# Temperature dependence of topological susceptibility using gradient flow

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for
WHOT QCD collaboration

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# Introduction

- Temperature dependence of topological susceptibility
  - axion cosmology
  - temperature dependence of axion mass

    Kitano-Yamada (2015), Bonati et.al(2016)

    Petreczky et.al(1606.03145), Frison et.al(1606.07175)

    K.Szabo (25Mon), S.Katz (25Mon), G.Martinelli (29Fri)

- Comparison between gauge and fermion definition
  - How to define topological charge density on lattice?

  Lüscher(82), (04), Phillips et.al(90)

  JL-QCD(15)

  Del Debbio et.al(05), Lüscher et.al(10), Ce et.al(15)

# How to calculate $\chi_T$ on lattice?

$$\chi_T = \frac{1}{V_4}\left(\langle Q^2\rangle - \langle Q\rangle^2\right)$$

Gauge definition

$$Q = \frac{1}{64\pi^2}\int d^4x\,\epsilon_{\mu\nu\rho\sigma}F^a_{\mu\nu}(x)F^a_{\rho\sigma}(x)$$

Fermion definition

$$\langle Q^2\rangle = \frac{m^2}{N_f^2}\left(\langle P^0P^0\rangle - N_f\langle P^aP^a\rangle\right) = \frac{m^2}{N_f^2}\langle P^0P^0\rangle_{\rm disc}$$

$$P^0 = \int d^4x\,\bar\psi(x)\gamma_5\psi(x) \qquad P^a = \int d^4x\,\bar\psi(x)T^a\gamma_5\psi(x)$$

$$\psi = \left(\psi_1,\ \cdots,\ \psi_{N_f}\right)$$

# A note on fermion definition

$$\langle Q^2 \rangle = \frac{m^2}{N_f^2}\left(\langle P^0 P^0 \rangle - N_f \langle P^a P^a \rangle\right) = \frac{m^2}{N_f^2}\langle P^0 P^0 \rangle_{\rm disc}$$

Use (integrated form of) Ward-Takahashi identity

non-Abelian $-2m\langle P^a \mathcal{O}\rangle = i\langle \delta^a \mathcal{O}\rangle$

Abelian $-2m\langle P^0 \mathcal{O}\rangle + 2N_f\langle Q\mathcal{O}\rangle = i\langle \delta^0 \mathcal{O}\rangle$

$\mathcal{O} = Q$ Abelian $N_f\langle Q^2\rangle = m\langle P^0 Q\rangle$

$\mathcal{O} = P^0$ Abelian $N_f\langle QP^0\rangle = m\langle P^0 P^0\rangle - \langle S^0\rangle$

$\mathcal{O} = P^b$ NA $-2m\langle P^a P^b\rangle = i\langle \delta^a P^b\rangle = -\left(\delta^{ab}\frac{2}{N_f}\langle S^0\rangle + d_{abc}\langle S^c\rangle\right)$

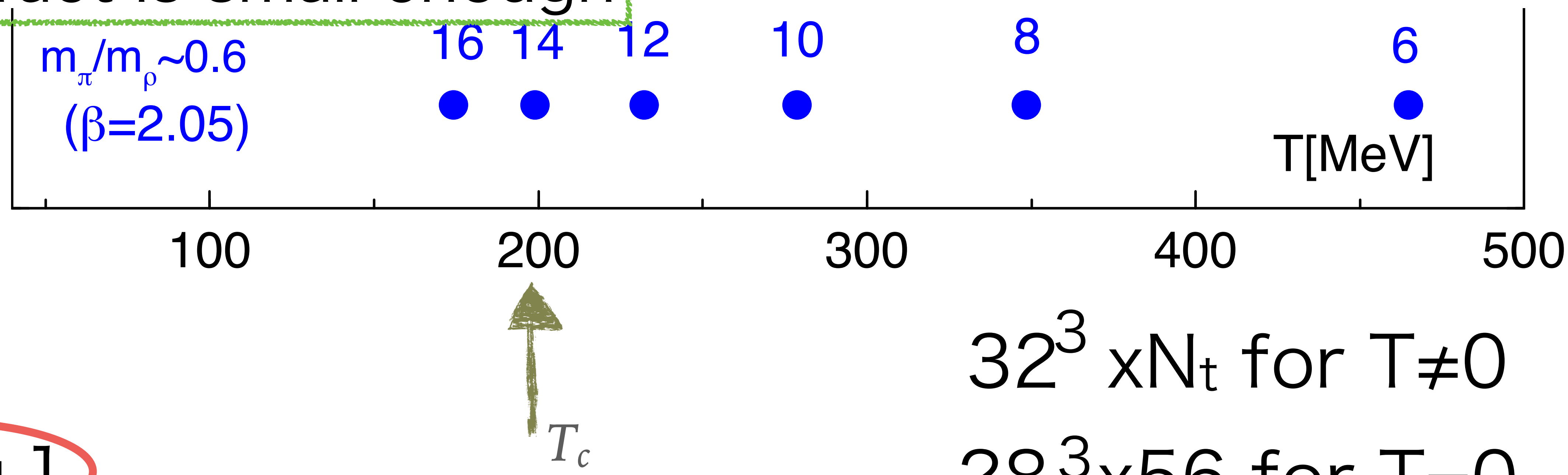
singlet scalar ($\langle S^0\rangle$)

non-singlet ($\langle S^c\rangle$) → 0

# Numerical setups

- Iwasaki gauge action
  - $\beta$=2.05 : a∼0.07 [fm], 1/a∼2.79[GeV]
- Fixed scale method    aT~1/Nt artifact may be severe
  - T=1/(aNt), Nt=16, 14, 12, 10, 8, 6, 4

aT~1/Nt artifact is small enough




$32^3$ xNt for T≠0

$28^3$x56 for T=0

- Nf=2+1
- NP improved Wilson fermion
- On an equal quark mass line

$$\frac{m_\pi}{m_\rho} \sim 0.6 \qquad \frac{m_{\eta_{ss}}}{m_\phi} \sim 0.74$$

# Measurement of gauge definition

## Gradient Flow

Narayanan-Neuberger(2006)
Lüscher(2009–)

**Flow the gauge field**

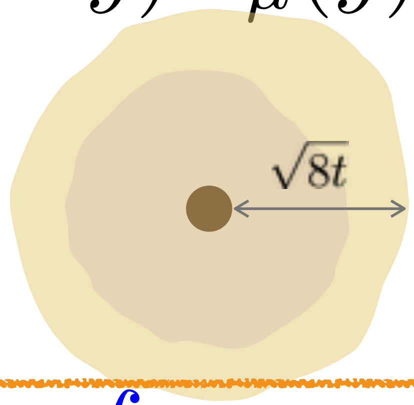
$$\partial_t A_\mu(t,x) = -\frac{\delta S_{\rm YM}}{\delta A_\mu} \qquad A_\mu(t=0,x) = A_\mu(x)$$

t: flow time, dim=[length$^2$]

**A kind of diffusion equation** $\partial_t A_\mu(t,x) = D_\nu G_{\nu\mu}$

**Solution** $A_\mu(t,x) = \int d^4y K_t(x-y)A_\mu(y) + \text{interactions}$

**heat kernel** $K_t(x) = \dfrac{e^{-x^2/4t}}{(4\pi t)^{D/2}}$




smear field within $\sqrt{8t}$

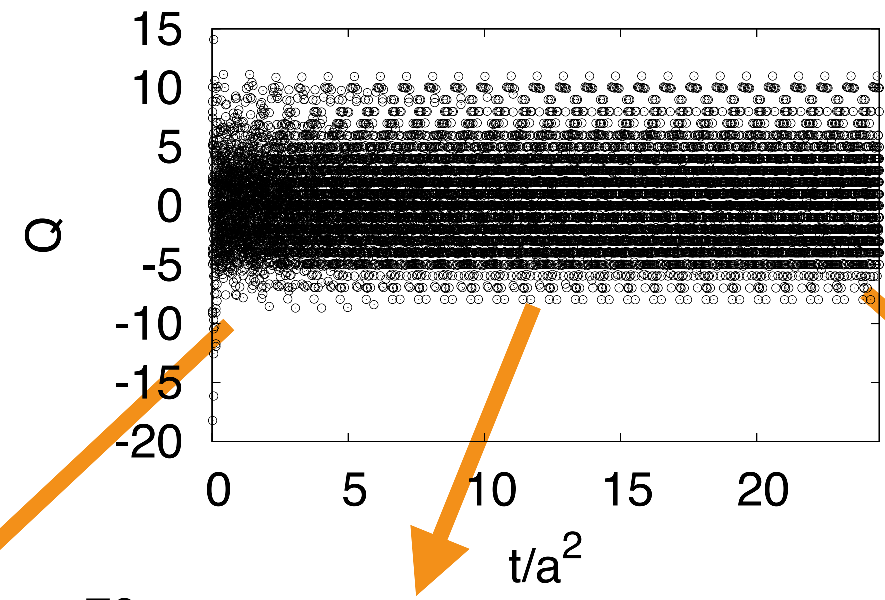
A good cooling step for $Q = \dfrac{1}{64\pi^2}\int d^4x \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$

# Results: Gauge definition

Flow of topological charge for all confs.

