The large N limit of the topological susceptibility of Yang-Mills gauge theory

arXiv:1607.05939

Marco Cè, **Miguel García Vera**, Leonardo Giusti and Stefan Schaefer

34th International Symposium on Lattice Field Theory
Southamptom, 24-30 July 2016

The $U(1)_A$ problem

Chiral symmetry breaking:
$SU(3)_L \times SU(3)_R \to SU(3)_V \to$ 8 Goldstone bosons π, K, η

What happens with $U(1)_A$?

Could the η' be the Goldstone boson associated to this symmetry?

$m_{NG} < \sqrt{3} m_\pi$, but $m_{\eta'} \approx 958\,\mathrm{MeV}$ [Weinberg (1975)]

The symmetry is explicitly broken by an anomaly: $\partial_\mu J_{\mu 5} = -i \frac{N_f g^2}{16\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}$

[Adler, Bell (1969)]

Witten - Veneziano, 1979 (Based on the $N \to \infty$, $g^2 N$ fixed limit)
At leading order in the $1/N$ expansion:

$\chi \neq 0$ for pure YM theoy, but $\chi = 0$ when massless fermions are added (?)

Solution: $m^2_{\eta'} \propto 1/N$, $\partial_\mu J_{\mu 5} \propto 1/N \to \eta'$ is a Goldstone boson at large N

$$\lim_{N\to\infty} m^2_{\eta'} = \lim_{N\to\infty} \frac{4 N_f}{f_\pi^2} \chi_{\mathrm{YM}}$$

Objective

Our goal: Compute the large N limit of χ_{YM}

Previous work:

- Cooling methods [Lucini et al. (2001), Del Debbio et al. (2002), Lucini et al. (2005)].
- Definition of χ using the index of the Dirac operator [Cundy et al. (2002)] $\longrightarrow$ expensive.
- Periodic boundary conditions (PBC) $\rightarrow$ large autocorrelations when approaching the continuum and large N limits [Del Debbio et al. (2002)].

This work:

- We use the theoretically clean definition of χ based on the Yang-Mills gradient flow [Narayanan, Neuberger (2006), Lüscher (2010)].
- We use open boundary conditions (OBC) to avoid the freezing of topology near the continuum [Lüscher, Schaefer (2010)].

Observables

The topological susceptibility χ^t at flow time t is defined as the two point function of the topological charge density $q^t(x)$

$$\chi^t = \int d^4x \left\langle q^t(x) q^t(0) \right\rangle$$

Provides a correct field theoretical definition of χ in the continuum [Cè et al. (2015)].

- Topological charge density

$$q^t(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\, G_{\mu\nu}(x) G_{\rho\sigma}(x)$$

- Yang-Mills Energy density

$$e^t(x) = \frac{1}{2} \mathrm{Tr}\, G_{\mu\nu}(x) G_{\rho\sigma}(x)$$

We use the clover definition of $G_{\mu\nu}$ on the lattice.

Definition of t_0

We want to compute the dimensionless quantity $t_0^2 \chi_{\mathrm{YM}}$.

In $\mathrm{SU}(3)$, the reference flow time t_0 is defined implicitly by the equation:

$$t^2 \langle e^t \rangle_{t=t_0} = 0.3$$

For general gauge group $\mathrm{SU}(N)$:

$$t^2 \langle e^t \rangle = \frac{3\left(N^2-1\right)}{128\pi^2 N} \lambda_t(q) \left[1 + c_1 \, \lambda_t(q) + \mathrm{O}\left(\lambda_t(q)^2\right)\right]$$

where $\lambda_t(q) = g^2(q) N$ at the scale $q = (8t)^{-1/2}$.

We define the scale t_0 as:

$$t^2 \langle e^t \rangle_{t=t_0} = 0.1125 \, \frac{\left(N^2-1\right)}{N}$$

Computation of the topological susceptibility with OBC

Why use OBC? $\rightarrow$ The freezing of the topology is worse at larger N

[Del Debbio et al. (2002), Amato et al. (2015)]

OBC have been shown to reduce τ_{int} for the slowly decaying topological modes

[Lüscher (2011,2013), Amato et al. (2015)]

With PBC:

$\chi = \langle Q^2 \rangle / V$ $\rightarrow$ Not possible with OBC as translation invariance in broken in the time direction.

