Spontaneous symmetry breaking induced by complex fermion determinant

--- yet another success of the complex Langevin method

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Y.I.-Nishimura, work in progress, arXiv:1608.XXXX[hep-lat]

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Introduction

- In many interesting physical system, the fermion determinant becomes complex. And its phase plays an important role in the determination of the vacuum.

speculations

- QCD at low temperature and high density — exotic fermion condensates
- superstring theory (type IIB matrix model) — SSB of SO(10) down to SO(4)

- If one applies the complex Langevin method (CLM), [Parisi '83] [Klauder '83]

the fermion determinant may cause the singular drift problem,
which is associated with the appearance of small eigenvalues of Dirac op..

⇨ wrong results

Our strategy

- We propose to avoid this problem
  by introducing a fermion bilinear term in the action and modify the Dirac op.

$$S_{\mathrm{f}} \to S_{\mathrm{f}} + \Delta S_{\mathrm{f}}\,(M)$$

$$D \to D'\,(M)$$

This modification deforms the eigenvalue distribution.

- After that, we get the results for the original system
  by extrapolating the parameter M to zero using reliable data only.

  new criterion (arXiv:1606.07627)

- We test this idea in a simple matrix model with rotational SO(4) symmetry.

  - The phase of the fermion determinant induces the SSB of SO(4) down to SO(2).

    cf) phase quenched model → no SSB

⇨ The obtained results agree well with the predictions obtained by the Gaussian expansion method.

[Nishimura-Okubo-Sugino '05]

Talk plan

1. Introduction
2. The SO(4) symmetric matrix model
3. The application of the CLM to the model
4. Results
5. Summary

The SO(4)-symmetric matrix model

[Nishimura '02]

- Partition function

$$Z = \int dX \, (\det D)^N \, e^{-S_\mathrm{b}}$$

$X_\mu : N \times N$ Hermitian matrices

$\mu = 1 \sim 4$

  - boson action

$$S_\mathrm{b} = \frac{1}{2} N \sum_{\mu=1}^{4} \mathrm{tr}\,(X_\mu^2)$$

$\Gamma^\mu : 2 \times 2$ gamma matrices

  - fermion determinant

$$D = \sum_{\mu=1}^{4} \Gamma^\mu \otimes X_\mu$$

$$\Gamma^\mu = \begin{cases} i\sigma_i & \text{for } \mu = i = 1,2,3 \\ \mathbf{1}_{2\times 2} & \text{for } \mu = 4 \end{cases}$$

This is exactly "massless" system

- $\det D$ is complex

- SO(4) rotational symmetry

- The fermion determinant induces the SSB of SO(4)

suggested by the Gaussian expansion method

[Nishimura-Okubo-Sugino '05]

Order parameters for the SSB of SO(4)

- In order to see the SSB
  1. introduce a external field in the action to break SO(4) sym.

$$S_{\rm b} \to S_{{\rm b}+\epsilon} = S_{\rm b} + \frac{N}{2}\sum_{\mu=1}^{4} \epsilon m_\mu {\rm tr}\left(X_\mu^2\right)$$

Here we chose

$$m_\mu = (1,2,4,8)$$

  2. $N \to \infty$
  3. extrapolate ϵ to zero.

- define the order parameters

$$\langle \lambda_\mu \rangle = \lim_{\epsilon \to 0} \langle \lambda_\mu \rangle_\epsilon$$

here $\langle \lambda_\mu \rangle_\epsilon = \left\langle \frac{1}{N}{\rm tr}\left(X_\mu^2\right) \right\rangle$

- In the Gaussian expansion method,

SO(2) case : $\langle \lambda_1 \rangle = \langle \lambda_2 \rangle \simeq 2.1,\ \langle \lambda_3 \rangle \simeq 1.0,\ \langle \lambda_4 \rangle \simeq 0.8$

SO(3) case : $\langle \lambda_1 \rangle = \langle \lambda_2 \rangle = \langle \lambda_3 \rangle \sim 1.75,\ \langle \lambda_4 \rangle \sim 0.75$

comparing the free energy

↓

SO(2)-sym. vacuum is favored.