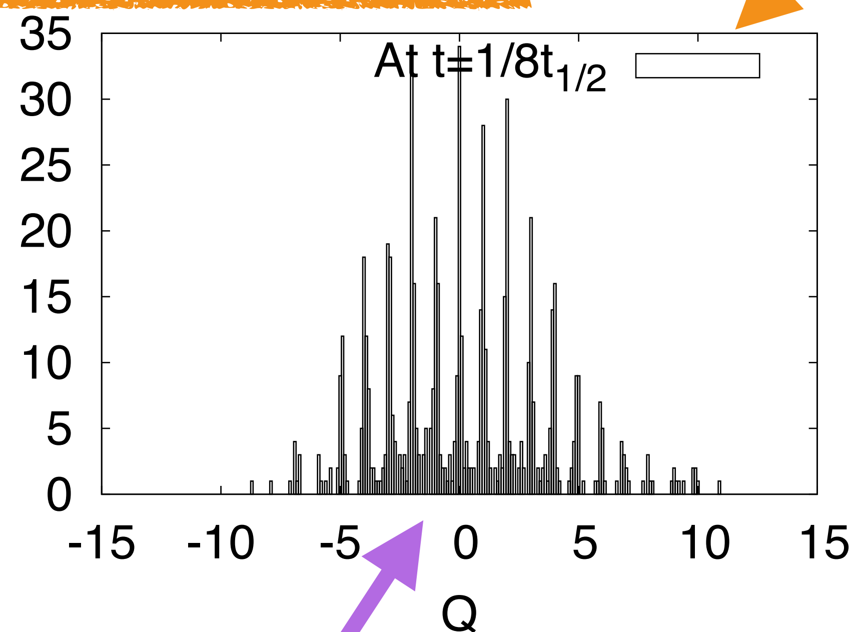
T/Tc=0

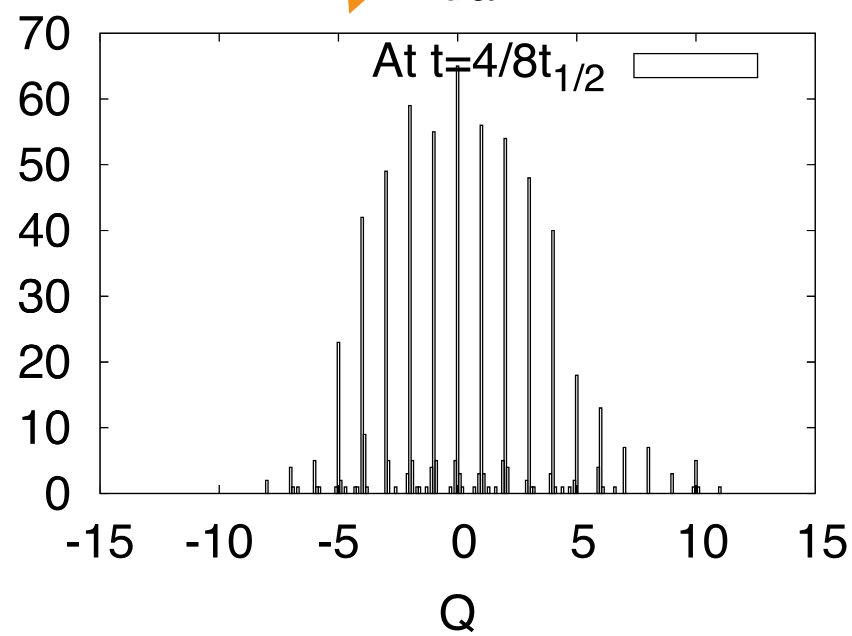
$$\sqrt{8t} = \frac{N_t}{2}$$

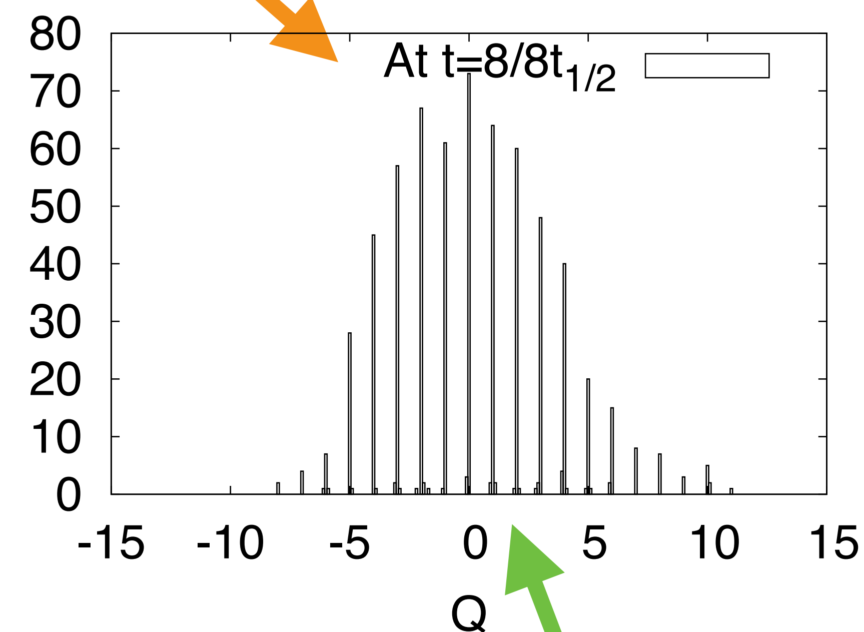



stop the flow before over smearing

Histogram




A lot of non-integer Q's

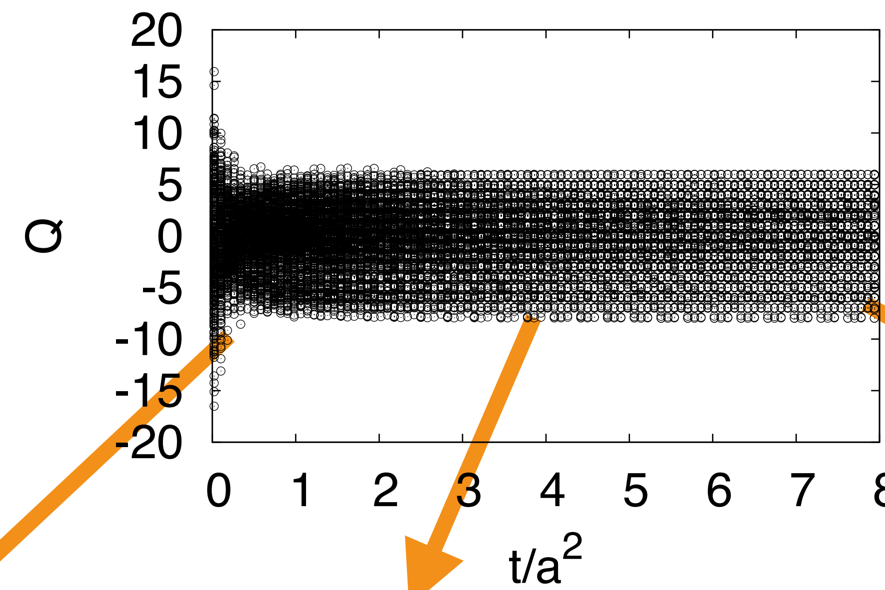
Works well as a cooling!

non-integer Q is rare.