With OBC: $\bar{q}^t(x_0) = \sum_{\vec{x}} q^t(\vec{x}, x_0)$

[Bruno et al. (2014)]

$$\bar{C}^t(\Delta) = \frac{1}{(T - 2d - \Delta)\, L^3} \sum_{x_0=d}^{T-1-d-\Delta} \left\langle \bar{q}^t(x_0)\bar{q}^t(x_0 + \Delta) \right\rangle$$

$$\chi^t_{\mathrm{YM}}(r) = \bar{C}^t(0) + 2 \sum_{\Delta=a}^{r} \bar{C}^t(\Delta)$$

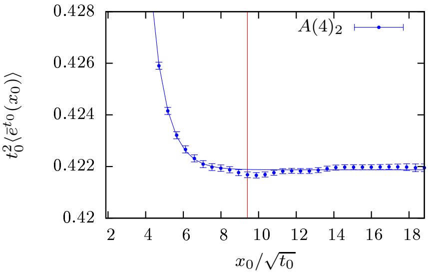
Ensembles

#run	N	T/a	L/a	a[fm]	#meas.	#it.
$A(4)_1$	4	64	16	0.096	22k	40
$A(4)_2$	4	80	20	0.078	41k	80
$A(4)_3$	4	96	24	0.065	21k	160
$A(5)_1$	5	64	16	0.095	15k	120
$A(5)_2$	5	80	20	0.077	27k	240
$A(5)_3$	5	96	24	0.064	14k	480
$A(6)_1$	6	64	16	0.095	30k	250
$A(6)_2$	6	80	20	0.076	17k	500
$A(6)_3$	6	96	24	0.063	16k	450

Table: The approximate lattice spacing using $\sqrt{t_0} = 0.166\,\mathrm{fm}$.

- 1 it. correspond to $\mathrm{n_{ov}} \propto a^{-1}$ overrelaxation sweeps followed by one heatbath sweep.
- The updates are done using the Cabibbo-Marinari stategy updating all the $N(N-1)/2\ \mathrm{SU}(2)$ subgroups of $\mathrm{SU}(N)$.

Open boundary effects



Sufficiently far away from the boundaries, observables assume their vacuum expectation values up to small exponential corrections.

We fit the data to a one excited state contribution from the boundary:

$f(x_0) = A + Be^{-mx_0}$

Plateau region:

$|f(d) - A| < 0.25\sigma$

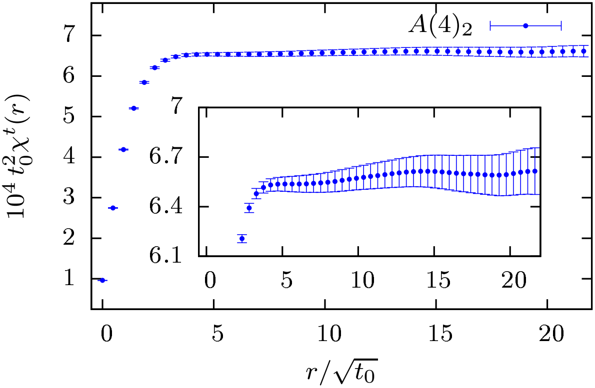
$$d_e = 9.5\sqrt{t_0}$$
$$d_\chi = 7.5\sqrt{t_0}$$

For both e and χ, the plateau region is larger or equal than $T/2a$.

Systematics from our definition of χ

Is it reasonable to compute $\sum_{\Delta=0}^{r} \langle \bar{q}^t(0)\bar{q}^t(\Delta)\rangle$ up $r = T - 2d$?

[Bazavov et al. (2010), Bruno et al. (2014)]

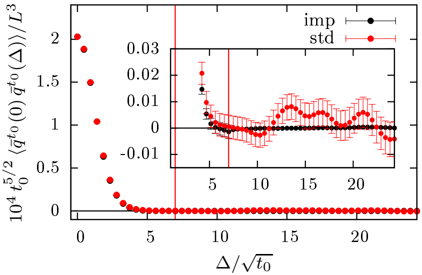


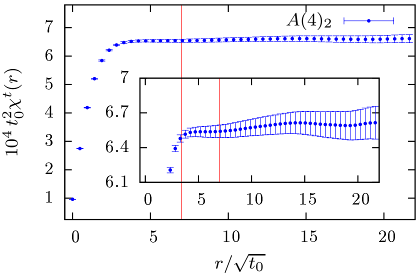
Systematics from our definition of χ

Is it reasonable to compute $\sum_{\Delta=0}^{r} \langle \bar{q}^t(0)\bar{q}^t(\Delta)\rangle$ up $r = T - 2d$?