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The application of the CLM to the model

- action

$$S = S_{\mathrm{b}+\epsilon} - N \ln\left(\det D\right)$$

- complex Langevin equation

$$\frac{dX_\mu}{dt} = -\frac{\partial S}{\partial X_\mu} + \eta_\mu(t)$$

$\eta_\mu(t)$: $N \times N$ Hermitian matrices

X_μ: $N \times N$ Hermitian matrices
$\rightarrow N \times N$ complex matrices

- drift term

$$\frac{\partial S}{\partial X_\mu} = N\left(1 + \epsilon m_\mu\right) X_\mu - N \mathrm{tr}_\alpha \left(D^{-1}\Gamma^\mu\right)$$

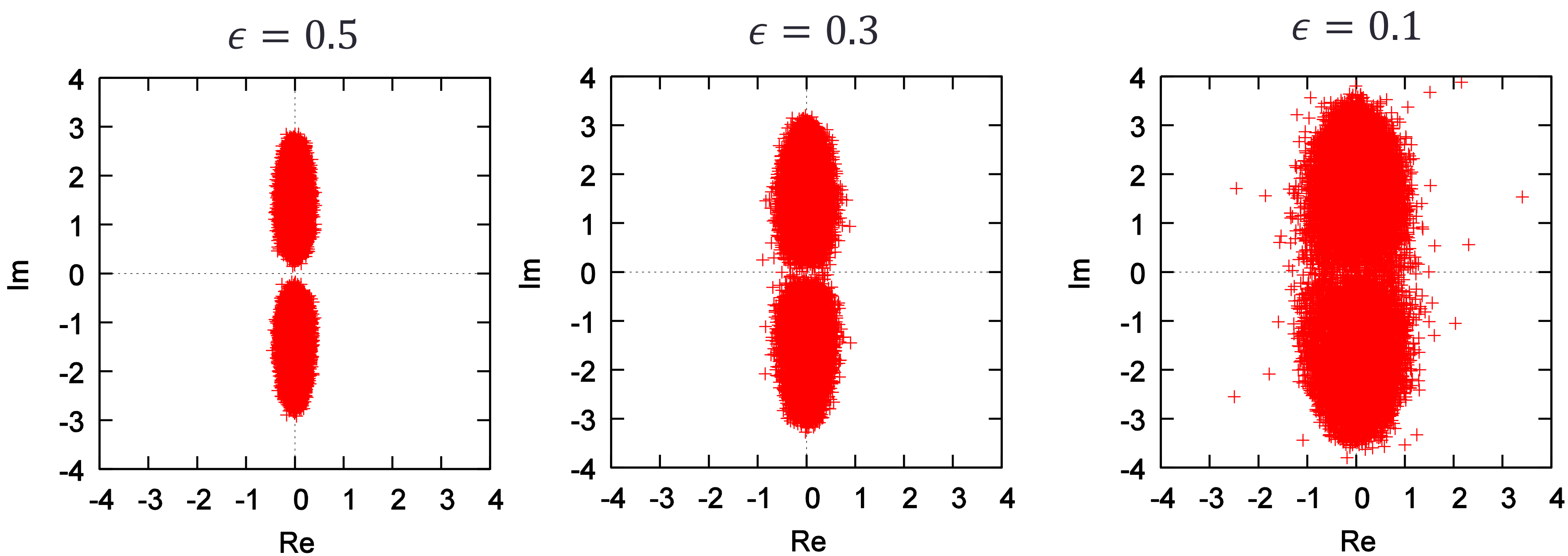
zero-eigenvalues of D correspond to a pole.

- The eigenvalues of D close to zero cause the singular-drift problem.

  ⇒ $P(X, t)$ needs to damp rapidly around the pole
  in order to justify the CLM.

The singular-drift problem

- The eigenvalue distribution of the Dirac op.



- Below some critical ϵ, the CLM does not work due to the singular-drift problem.
- We try to solve this problem by introducing a fermion bilinear term in the action.