# Results: Gauge definition

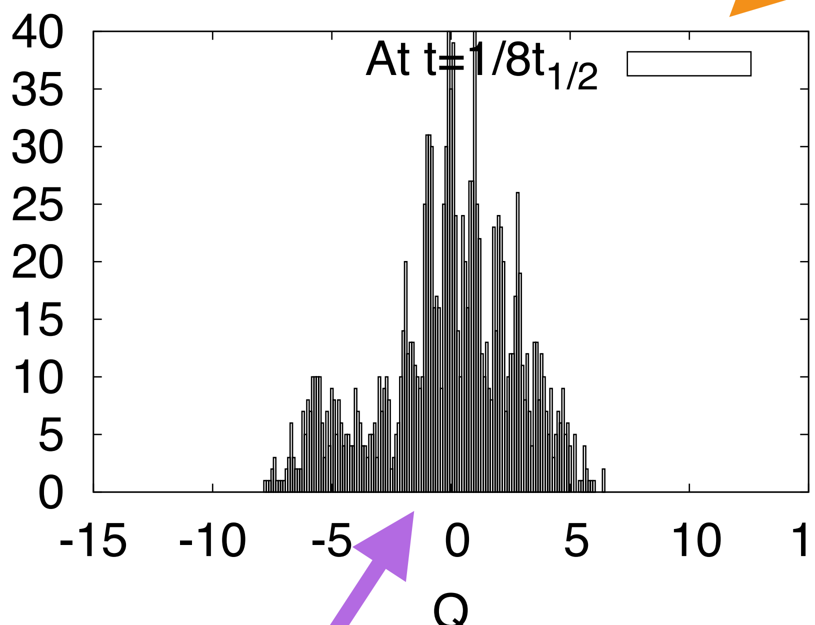
$T/T_c$=0.92

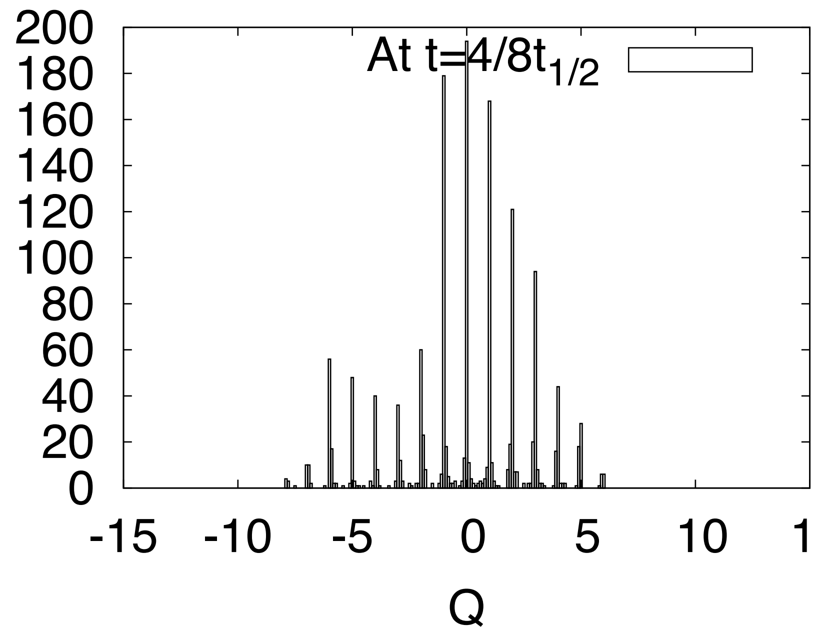
Flow of topological charge Q

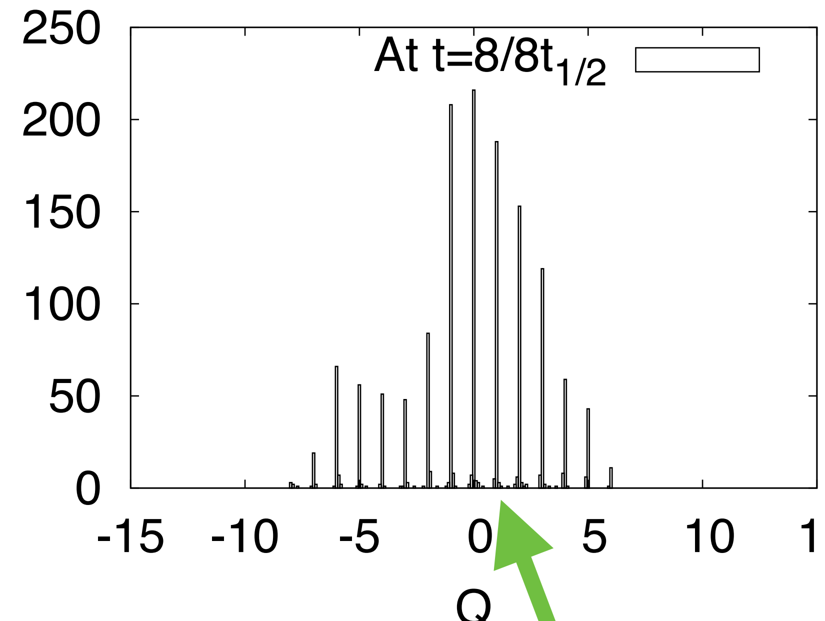



$$\sqrt{8t} = \frac{N_t}{2}$$

Histogram



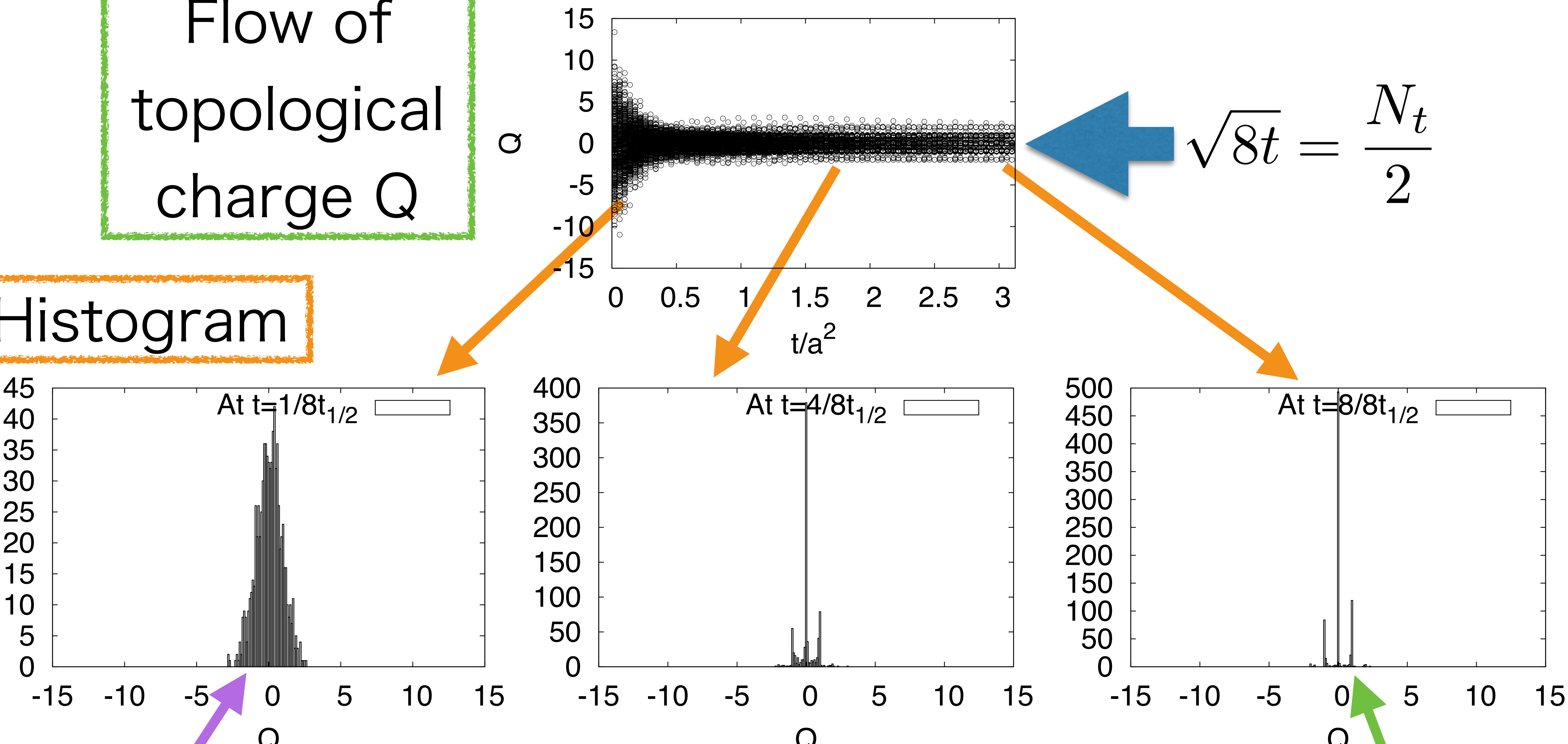

A lot of non-integer Q's

non-integer Q is rare.