[Bazavov et al. (2010), Bruno et al. (2014)]

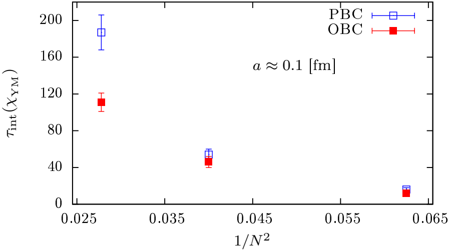
$SU(3)$, $\beta = 6.11$, $t_0 = 4.5776(15)$

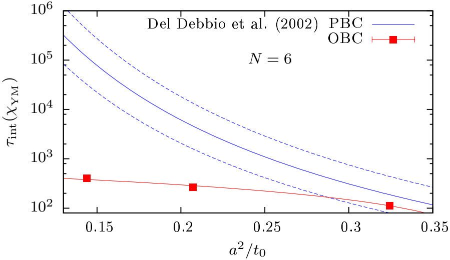


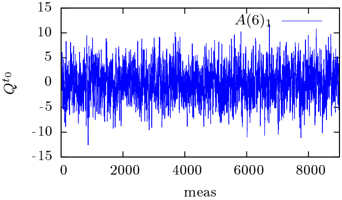


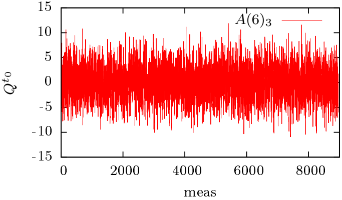
Using multilevel algorithms [MGV, Schaefer (2016)] $\rightarrow$ $r = 7.0\sqrt{t_0}$

Autocorrelations





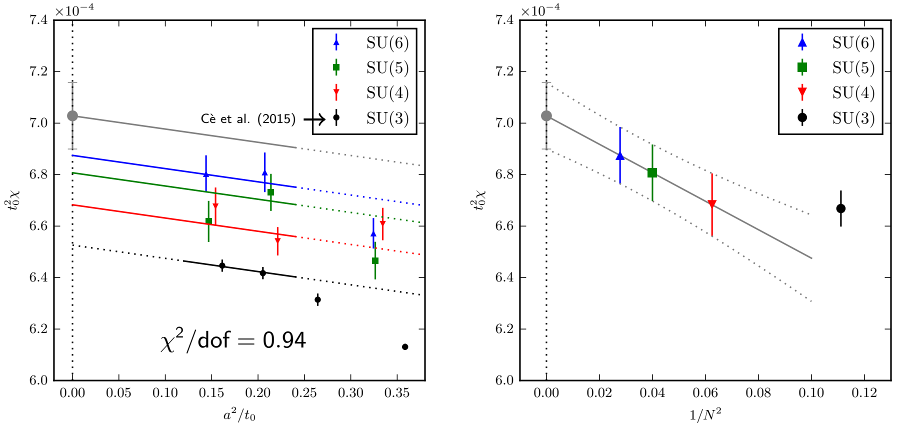




Simulations at fine lattice spacings are only possible due to the use of OBC.

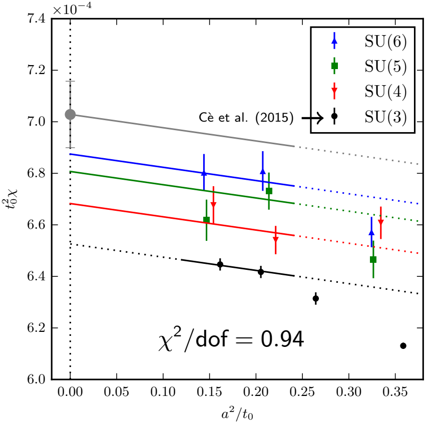
Large-N and continuum limits

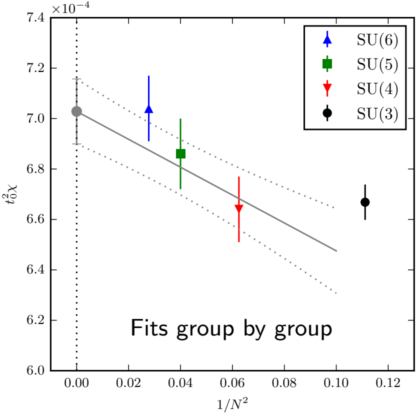
$$t_0^2\chi_{\rm YM}(1/N,a) = \begin{cases} \chi_{\rm YM}(0,0) + c_1\frac{1}{N^2} + c_2\frac{a^2}{t_0} & \text{if} \quad N > 3 \\ c_3 + c_2\frac{a^2}{t_0} & \text{if} \quad N = 3 \end{cases}$$