Introduction of a fermion bilinear term

- introduce a fermion bilinear term ΔS_f in the action

$$\Delta S_{\rm f} = -N M_\mu {\rm tr}\left(\bar{\psi}\Gamma^\mu \psi\right)$$

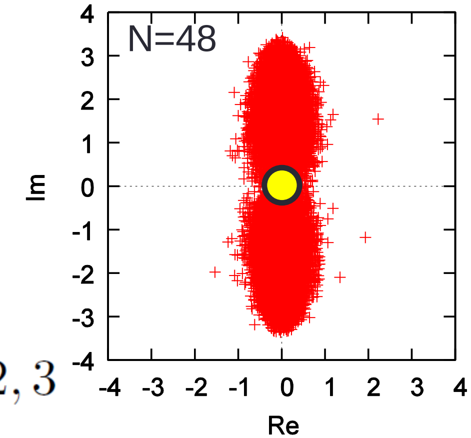
$$D \to D' = \Gamma^\mu \otimes (X_\mu + M_\mu \mathbf{1})$$

M_μ : parameter

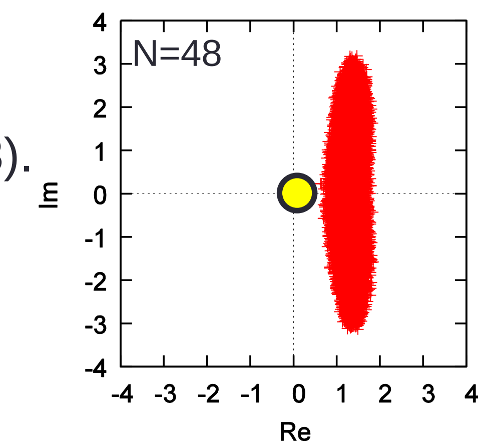
here we used

$$M_\mu = (0,0,0,m_{\rm f}) \quad \Gamma^\mu = \begin{cases} i\sigma_i & \text{for } \mu = i = 1,2,3 \\ \mathbf{1}_{2\times 2} & \text{for } \mu = 4 \end{cases}$$

$\epsilon = 0.2, \ m_f = 0$



$\epsilon = 0.2, \ m_f = 1.0$



- The eigenvalue distribution shifts to the real direction.

  $\longrightarrow$ The CLM becomes valid even at small ϵ.

- This breaks the SO(4) symmetry minimally down to SO(3).

  - First we investigate the SSB of this SO(3).

Validity of the CLM

- From the histogram of the magnitude of the drift, the validity of the CLM can be judged. → Shimasaki's talk, Nagata's talk [Nagata-Nishimura-Shimasaki '16]

n : the magnitude of the drift term

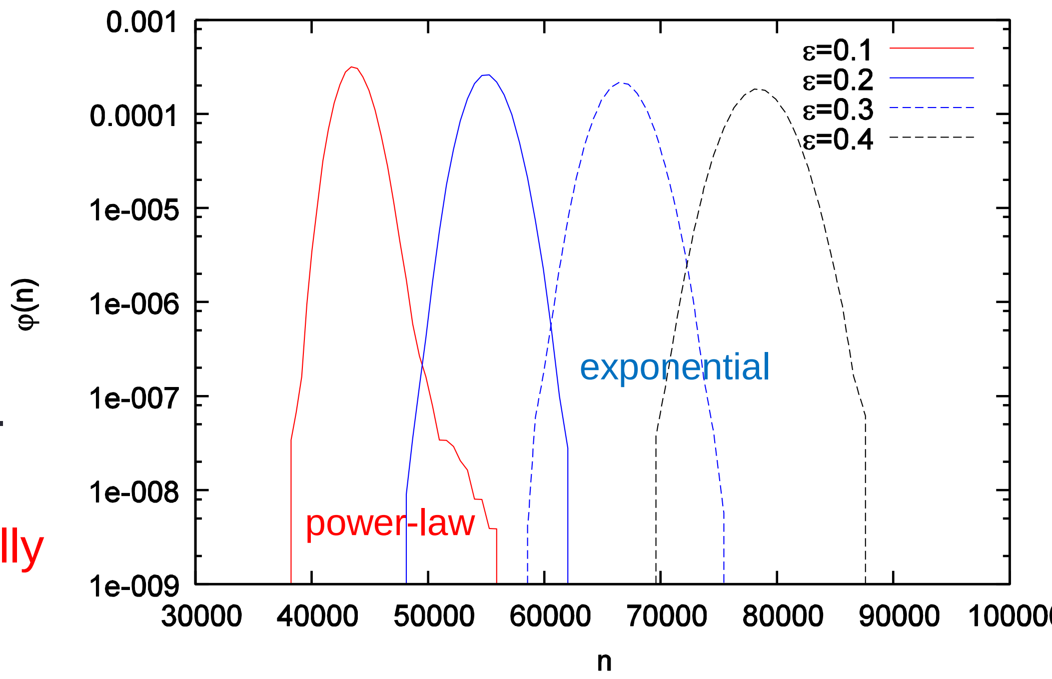
$$n = \frac{1}{4}\sum_{\mu=1}^{4} \mathrm{tr} \left| \left(\frac{\partial S}{\partial X_\mu} \right)_{ij} \right|^2$$

