Up and down quark masses and corrections to Dashen’s theorem from lattice QCD and quenched QED

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The main contribution to hadron masses comes from the energy associated with the non-perturbative QCD interactions inside the particles. It is common to perform lattice computations at a common light quark mass $m_{ud} = \frac{1}{2}(m_u + m_d)$ and a strange quark mass $m_s$. Such calculations are commonly referred to as $N_f = 2 + 1$.

We have to consider $m_u \neq m_d$ and include QED in order to properly account for isospin breaking.
Dashens’ theorem [1] states that in the $SU(3)$ flavor symmetric limit the electromagnetic mass splittings of the pseudoscalar mesons behave as

$$
(M_{\pi^\pm}^2 - M_{\pi^0}^2)_{EM} = (M_{K^\pm}^2 - M_{K^0}^2)_{EM},
$$

$$
(M_{\pi^0}^2)_{EM} = (M_{K^0}^2)_{EM},
$$

Violations to Dashens’ theorem can be parametrized by

$$
\varepsilon = \frac{\Delta_{QED} M_K^2 - \Delta_{QED} M_{\pi}^2}{\Delta M_{\pi}^2}
$$

The $K$ mass splitting can be at leading $\mathcal{O}(\alpha, \delta m)$ order expanded:

$$
\Delta M_K^2 = C_K \alpha + D'_K \delta m
$$

where $\delta m = m_u - m_d$. Knowing $\Delta_{QED} M_K^2$ and the PQ$\chi$PT parameter $B_2$ we could determine $\delta m$.

Technical details of the lattice calculation:

- Tree-Level improved Symanzik gauge action.
- Tree-Level Clover-improved Wilson fermion action with two levels of gauge smearing.
- Coulomb gauge for the non-compact $U(1)$ gauge fields.
- Zero mode subtraction for the $U(1)$ field according to $a^4 \sum_x A_\mu(x) = 0$

In this work: Two approximations:

\[ \text{QCD} \leftrightarrow \text{QED} \]
Finite-Volume-Corrections

QED in a box with periodic boundary condition $\rightarrow$ mirror particle. Full calculation for e.g. charged scalar particles yields ($\kappa = 2.837\ldots$) [1,2]

$$\frac{M^L}{M_\infty} = -\frac{\kappa}{M_\infty L} \left[ 1 + \frac{2}{M_\infty L} \left( 1 - \frac{\pi}{2\kappa} \frac{T}{L} \right) \right] + \frac{1}{L^3}$$

QCD has a mass gap $\rightarrow$ Finite volume corrections to masses scale exponential with the decay length determined by the lightest propagating degree of freedom. E.g. for Pion

$$\frac{M^L_\pi}{M_\pi^\infty} = 1 + c \sqrt{\frac{M_\pi}{L^3}} \exp(-M_\pi L)$$


Finite-Volume-Corrections

\[
\frac{M^L}{M^\infty} = -\frac{\kappa M^\infty}{L} \left[ 1 + \frac{2}{M^\infty L} \left( 1 - \frac{\pi}{2\kappa} \frac{T}{L} \right) \right] + c \frac{1}{L^3}
\]

First two orders are universal, third order is structure dependent.

The behavior depends on the dimensionless parameter \( T/L \). This is related to the zero mode subtraction scheme chosen.

Faded points: No finite volume correction, Solid points: First two orders corrected.
The fit function

If one defines

\[ \Delta M = M_{uu}^2 - M_{dd}^2 \]
\[ \Delta M_K^2 = M_{K+}^2 - M_{K0}^2 \]

one can write

\[ \Delta M_K^2 = C_K \alpha + D_K \Delta M^2 \]

Here \( C_K \) parametrizes the QED contributions to the Kaon isospin splitting and \( D_K \) parametrizes the QCD contributions.

\[ C_K = c_0 + c_1 M_\pi^2 + c_2 (M_K^2 - 0.5 M_\pi^2) + c_3 a + c_4 \frac{1}{L^3} \]

\[ D_K = d_0 + d_1 M_\pi^2 + d_2 (M_K^2 - 0.5 M_\pi^2) + d_3 f(a) \]

with \( f(a) \) being either \( a^2 \) or \( \alpha a \).
The fit function

A fully correlated fit is performed:

\[ M_{\pi^+}^2 \ [\text{MeV}^2] \propto m_{ud} \]

\[ \alpha C_K \ [\text{MeV}^2] \]