# Results: Gauge definition

$T/T_c$=1.47

Flow of topological charge Q




$$\sqrt{8t} = \frac{N_t}{2}$$

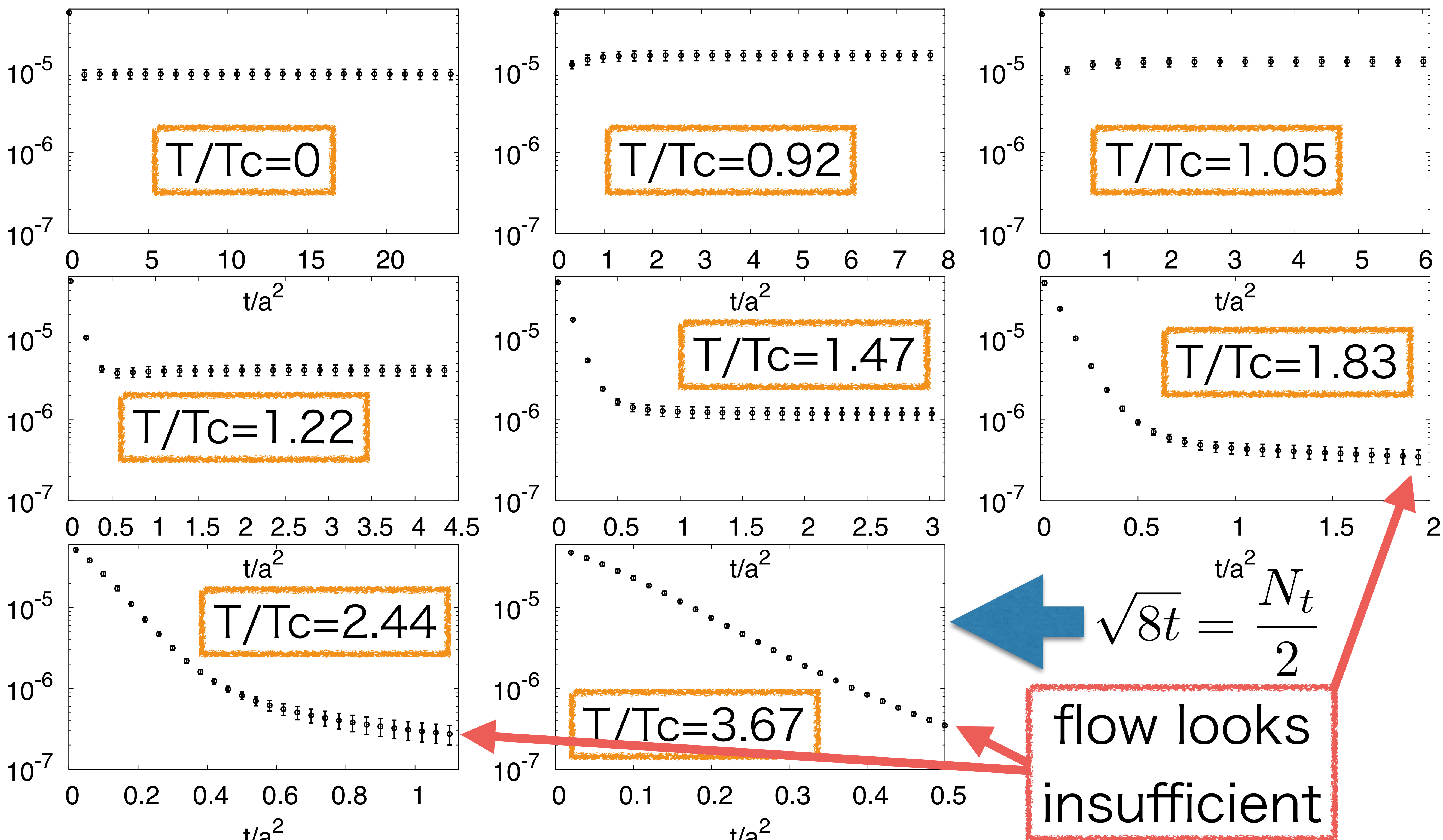
Histogram

A lot of non-integer Q's

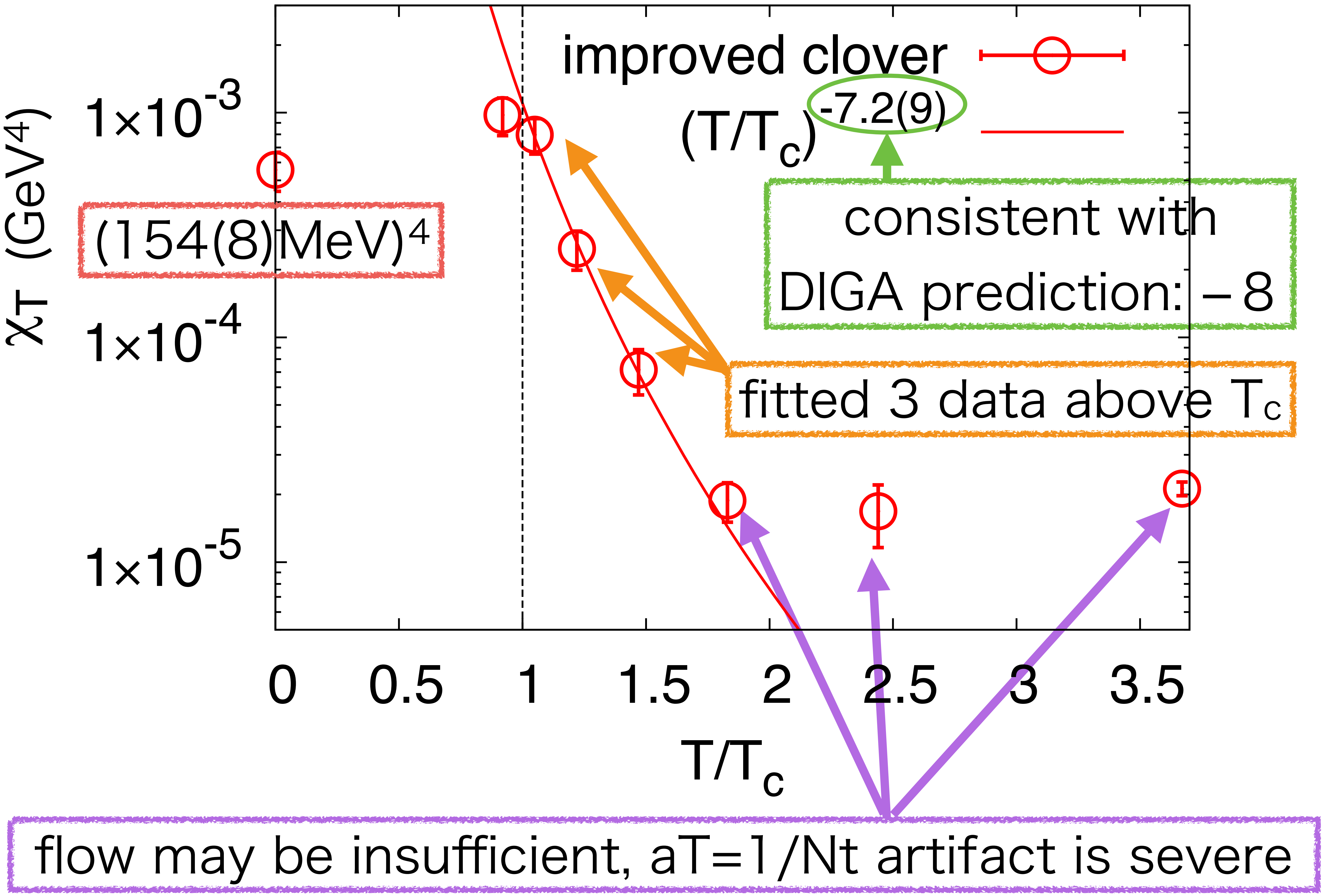
non-integer Q is rare.

