Electromagnetic pion form factor near physical point in $N_f = 2 + 1$ lattice QCD

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(PACS Collaboration)
Motivation

- Electromagnetic form factor
  \( = \) deviation from charged point particle
  \( \rightarrow \) Hadron’s structure (cf. charge radius)

However, errors of charge radii in lattice calculation are larger than experimental one

- chiral extrapolation \( \rightarrow \) error increases
- small momentum transfer and suppressing finite size effect
  \( \rightarrow \) need large box size

Purpose

calculation near physical point

\[ m_\pi = 0.145 \text{(GeV)}, \ L = 8.1 \text{(fm)} \]
Electromagnetic form factor (ff)

Definition \[ \langle \pi^+(p_f) | V_\mu | \pi^+(p_i) \rangle = (p_f + p_i)_\mu f_{\pi\pi}(q^2) \]

Electromagnetic current \[ V_\mu = \sum_f Q_f \bar{\psi}_f \gamma_\mu \psi_f \quad \psi_f = u, d, s \quad Q_f: \text{charge of flavor } f \]

\[ f_{\pi\pi}(0) = 1 \quad (\text{normalization condition relating to pion’s charge}) \]

\[ q^2 = -(p_f - p_i)^2 \geq 0 \quad (\text{space-like momentum transfer } \rightarrow \text{calculating ff directly}) \]

calculating 3-pt function and 2-pt function by Lattice QCD

\[ C_{\pi\pi} = Z_V \left\langle 0 | O_\pi(t_f, \vec{p}_f = \vec{0}) V_4(t, \vec{p} = \vec{p}_f - \vec{p}_i) O_\pi^\dagger(0, \vec{p}_i) | 0 \right\rangle \]
\[ = Z_V \frac{Z_\pi(0)Z_\pi(p)}{4E_\pi(p)m_\pi} \langle \pi(0) | V_4(0, p) | \pi(p) \rangle e^{-E_\pi(p)t} e^{-m_\pi(t_f - t)} \]
\[ = Z_V \frac{Z_\pi(0)Z_\pi(p)}{4E_\pi(p)m_\pi} (E_\pi(p) + m_\pi) f_{\pi\pi}(q^2) e^{-E_\pi(p)t} e^{-m_\pi(t_f - t)} \]

\[ R(t, p) = \frac{2m_\pi Z_\pi(0)}{(E_\pi(p) + m_\pi)Z_\pi(p)} \frac{C_{\pi\pi} V_\pi(t, t_f; p)}{C_{\pi\pi} V_\pi(t, t_f, 0)} e^{(E_\pi(p) - m_\pi)t} \]

in \( 0 \ll t \ll t_f \) we can extract form factor from \( R \)
Calculation 3-point function

- connected 3-point function is
  \[ C_{\pi\pi} = Z_V \langle 0 | O_\pi(t_f, \vec{p}_f) V_4(t, \vec{q}) O_\pi^\dagger(t_i = 0, \vec{p}_i) | 0 \rangle \]
  consist of 3 quark propagators

- 1 random wall source
  \[ \vec{p}_i = 0 \]

- 2 sequential source
  \[ \vec{p}_f = 0 \]

- 3 random wall source
  \[ \vec{p}_i \neq 0 \]

random wall source \((A,B:\text{color\&spinor index})\)

\[
\eta_B(\vec{y}, t_i) = \left\{ \frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}} \right\} \lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N} \eta_A^j(\vec{x}, t_i) \eta_B^{ij}(\vec{y}, t_i) = \delta(\vec{x} - \vec{y}) \delta_{AB}
\]

\[ \in \mathbb{Z}(2) \otimes \mathbb{Z}(2) \]

calculation cost is reduced by random wall source

RBC&UKQCD: JHEP( 0807 (2008) 112)

disconnected term is vanished by charge symmetry
Simulation details
All the results are preliminary

gauge configuration (HPCI Strategic Program Field 5)