$\varphi(n)$: probability distribution of the magnitude n

□ new criterion

$\varphi(n)$ should damp faster than exponential.

$m_f = 1.0$



- $\varphi(n)$ for $\epsilon \geq 0.2$ damps exponentially (but not for $\epsilon \leq 0.1$).

⇨ This enables us to extrapolate ϵ using correct data only.

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Extrapolation to $\epsilon = 0$

Y.I.-Nishimura, work in progress

- order parameters for finite m_f

$$\langle \lambda_i \rangle_{m_f} = \lim_{\epsilon \to 0} \langle \lambda_i \rangle_{\epsilon, m_f}$$

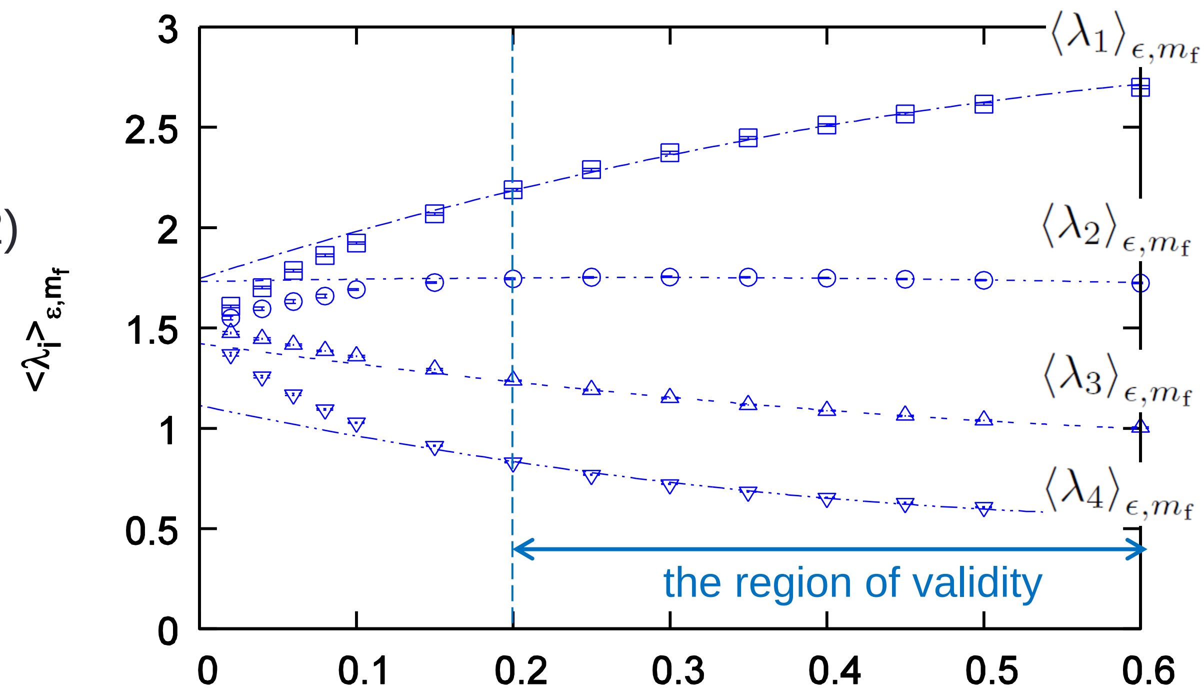
$$\langle \lambda_1 \rangle_{m_f} = 1.74(2)$$
$$\langle \lambda_2 \rangle_{m_f} = 1.731(8)$$
$$\langle \lambda_3 \rangle_{m_f} = 1.422(9)$$
$$\langle \lambda_4 \rangle_{m_f} = 1.11(2)$$

