \( t \in [t_{\text{min}}, \ldots, 17] \) with fixed energies \( E_\rho = 770 \text{ MeV} \) and \( E_\phi = 1020 \). \( C_s(t) \) is the strange connected correlator.

We fit to \( C(t) + C_s(t) \) instead of \( C(t) \) since the former has a spectral representation.

We could use this model alone for the long-distance tail to help identify a plateau but it would miss the two-pion tail.
We therefore additionally calculate the two-pion tail for the disconnected diagram in ChPT:
We then pick a point in the potential plateau region such as $T = 20$ and use a combined estimate of the resonance model and the two-pion tail to estimate $\sum_{t=T+1}^{\infty} \omega_t C(t)$ as a systematic uncertainty.

Combined with an estimate of discretization errors, we find

$$a_{\mu}^{\text{HVP (LO) DISC}} = -9.6(3.3)_{\text{stat}}(2.3)_{\text{sys}} \times 10^{-10}. \quad (7)$$
Status and prospects:

- Improved statistical estimators both for connected light and disconnected contributions at physical point.

- For the connected light contribution our method reduces noise in the long-distance part of the correlator by an order of magnitude compared to same-cost $Z_2$ sources.

- For the disconnected contributions our method allowed for a precise calculation at physical pion mass with a total of 45 propagators on 20 configurations.
Disconnected HVP contribution at physical point calculated at $\Delta a_{\mu}^{\text{HVP, disc}} = 4 \times 10^{-10}$ (BNL E821 has $\Delta a_{\mu} = 6.3 \times 10^{-10}$, FNAL E989 aims at $\Delta a_{\mu} = 1.6 \times 10^{-10}$). Further improvements with our methodology straightforward.

A combined analysis of lattice and R-ratio data yields a very precise result with current data (0.7% statistical uncertainty); $O(1\%)$ continuum limit lattice-only result is also within reach.

Continuum limit, charm contribution, QED/isospin-breaking corrections are still missing.

Finite-volume behavior so far is consistent with NLO FV ChPT prediction within errors, designated study of fixed $T$ and different spatial volume in progress.
Thank you
A closer look at the NLO FV ChPT prediction (1-loop sQED):

We show the partial sum $\sum_{t=0}^{T} w_t C(t)$ for different geometries and volumes:
The dispersive approach to HVP LO

The dispersion relation

\[ \Pi_{\mu\nu}(q) = i \left( q_{\mu}q_{\nu} - g_{\mu\nu}q^2 \right) \Pi(q^2) \]

\[ \Pi(q^2) = -\frac{q^2}{\pi} \int_{4m_{\pi}^2}^{\infty} ds \frac{\text{Im}\Pi(s)}{s} \frac{1}{q^2 - s} . \]

allows for the determination of \( a_{\mu}^{\text{HVP}} \) from experimental data via

\[ a_{\mu}^{\text{HVP LO}} = \left( \frac{\alpha m_{\mu}}{3\pi} \right)^2 \left[ \int_{4m_{\pi}^2}^{E_0^2} ds \frac{R_{\gamma}(s)\hat{K}(s)}{s^2} + \int_{E_0^2}^{\infty} ds \frac{R_{\gamma}^{\text{QCD}}(s)\hat{K}(s)}{s^2} \right] , \]

\[ R_{\gamma}(s) = \sigma^{(0)}(e^+e^- \to \gamma^* \to \text{hadrons})/\frac{4\pi\alpha^2}{3s} . \]

Experimentally with or without additional hard photon (ISR: \( e^+e^- \to \gamma^* (\to \text{hadrons})\gamma \))
Experimental setup: muon storage ring with tuned momentum of muons to cancel leading coupling to electric field

\[
\tilde{\omega}_a = -\frac{q}{m} \left[ a_\mu \vec{B} - \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \frac{\vec{\beta} \times \vec{E}}{c} \right]
\]

Because of parity violation in weak decay of muon, a correlation between muon spin and decay electron direction exists, which can be used to measure the anomalous precession frequency \( \omega_a \):
FIGURE 7: Our calculation of the leading-order (LO) hadronic vacuum polarization contributions to $(g-2)_\mu$ in the energy range 600 - 900 MeV from BESIII and based on the data from KLOE 08 [6], 10 [7], 12 [8], and BaBar [10], with the statistical and systematic errors. The statistical and systematic errors are added quadratically. The band shows the range of the BESIII result.
### Jegerlehner FCCP2015 summary:

<table>
<thead>
<tr>
<th>final state range (GeV)</th>
<th>$a_\mu^{\text{had}(1)} \times 10^{10}$</th>
<th>rel</th>
<th>abs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$ (0.28, 1.05)</td>
<td>507.55 (0.39) (2.68)[2.71]</td>
<td>0.5%</td>
<td>39.9%</td>
</tr>
<tr>
<td>$\omega$ (0.42, 0.81)</td>
<td>35.23 (0.42) (0.95)[1.04]</td>
<td>3.0%</td>
<td>5.9%</td>
</tr>
<tr>
<td>$\phi$ (1.00, 1.04)</td>
<td>34.31 (0.48) (0.79)[0.92]</td>
<td>2.7%</td>
<td>4.7%</td>
</tr>
<tr>
<td>$J/\psi$</td>
<td>8.94 (0.42) (0.41)[0.59]</td>
<td>6.6%</td>
<td>1.9%</td>
</tr>
<tr>
<td>$\Upsilon$</td>
<td>0.11 (0.00) (0.01)[0.01]</td>
<td>6.8%</td>
<td>0.0%</td>
</tr>
<tr>
<td>had (1.05, 2.00)</td>
<td>60.45 (0.21) (2.80)[2.80]</td>
<td>4.6%</td>
<td>42.9%</td>
</tr>
<tr>
<td>had (2.00, 3.10)</td>
<td>21.63 (0.12) (0.92)[0.93]</td>
<td>4.3%</td>
<td>4.7%</td>
</tr>
<tr>
<td>had (3.10, 3.60)</td>
<td>3.77 (0.03) (0.10)[0.10]</td>
<td>2.8%</td>
<td>0.1%</td>
</tr>
<tr>
<td>had (3.60, 9.46)</td>
<td>13.77 (0.04) (0.01)[0.04]</td>
<td>0.3%</td>
<td>0.0%</td>
</tr>
<tr>
<td>had (9.46, 13.00)</td>
<td>1.28 (0.01) (0.07)[0.07]</td>
<td>5.4%</td>
<td>0.0%</td>
</tr>
<tr>
<td>pQCD (13.0, $\infty$)</td>
<td>1.53 (0.00) (0.00)[0.00]</td>
<td>0.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>data (0.28, 13.00)</td>
<td>687.06 (0.89) (4.19)[4.28]</td>
<td>0.6%</td>
<td>0.0%</td>
</tr>
<tr>
<td>total</td>
<td>688.59 (0.89) (4.19)[4.28]</td>
<td>0.6%</td>
<td>100.0%</td>
</tr>
</tbody>
</table>

Results for $a_\mu^{\text{had}(1)} \times 10^{10}$. **Update August 2015**, incl SCAN[NSK]+ISR[KLOE10,KLOE12,BaBar,BESIII]

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**F. Jegerlehner FCCP 2015, Capri, Sept. 10-12, 2015**
Jegerlehner FCCP2015 summary ($\tau \leftrightarrow e^+ e^-$):

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Excl. $\tau$ (1σ)</th>
<th>Incl. $\tau$ (1σ)</th>
<th>All (1σ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NSK ($e^+ e^-$)</td>
<td>177.8 ± 6.9</td>
<td></td>
<td>[3.3 σ]</td>
</tr>
<tr>
<td>NSK+KLOE ($e^+ e^-$)</td>
<td>173.8 ± 6.6</td>
<td></td>
<td>[3.9 σ]</td>
</tr>
<tr>
<td>NSK+BaBar ($e^+ e^-$)</td>
<td>181.7 ± 6.3</td>
<td></td>
<td>[3.1 σ]</td>
</tr>
<tr>
<td>NSK+BESIII ($e^+ e^-$)</td>
<td>177.6 ± 6.8</td>
<td></td>
<td>[3.4 σ]</td>
</tr>
<tr>
<td>ALL ($e^+ e^-$)</td>
<td>177.8 ± 6.2</td>
<td></td>
<td>[3.5 σ]</td>
</tr>
<tr>
<td>NSK ($e^+ e^-+\tau$)</td>
<td>178.1 ± 5.9</td>
<td></td>
<td>[3.6 σ]</td>
</tr>
<tr>
<td>NSK+KLOE ($e^+ e^-+\tau$)</td>
<td>174.1 ± 5.6</td>
<td></td>
<td>[4.1 σ]</td>
</tr>
<tr>
<td>NSK+BaBar ($e^+ e^-+\tau$)</td>
<td>182.0 ± 5.4</td>
<td></td>
<td>[3.3 σ]</td>
</tr>
<tr>
<td>NSK+BESIII ($e^+ e^-+\tau$)</td>
<td>177.9 ± 5.8</td>
<td></td>
<td>[3.7 σ]</td>
</tr>
<tr>
<td>ALL ($e^+ e^-+\tau$)</td>
<td>178.1 ± 5.3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

BNL-E821 (world average) 208.9 ± 6.3

$\mu \times 10^{10} - 11659000$

$J_{e\tau}$
The anomalous magnetic moment $a$ can be expressed in terms of scattering of particle off a classical photon background

For external photon index $\mu$ with momentum $q$ the scattering amplitude can be generally written as

$$(-ie) \left[ \gamma_\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q^\nu}{2m} F_2(q^2) \right]$$

with $F_2(0) = a$. 

\[ (8) \]
The muon anomalous magnetic moment promises to be useful to discover new physics beyond the standard model (SM) of particle physics.

In general, new physics contributions to $a_\ell$ are given by $a_\ell - a_\ell^{\text{SM}} \propto (m_\ell^2/\Lambda_{\text{NP}}^2)$ for lepton $\ell = e, \mu, \tau$ and new physics scale $\Lambda_{\text{NP}}$.

With $\ell = \tau$ being experimentally inaccessible, $\ell = \mu$ promises good sensitivity to new physics.

Example contributions: one-loop MSSM neutralino/smuon and chargino/sneutrino contributions to $a_\mu$. 

![Diagram of one-loop MSSM neutralino/smuon and chargino/sneutrino contributions to $a_\mu$]