\( \beta = 3.31, a = 0.114 \text{ fm} \)

\( \beta = 3.50, a = 0.088 \text{ fm} \)

\( \beta = 3.61, a = 0.075 \text{ fm} \)

\( \beta = 3.70, a = 0.064 \text{ fm} \)

\( \beta = 3.80, a = 0.053 \text{ fm} \)
The fit function

\[ M_{\pi^+}^2 \propto m_{ud} \]

\[ M_{K_X}^2 \propto m_{ud} \]

\[ \Delta M(\Phi)^2 D_K \]

\[ \alpha C_K \]

\[ \beta = 3.31, a = 0.114 \text{ fm} \]

\[ \beta = 3.50, a = 0.088 \text{ fm} \]

\[ \beta = 3.61, a = 0.075 \text{ fm} \]

\[ \beta = 3.70, a = 0.064 \text{ fm} \]

\[ \beta = 3.80, a = 0.053 \text{ fm} \]
To estimate systematic errors all higher order terms that can not be properly dealt with are treated in two different ways:

- Two different plateau ranges for mass extractions.
- Scale setting either with $\Omega$ or $\Xi^-$.  
- Different cuts on $M^2_\pi$:
  - 400 MeV or 450 MeV for the scale setting.
  - 350 MeV or 400 MeV for $\Delta M^2_K$.
- Replacing Taylor expansions in $C_K$ and $D_K$ by Padé approximations.
- Using $\alpha a$ or $a^2$ continuum extrapolation for the QCD part.

Resulting in 128 different full analysis. Systematic error is the spread of these analysis. Each analysis is weighted with its $Q$ value.

The statistical error is estimated with a 2000 samples bootstrap procedure.
Results

\[ \varepsilon = \frac{\Delta_{\text{QED}} M_K^2 - \Delta_{\text{QED}} M_\pi^2}{\Delta M_\pi^2} = \frac{\Delta_{\text{QED}} M_K^2 - \Delta M_\pi^2}{\Delta M_\pi^2} \]

at leading order in \( \delta m \) one can show that \( \Delta_{\text{QED}} M_\pi^2 = \Delta M_\pi^2 \). \( \Delta_{\text{QED}} M_\pi^2 \) is expensive to calcualte but \( \Delta M_\pi^2 \) is known experimentally.

We can use \( \Delta_{\text{QED}} M_K^2 = \alpha(\phi) C_K \) to calculate \( \varepsilon \):

\[ \varepsilon = 0.73(2)(5)(17) \]

where the errors are statistical, lattice systematics and quenching uncertainty (10\% in \( \Delta_{\text{QED}} M_K^2 \))

Violation of Dashen’s theorem

\[ \varepsilon = 0.73(2)(5)(17) \]
We use \( \Delta M^2 = 2B_2\delta m + \mathcal{O}(m_{ud}\alpha, m_{ud}\delta m, \alpha^2, \alpha\delta m, \delta m^2) \) with \( B_2 \) from [1] to get

\[
\delta m = \frac{\Delta M}{2B_2} = -2.41(6)(4)(9) \text{ MeV}
\]

Using \( m_{ud} \) from [2] we arrive at

\[
\frac{m_u}{m_d} = 0.485(11)(8)(14)
\]


Using $m_s/m_{ud} = 27.53(20)(8)$ [1] we can calculate the flavor braking ratios $R$ and $Q$:

**Flavor breaking ratios**

\[
R = \frac{m_s - m_{ud}}{m_d - m_s} = 38.2(1.1)(0.8)(1.4)
\]

\[
Q = \sqrt{\frac{m_s^2 - m_{ud}^2}{m_d^2 - m_u^2}} = 23.4(0.4)(0.3)(0.4)
\]

We calculated the violation to Dashens’ theorem
\[ \varepsilon = 0.73(2)(5)(17) \]
and the light quark mass ratio
\[ \frac{m_u}{m_d} = 0.485(11)(8)(14). \]
All systematics from the lattice calculation have been systematically estimated. The quenching uncertainty is taken into account.
\( m_u = 0 \) can be excluded with 24\( \sigma \) by these results.