SO(2)

SSB

SO(3) → SO(2)

the order parameter for $m_f = 1.0$



ϵ (parameter of the symmetry breaking term)

Extrapolation to $m_f = 0$

Y.I.-Nishimura, work in progress

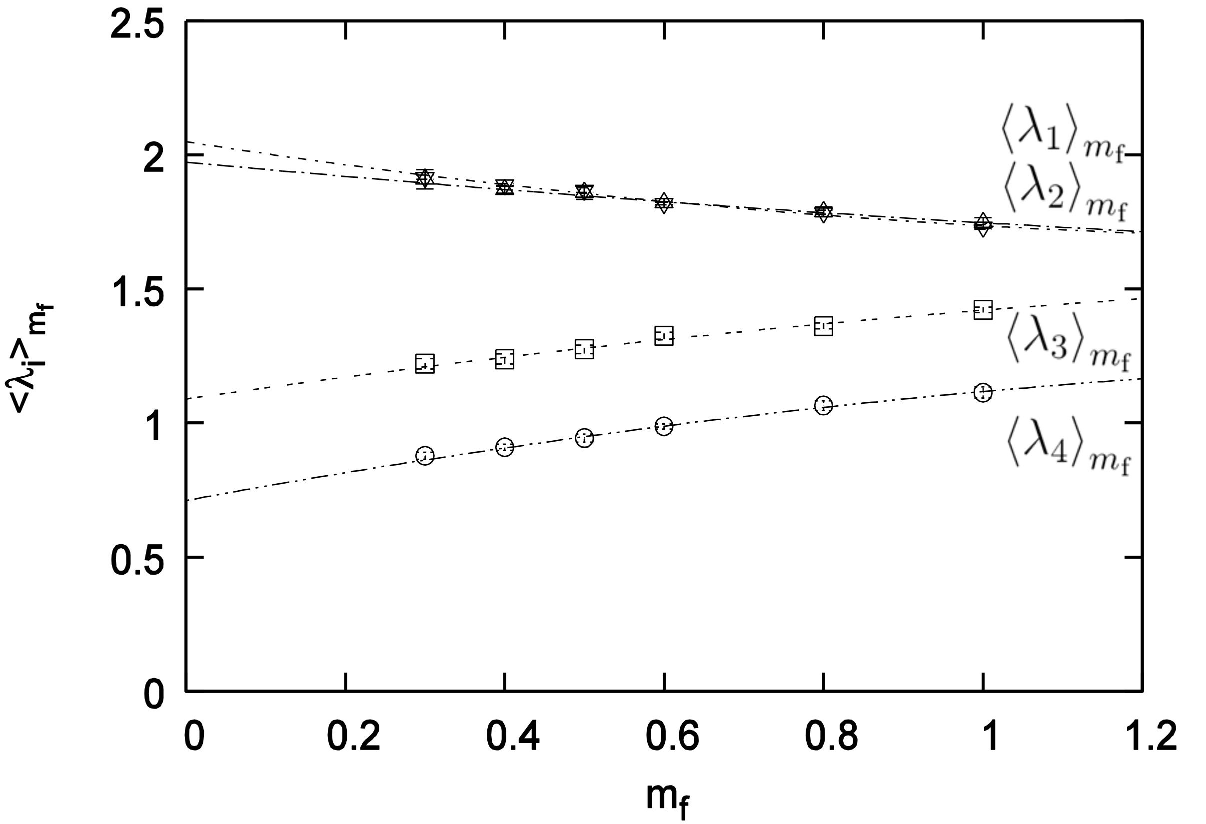
- order parameters

$$\langle \lambda_i \rangle = \lim_{m_f \to 0} \langle \lambda_i \rangle_{m_f}$$

$$\langle \lambda_1 \rangle = 1.97(4)$$
$$\langle \lambda_2 \rangle = 2.04(5)$$
$$\langle \lambda_3 \rangle = 1.08(8)$$
$$\langle \lambda_4 \rangle = 0.71(3)$$

SO(2)

SSB from SO(4) to SO(2)



previous results (Gaussian expansion method)

$$\langle \lambda_1 \rangle = \langle \lambda_2 \rangle \simeq 2.1$$

$$\langle \lambda_3 \rangle \simeq 1.0 \qquad \langle \lambda_4 \rangle \simeq 0.8$$

Summary

- For the success of the CLM in the present case,
  it was crucial to overcome the singular-drift problem.

- Our strategy:
  we deformed the Dirac op. while maintaining the qualitative feature of vacuum as much as possible.
  Then, we extrapolated the deformation parameter to zero.

- In the SO(4)-sym. matrix model,
  we had to introduce infinitesimal symmetry breaking terms to probe the SSB.

- By using the criterion to justify the method, we can extrapolate the parameter using reliable data only.

  → The results agree with the prediction obtained by the GEM.

- This strategy would be useful in finite density QCD at low temperature and high density.
- Since various exotic fermion condensates are expected there,
  deforming the Dirac op. with the corresponding bilinear term
  would not disturb the vacuum significantly.

Future Works

- Application to the IKKT matrix model which has the SO(10) symmetry.
  - The non-perturbative formulation of superstring theory.