Large-N and continuum limits

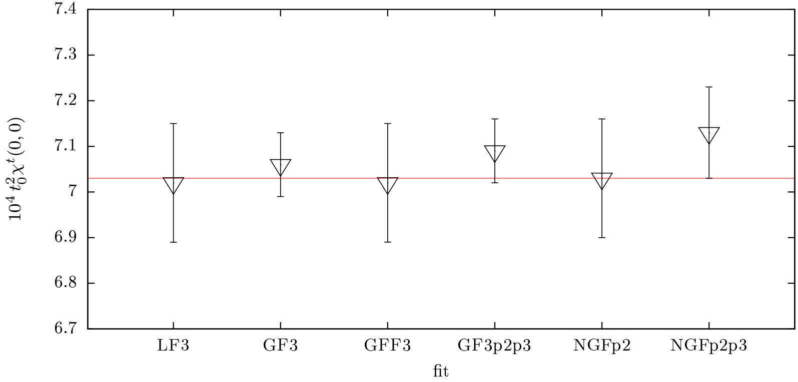
$$t_0^2\chi_{\rm YM}(1/N,a) = \begin{cases} \chi_{\rm YM}(0,0) + c_1\frac{1}{N^2} + c_2\frac{a^2}{t_0} & \text{if} \quad N>3 \\ c_3 + c_2\frac{a^2}{t_0} & \text{if} \quad N=3 \end{cases}$$





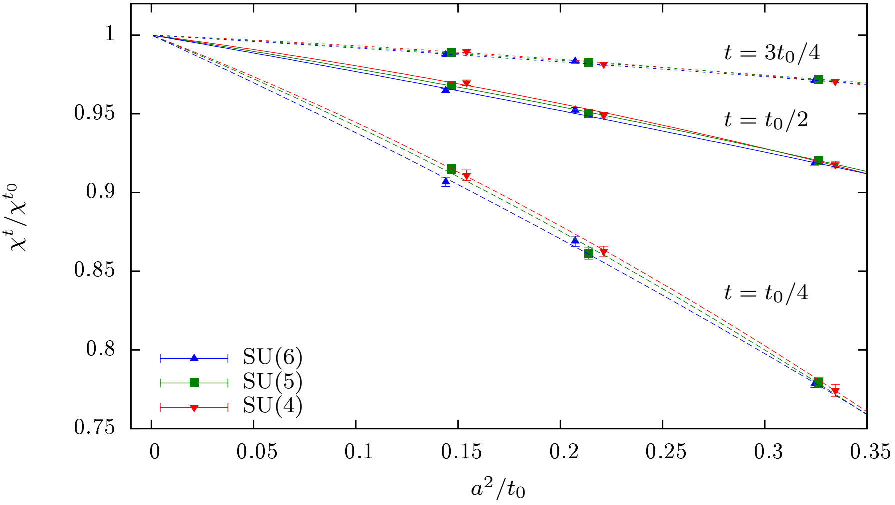
Large-N and continuum limits

Different fit strategies give compatible results.



$$t_0^2 \chi_{\mathrm{YM}}(0,0) = 7.03(13) \times 10^{-4}$$

t dependent discretization effects



Conclusions

- We have computed the large N limit of $\chi_{\rm YM}$ with an unprecedented accuracy thanks to a solid definition through the YM Gradient flow and the use of open boundary conditions (OBC).
- By using OBC we were able to go to finer lattice spacings and keep the autocorrelations under control.
- Through a careful study of all the systematic effects we quote a result in the large N and continuum limit with a percent level accuracy.
- The value computed for $t_0^2\chi_{\rm YM} = 7.03(13) \times 10^{-4}$ is a new verification of the Witten-Veneziano relation that gives mass to the η' meson.
- We find the large N effects to be small at our level of accuracy.

Thank you very much for your attention!