# Results: Gauge definition

## Flow of topological susceptibility



# Results: Gauge definition



# Measurement of fermion definition

**Flow of quark field**

Lüscher, JHEP 1304, 123 (2013)

$$\partial_t \chi(t,x) = D_\mu D_\mu \chi(t,x) \qquad \chi(t=0,x) = \psi(x)$$

$$\partial_t \bar{\chi}(t,x) = \bar{\chi}(t,x) \overleftarrow{D}_\mu \overleftarrow{D}_\mu \qquad \bar{\chi}(t=0,x) = \bar{\psi}(x)$$

**flow the gauge field simultaneously**

**Renormalization is needed for quark field**

$$\chi_R(t,x) = Z_\chi \chi_0(t,x)$$

**No more renormalization is needed for composite op.**

$$\left(\bar{\chi}(t,x)\chi(t,x)\right)_R = Z_\chi^2 \left(\bar{\chi}(t,x)\chi(t,x)\right)_0$$

# Measurement of fermion definition

A great view point:

## Gradient Flow as a renormalization scheme

Narayanan-Neuberger(2006), Lüscher(2010), Lüscher-Weisz(2011)

Operators with flowed field $\chi(t,x),\ \bar{\chi}(t,x)$

- does not have contact term singularity
- already renormalized except for wave function renormalization

scale: $\sqrt{8t}$

absorbed

NP renormalized operator ← flow + a→0 ← lattice operator

$Z_\chi^2 \bar{\chi}(t,x)\gamma_5\chi(t,x)$ $\bar{\psi}(x)\gamma_5\psi(x)$

universal

finite ren.

$\overline{\text{MS}}$ scheme

differences in lattice action, lattice operator

# Measurement of fermion definition

From flowed operator to proper operator

NP renormalized in flow scheme

operator which satisfies chiral WT identity

does not satisfy chiral WT id.

$m_R \left(\bar{\psi}\gamma_5\psi\right)_R$ in WT id.

Need a matching factor like $Z_A$

Matching coefficient is calculable perturbatively for $t \to 0$ limit in any scheme

Hieda-Suzuki, arXiv:1606.04193

Adopt $\overline{\mathrm{MS}}$ scheme with scale matching $\mu = \frac{1}{\sqrt{8t}}$

# Measurement of fermion definition

**From flowed operator to proper operator**

$$m_R\left(\bar{\psi}\gamma_5\psi\right)_R=\lim_{t\to 0} c_S(t)\, m_{\overline{\rm MS}}(1/\sqrt{8t})\,\varphi(t)\,\bar{\chi}(t,x)\gamma_5\chi(t,x)$$

flowed operator

$$\varphi(t)=\frac{-6}{(4\pi)^2t^2\left\langle \bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\right\rangle_{T=0}}$$

- introduced to cancel wave function renormalization
- a part of flow scheme

$$c_S(t)=\left\{1+\frac{\bar{g}_{\overline{\rm MS}}(1/\sqrt{8t})^2}{(4\pi)^2}\left(4\gamma-8\ln 2+8+\frac{4}{3}\ln(432)\right)\right\}$$

- The matching coefficient

# Measurement of fermion definition

Only three steps to calculate $m^2\langle P^0 P^0 \rangle$

1. Flow the gauge and quark field

2. Calculate VEV of flowed operators

3. Multiply the coefficients and take $t \to 0$ limit

# Measurement of fermion definition

3. Multiply the coefficients and take t→0 limit

$$c_S(t) m_{\overline{\rm MS}}(1/\sqrt{8t})\varphi(t)\bar{\chi}(t,x)\gamma_5\chi(t,x) = m_R\left(\bar{\psi}\gamma_5\psi\right)_R$$

+t(dim. 6 operators) +O($t^2$) → 0

Possible form of lattice artifact

flowed operator on lattice

$$c_S(t) m_{\overline{\rm MS}}(1/\sqrt{8t})\varphi(t)\bar{\chi}(t,x,a)\gamma_5\chi(t,x,a)$$

$$= m_R\left(\bar{\psi}\gamma_5\psi\right)_R + \frac{a^2}{t}\text{(dim4 operator)}$$ → tamed at large t

+t(dim6 operator) → we need a window region

$$+O(a^2T^2, a^2m^2, a^2\Lambda_{\rm QCD}^2)$$ → need to take a→0 limit

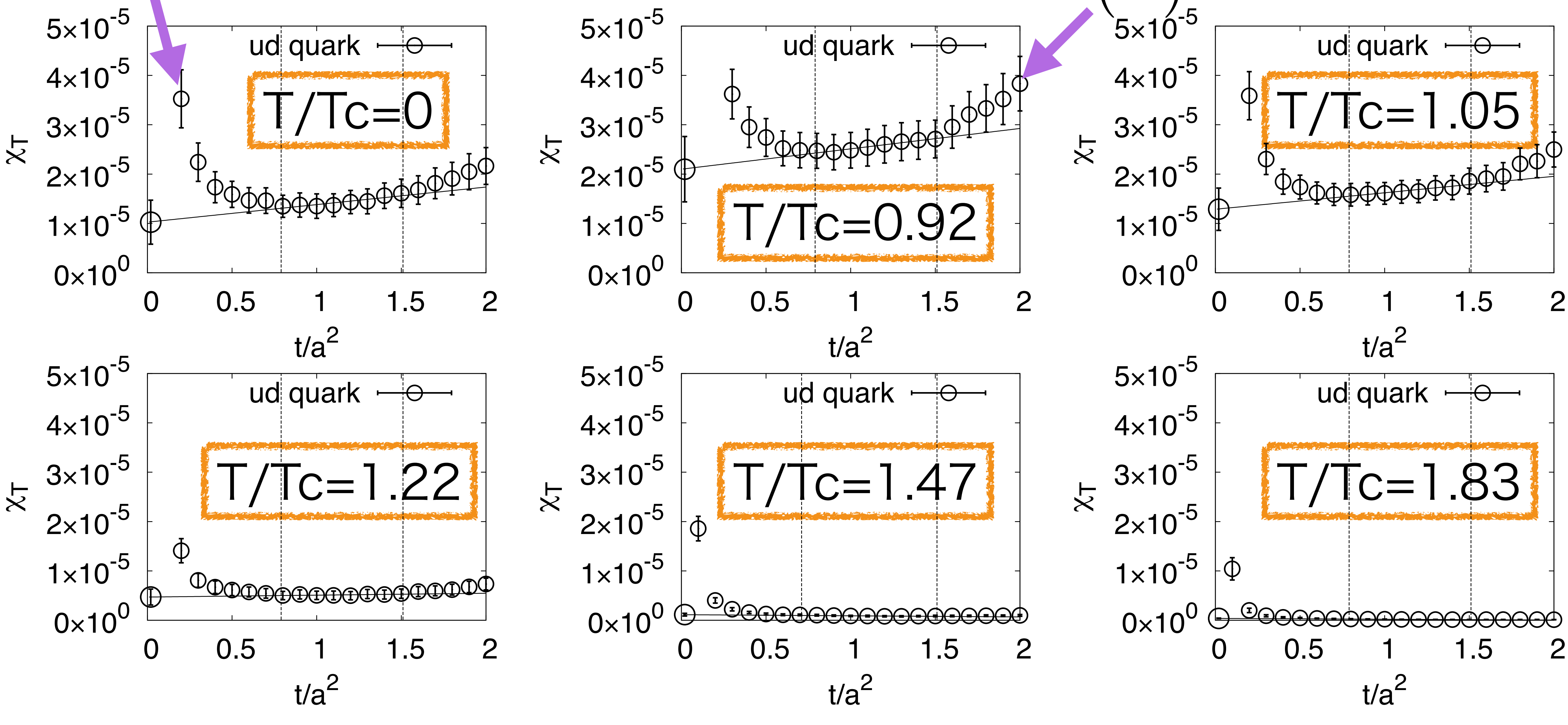
# Results: Fermion definition

$$\langle Q^2 \rangle = \frac{m^2}{N_f^2} \langle P^0 P^0 \rangle_{\text{disc}}$$