\[ m_\pi = 0.145 \text{(GeV)}, \quad L = 8.1 \text{(fm)} \]
\[ m_\pi L \approx 6 \quad (\kappa_{ud}, \kappa_s) = (0.126117, 0.124790) \]
\[ L^3 \times T = 96^3 \times 96 \quad a^{-1} = 2.333 \text{(GeV)} \rightarrow a = 0.084 \text{(fm)} \]
\[ m_K \approx 0.525 \text{GeV} \]
\[ N_f = 2 + 1 \quad \text{Iwasaki gauge + stout smeared link Wilson clover action} \]
\[ \beta = 1.82, \quad n_{stout} = 6, \quad \rho = 0.1, \quad c_{sw} = 1.11 \]

measurement parameter

40 configs in total at present

4 sources \times 4 \text{ directions (x, y, z, t)} \times

2 random sources = 32 meas. per config.

periodic boundary condition for all directions

\[ t_f - t_i = 36 \]

\[
\vec{\mathbf{p}} = \frac{2\pi}{L} \cdot \hat{n}
\]

resources
PRIMRGY cx400 (tatara), RIIT, Kyusyu University and HAPACS, CCS, University of Tsukuba
preliminary result: effective energy

\[ q^2 = 2m_\pi (E_\pi - m_\pi) \]

\[ q^2(GeV^2) \]
\[
\begin{array}{|c|c|}
\hline
n & q^2(GeV^2) \\
\hline
n = 1 & 0.01909(7) \\
\hline
n = 2 & 0.03363(22) \\
\hline
n = 3 & 0.04580(45) \\
\hline
n = 4 & 0.05633(54) \\
\hline
n = 5 & 0.06688(62) \\
\hline
n = 6 & 0.07593(115) \\
\hline
\end{array}
\]

previous studies used Twisted Boundary Condition (TBC)
for small momentum transfers
→ we obtain small momentum transfers without TBC
(no extra finite size effect by TBC and not partially quenched QCD)
Extraction of form factor

heavy pion mass
$m_\pi = 0.51\text{(GeV)}, \ L = 2.9\text{(fm)}$
when $0 \ll t \ll t_f$
ff is extracted by fitting plateau of R as constant

Diagrammatically,
one pion from source
the other pion from sink

(using periodic boundary condition in temporal direction)
While, light pion mass

\[ m_\pi = 0.145 \text{(GeV)}, \ L = 8.1 \text{(fm)} \]

t dependence appears in R

no plateau in R

we should not only take usual form factor but also other effect into account

pion wrapping around effect

the opposite direction pion from sink + pion from source + finite size effect of 2 pions

\[ e^{-E_\pi_f \times (T-t_f + t)} e^{-E_\pi_i (t-t_i)} e^{-\Delta E(t-t_i)} \]

another 2pions effect; source \( \Leftrightarrow \) sink

we fit $R$ with

$$f_0 + f_1 \times \frac{e^{-E_{2\pi} \times t} e^{-E_{\pi_f} \times (T-t_f)} - e^{-E_{2\pi} \times (t_f-t)} e^{-E_{\pi_i} \times (T-t_f)}}{e^{-E_{\pi_i} \times t} e^{-E_{\pi_f} \times (t_f-t)}}$$

form factor

$E_{2\pi} = E_{\pi_i} + E_{\pi_f} + \Delta E$

$E_{\pi_i}, E_{\pi_f}$: single pion energy

$\Delta E$: finite size effect by 2 pions interaction

thanks to large box calculation, we assume that finite size effect is negligible $\rightarrow \Delta E = 0$
preliminary result: analysis of form factor

fit form $f_0 + f_1 \times \frac{e^{-E_2 \pi \times t} e^{-E_{\pi f} \times (T-t_f)} - e^{-E_2 \pi \times (t_f-t)} e^{-E_{\pi i} \times (T-t_f)}}{e^{-E_{\pi i} \times t} e^{-E_{\pi f} \times (t_f-t)}}$

$$R(t, p) = \frac{2m_\pi Z_\pi(0)}{(E_\pi(p) + m_\pi)Z_\pi(p)} \frac{C_\pi V_\pi(t, t_f; \overrightarrow{p})}{C_\pi V_\pi(t, t_f; 0)} e^{(E_\pi(p) - m_\pi)t}$$