Applied to ud quarks system

$O\left(\frac{a^2}{t}\right)$

$O\left(t^2\right)$



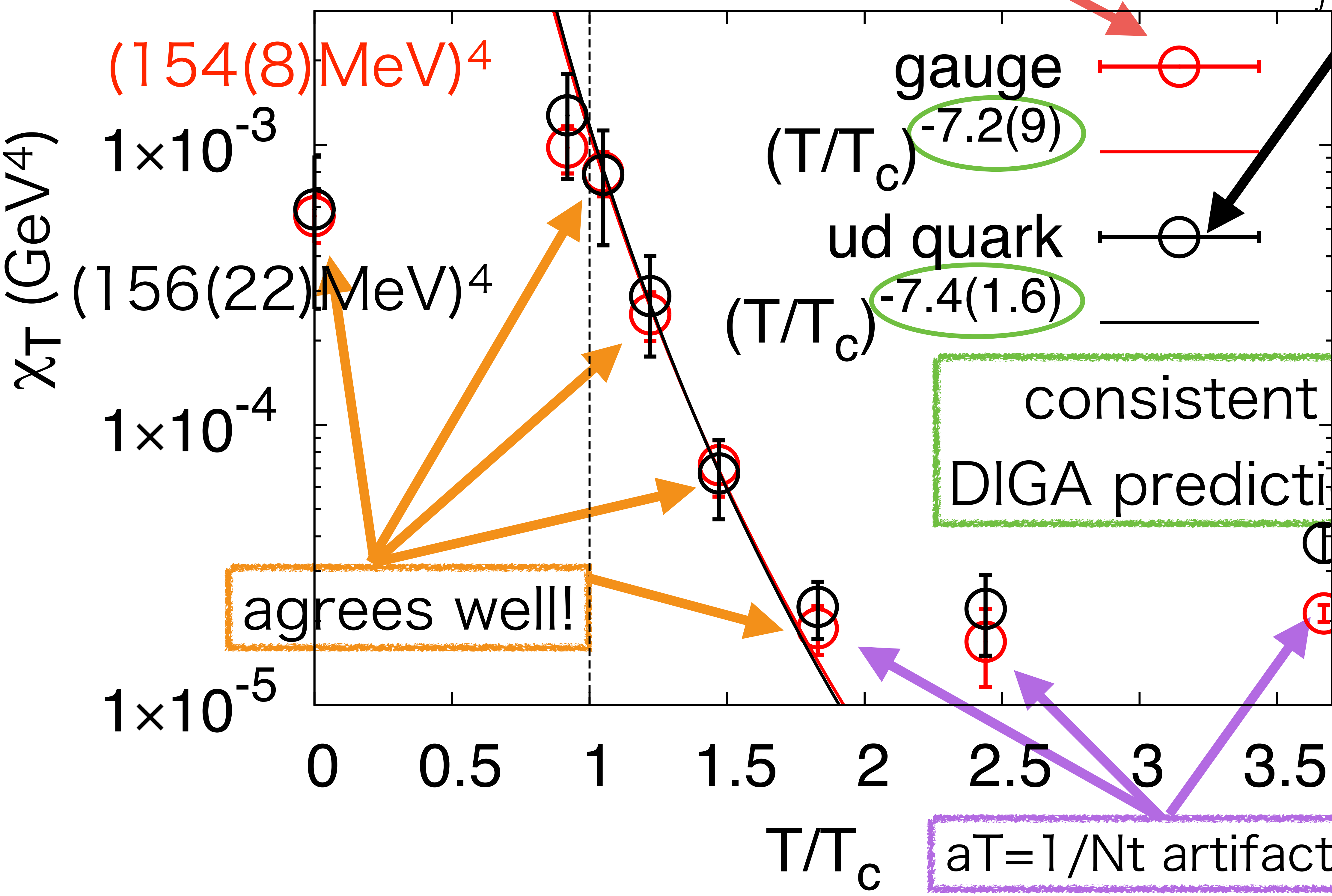
# Conclusion

$$\chi_T = \frac{1}{V_4}\left(\langle Q^2\rangle - \langle Q\rangle^2\right)$$

$$Q = \frac{1}{64\pi^2}\int d^4x \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$$

singularity free definition

$$\langle Q^2\rangle = \frac{m^2}{N_f^2}\langle P^0 P^0\rangle_{\rm disc}$$



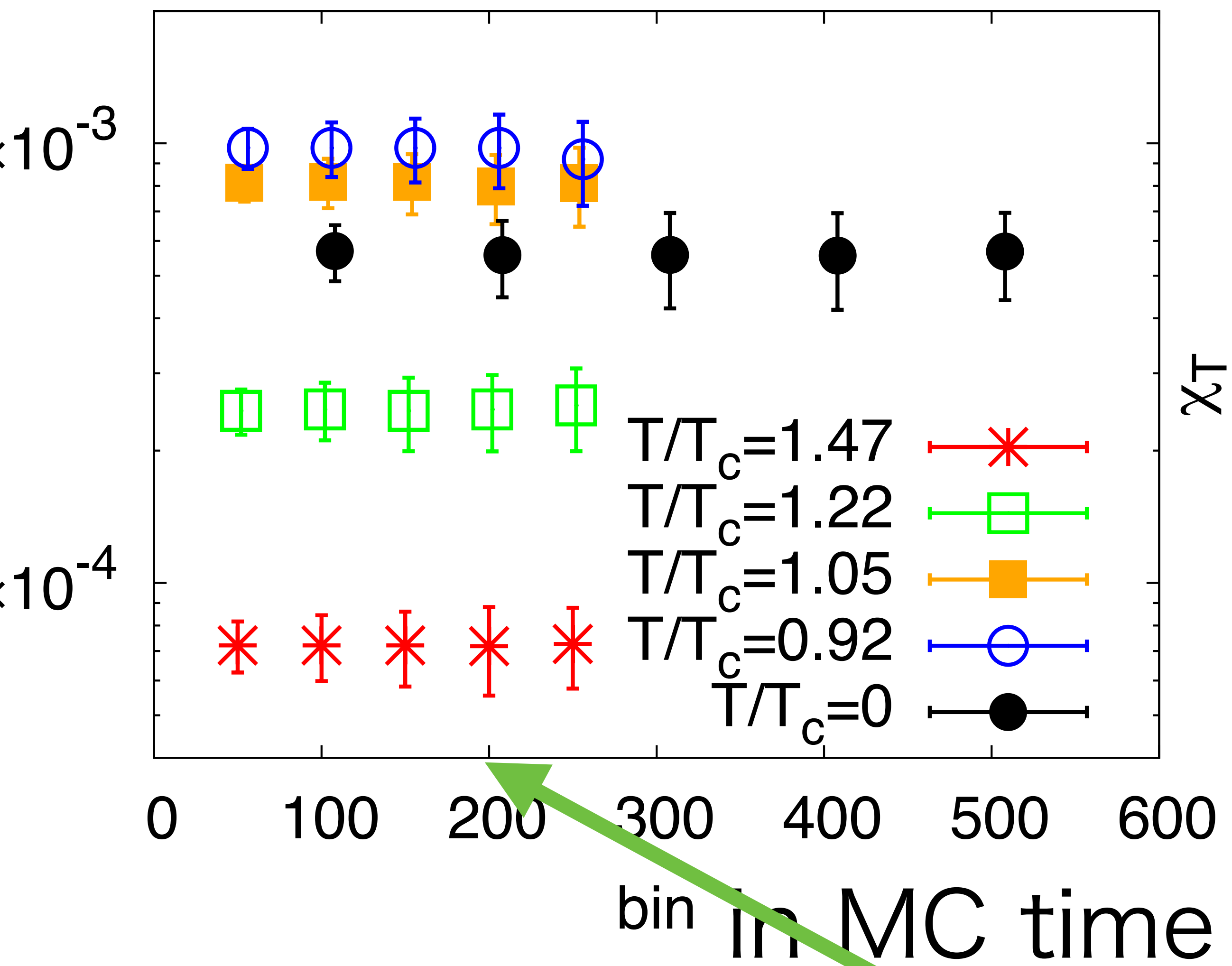

consistent with DIGA prediction: −8

agrees well!