$q^2 = 0.01909 \text{GeV}^2$

red: fit including 2 pions effect
blue: form factor including 2 pions effect (f_0)
green: form factor with constant fit (f_0 only)

$q^2 = 0.03363 \text{GeV}^2$

fits including 2 pions effect are better than constant ones
preliminary result: form factor vs $q^2$

\[ f_{\pi\pi}(q^2) = \frac{1}{1 + q^2/M_{\text{pole}}^2} \]

\[ \langle r^2 \rangle = -6 \frac{d}{dq^2} f_{\pi\pi}(q^2) \bigg|_{q^2=0} \]

\[ = \frac{6}{M_{\text{pole}}^2} \]

\[ \langle r^2 \rangle_{\text{exp}} = 0.451(11)(f m^2) \]

(from PDG)

roughly consistent with experiment
preliminary result: NLO SU(2) ChPT fit

\[ f_{\pi\pi}(q^2) = 1 + \frac{1}{f^2} \left[ 2l_6q^2 + \frac{m^2H(x)}{8\pi^2} + \frac{q^2}{48\pi^2}\log\left( \frac{m^2}{\mu^2} \right) \right] \]

\[ \langle r^2 \rangle = \frac{-12l_6}{f^2} - \frac{1}{8\pi^2f^2} \left( \log \left( \frac{m_{\pi}^2}{\mu^2} \right) + 1 \right) \]

\[ H(x) = -\frac{4}{3} + \frac{5}{18}x - \frac{x-4}{6}\sqrt{\frac{x-4}{x}}\log\left( \frac{\sqrt{\frac{x-4}{x}} + 1}{\sqrt{\frac{x-4}{x}} - 1} \right) \quad (x = -\frac{q^2}{m^2}) \]

\[ \mu = m_{\rho} = 0.77 \text{(GeV)} \]

\[ f = 0.12925 \text{(GeV)} \quad (m_u, m_d \to 0) \]

there is one fit parameter

: Low Energy Constant \( l_6 \)

<table>
<thead>
<tr>
<th></th>
<th>( l_6 )</th>
<th>( \chi^2/d.o.f )</th>
</tr>
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<tbody>
<tr>
<td>2( \pi ) fit (n=6)</td>
<td>-0.01223(75)</td>
<td>0.42</td>
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<td>2( \pi ) fit (n=3)</td>
<td>-0.01234(72)</td>
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<td>FLAG(2015)</td>
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fit n=6: all 6 momentum transfers

n=3: smaller 3 momentum transfers

in FLAG’s result, we used

\[ f = 0.122553 \text{(GeV)} \]

\( (m_u, m_d \to 0) \)

ChPT fit works well

thanks to small \( q^2 \) and almost physical pion mass
Pion mass dependence of $\langle r^2 \rangle$

FLAG’s radius is estimated by SU(2)ChPT form with physical pion mass and
$f = 0.122553(\text{GeV})$
and
$l_6 = -0.01234(127)$

preliminary result $\langle r^2 \rangle = 0.408(21)(\text{fm}^2)$ (NLO SU(2) ChPT)
consistent with HPQCD and FLAG(2015)
comparison with other calculation at physical point

\[ m_\pi = 0.13957 \text{(GeV)} \]

preliminary result

\[ \langle r^2 \rangle = 0.412(21) \text{(fm}^2 \text{)} \]

extrapolation with NLO SU(2) ChPT formula

\[ \langle r^2 \rangle = \frac{-12l_6}{f^2} - \frac{1}{8\pi^2 f^2} \left( \log \left( \frac{m_\pi^2}{\mu^2} \right) + 1 \right) \]

consistent with previous results and roughly consist with experiment
conclusion
We calculated pion electromagnetic form factor in N=2+1 Lattice QCD

\( m_\pi = 0.145 \text{GeV} \)

• smaller momentum transfers by large box size
• fits including 2pions effect to extract form factor

preliminary result: \( \langle r^2 \rangle = 0.412(21)(\text{fm}^2) \)
(NLO SU(2) ChPT at physical point)

roughly consistent with experiment and consistent previous results

future works
• increasing statistics (~100 configs)
• analysis with other fit forms
  • NNLO SU(2) ChPT
  • NLO SU(3) ChPT
• reweighting strange quark mass +small extrapolation

\[ m_K = 0.525(\text{GeV}) \rightarrow m_K = 0.493(\text{GeV}) \]
back up
Charge radius

In non-relativistic limit, $f_f$ is regarded as 3D Fourier transformation of charge density.