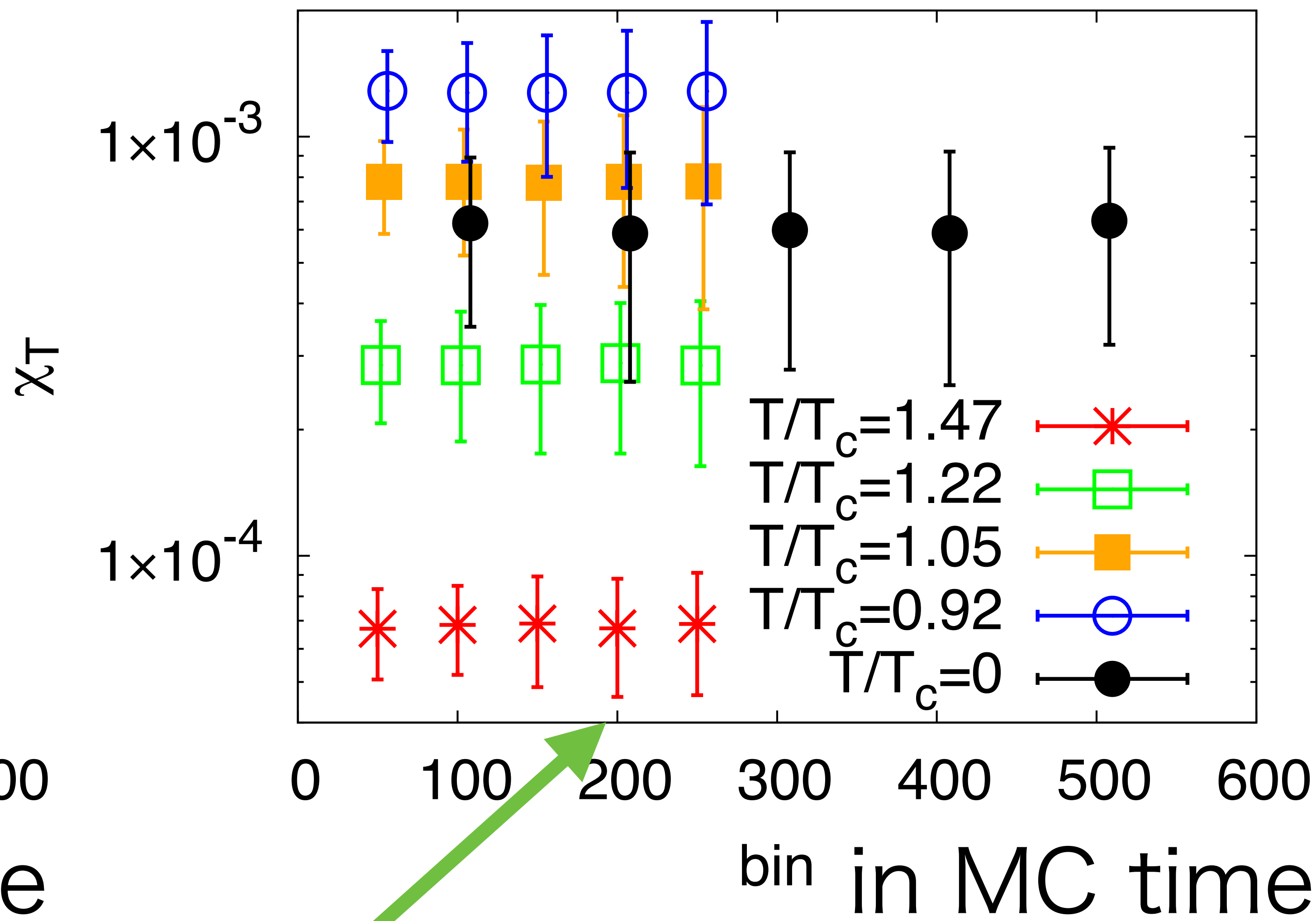
aT=1/Nt artifact is severe

# Autocorrelation: bin size analysis

gauge definition



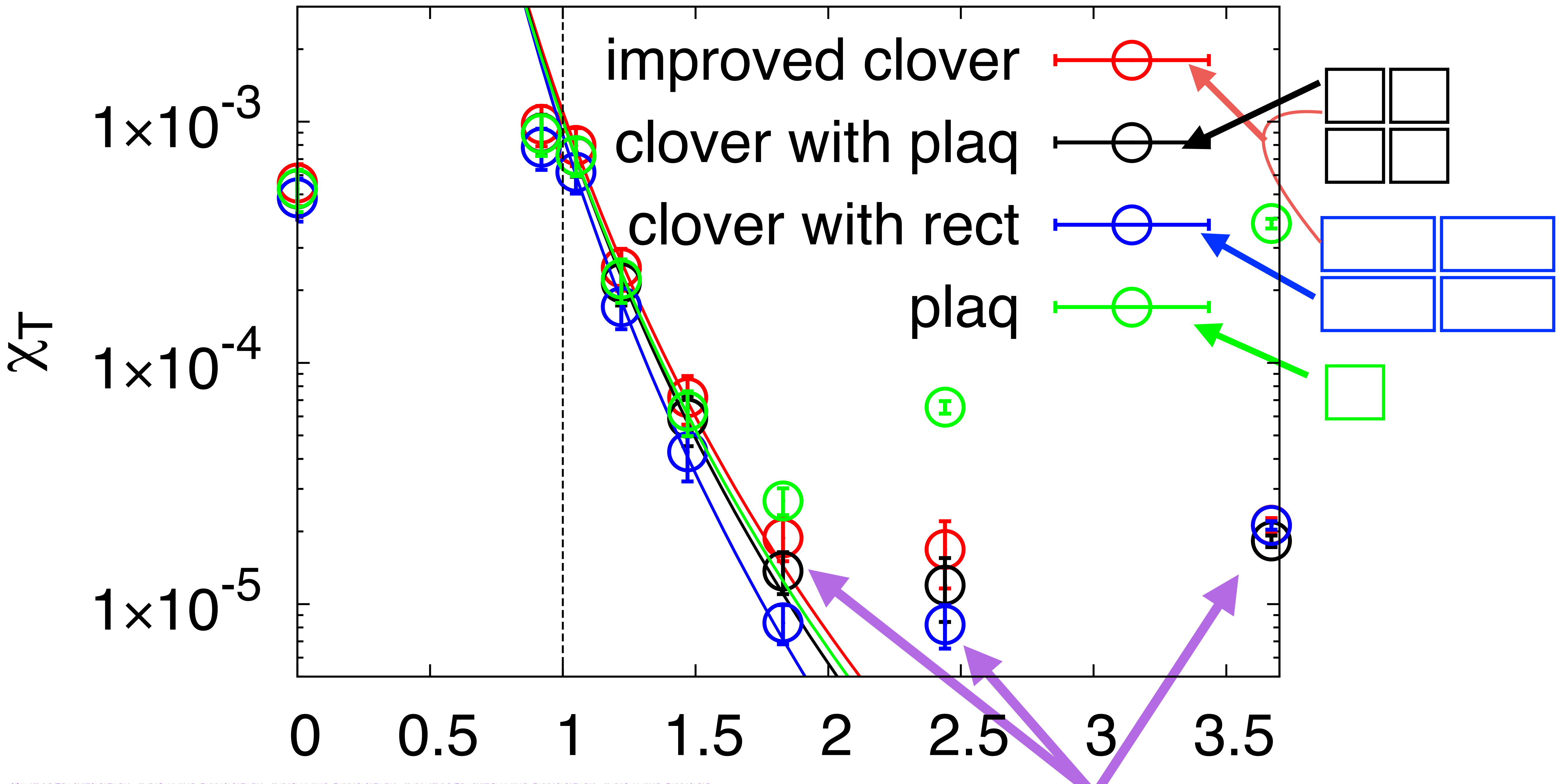

fermion definition




bin=200 looks enough

# Lattice artifact

gauge definition $Q = \frac{1}{64\pi^2}\int d^4x \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$



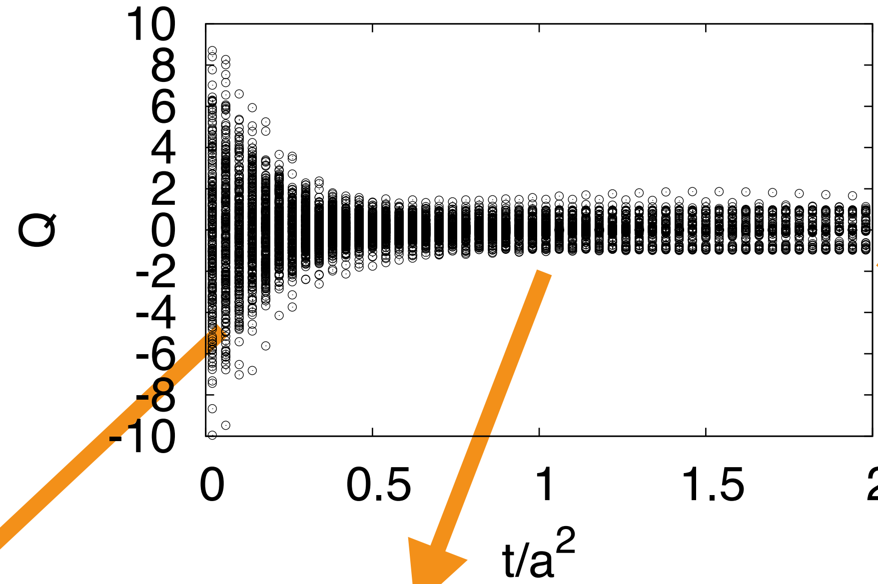

flow may be insufficient

aT=1/Nt artifact is severe

# Results: Gauge definition

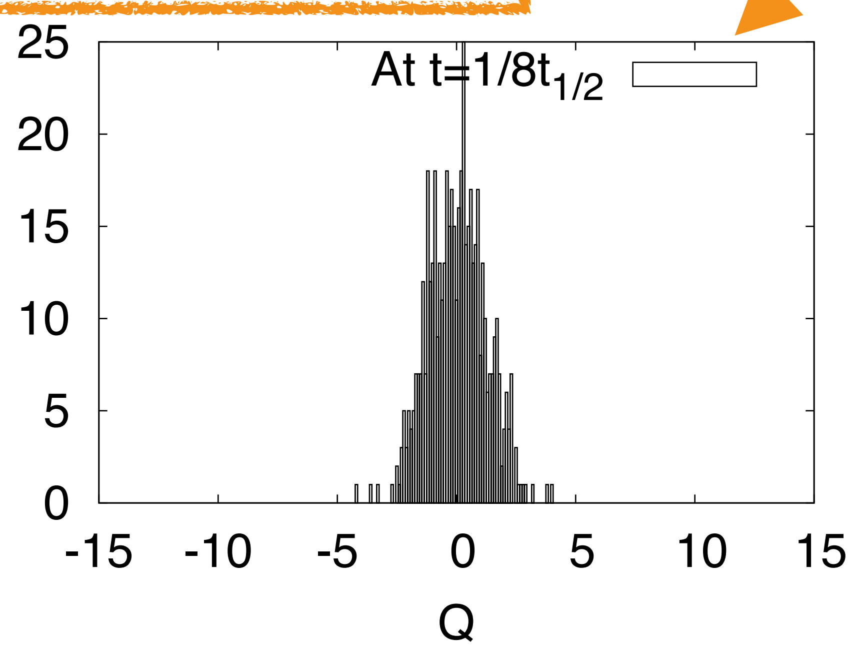
$T/T_c$=1.83

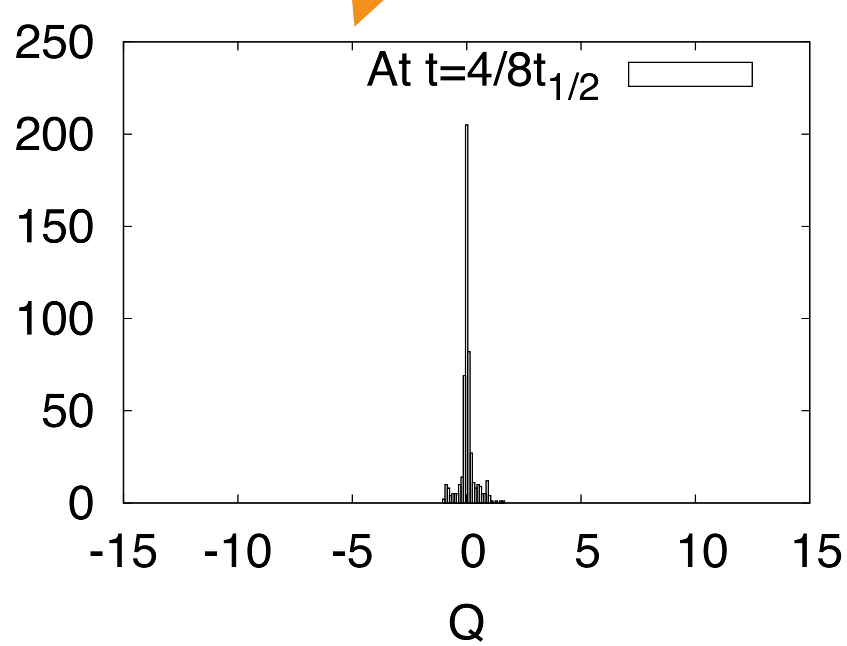
Flow of topological charge Q

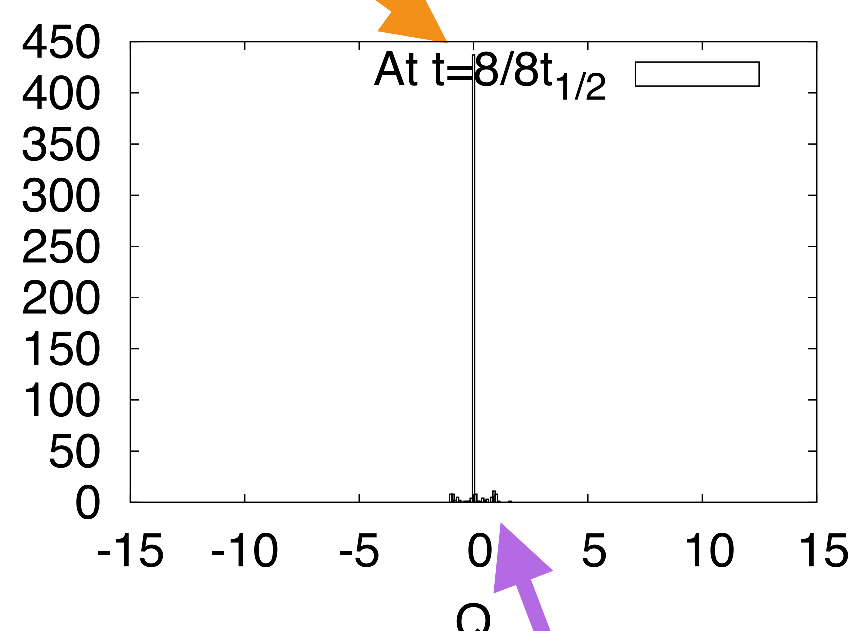



$$\sqrt{8t} = \frac{N_t}{2}$$

Histogram




fluctuation is small

# Measurement of fermion definition

Only three steps to calculate $m^2 \langle P^0 P^0 \rangle$

1. Flow the gauge and quark field

2. Calculate VEV of flowed operators

Wick contraction is a complication

$$\langle \chi(t,x)\bar{\chi}(t,y)\rangle_{\mathrm{Wick}} \neq (D(A_\mu(t,x)) + m)^{-1}$$