Assuming spherical symmetry of density and $\int d^3x \rho_\pi(x) = 1$.

Expand $f_f$ by $\vec{q} \cdot \vec{x} \ll 1$

$$f_{\pi\pi}(q^2) = \int d^3x e^{i \vec{q} \cdot \vec{x}} \rho_\pi(\vec{x})$$

$$f_{\pi\pi}(q^2) = \int_0^\infty r^2 dr \int_{-1}^1 d(cos \theta) \int_0^{2\pi} d\phi (1 - \frac{1}{2} q^2 \cdot r^2 \cos^2 \theta + \ldots) \rho_\pi(r)$$

$$= 1 - \frac{1}{6} q^2 \int_0^\infty 4\pi r^2 dr r^2 \rho_\pi(r) + \ldots$$

$$f_{\pi\pi}(q^2) = 1 - \frac{\langle r^2_\pi \rangle}{6} q^2 + \ldots$$

We can consider mean square of charge radius as 1st differential coefficient of $f_f$:

$$\langle r^2_\pi \rangle = -6 \frac{d}{dq^2} f_{\pi\pi}(q^2) \bigg|_{q^2=0}$$

(so we need large box size for small momentum)
disconnect diagram of 3-point function

\[ C^{\text{disc}}_{\pi J_\mu \pi}(U) = \frac{1}{2} C^{\text{disc}}_{\pi J_\mu \pi}(U) + \frac{1}{2} C^{\text{disc}}_{\pi J_\mu \pi}(U^*) \]

from charge conjugation

\[ C\gamma_\mu C^{-1} = -\gamma^T_\mu, \quad CUC^{-1} = U^* = (U^\dagger)^T \]

\[ C^{\text{disc}}_{\pi J_\mu \pi}(U) = -C^{\text{disc}}_{\pi J_\mu \pi}(U^*) \]

\[ C^{\text{disc}}_{\pi J_\mu \pi}(U) = \frac{1}{2} C^{\text{disc}}_{\pi J_\mu \pi}(U) - \frac{1}{2} C^{\text{disc}}_{\pi J_\mu \pi}(U) = 0 \]

disconnected term is vanished by charge symmetry
R in Dirichlet BC

there is no t dependence in Dirichlet BC

\[ R(t,p) \]

- Dirichlet
  \[ 11 \text{conf}, q^2 \approx 0.019(\text{GeV}^2) \]
  \[ |t_f - t_i| = 25 \]

- periodic
  \[ 19 \text{conf}, q^2 \approx 0.019(\text{GeV}^2) \]
  \[ |t_f - t_i| = 36 \]
preliminary result: analysis of form factor

$q^2 = 0.04580(45)$

$q^2 = 0.05633(54)$

$q^2 = 0.06688(62)$

$q^2 = 0.07593(115)$
**preliminary result: charge radii & LECs**

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<td>const fit</td>
<td>0.366(16)</td>
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$\langle r^2 \rangle_{\text{exp}} = 0.451(11)(fm^2)$

$\langle r^2 \rangle_{\text{FLAG}} = 0.461(39)(fm^2)$

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FLAG’s radius is estimated by SU(2)ChPT form and $f = 0.122553(GeV)$

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the phenomenological result is estimated by $\tilde{l}_6 = 16.0(0.9)$

JHEP 9805, 014 (1998)

with conversion formula

$$l_6 = -\frac{1}{6 \times (4\pi)^2} \left( \tilde{l}_6 + \ln \left( \frac{m_{\text{phys}}}{\mu} \right)^2 \right)$$

and using $f = 0.12275(GeV)$

fit n=3: smaller 3 momentum transfers
n=6: all the data (6 momentum transfers)