  - It is expected that 4d space emerges from compact 10d space.

It is suggested that the SO(10) breaks down to SO(3)
by the Gaussian expansion method.

It is based on the systematic calculation, and using approximations.

[Nishimura, Okubo, Sugino '05]

We need to study this from first-principle calculation using the CLM.

- The finite density QCD

In the high density low temperature phase, introducing an external source as we did here may help reduce the singular-drift problem.

Introduction

- Complex action problem

$$Z = \int dx\, e^{-S(x)} \qquad S(x) \in \mathbb{C}$$

  - e^{-S} is no longer regarded as the Boltzmann weight factor.
  - It appears in finite density QCD, SYM theory, real time dynamics etc.

- The complex Langevin method

  - certain problems often appear in the CLM, which leads to wrong results.
  - The CLM works successfully in finite density QCD and Random matrix theory.

at deconfined phase $T > T_c$ [Seiler, Sexty, Stamatescu '12][Sexty '13]

with quark mass [Mollgaard, Splittorff '14]

The complex Langevin method

[Parisi '83] [Klauder '83]

For the partition function with a complex action

$$Z = \int dx\, e^{-S(x)} \qquad S(x) \in \mathbb{C}$$

- complexify x and consider holomorphic extension of S

$$x \to z = x + iy \qquad S(x) \to S(x, y)$$

- the complex Langevin equation

$$\begin{cases} \dfrac{d}{dt}x = -\mathrm{Re}\partial_z S(z) + \underbrace{\eta(t)}_{\text{real}} \\ \dfrac{d}{dt}y = -\mathrm{Im}\partial_z S(z) \end{cases}$$

t : the Langevin time

η : white noise

$$\langle \eta(t) \rangle = 0$$

$$\langle \eta(t)\, \eta(t') \rangle = 2\delta(t - t')$$

- The probability distribution satisfies the Fokker-Planck like equation

$$\partial_t P(x, y; t) = \partial_x \left(\partial_x + \mathrm{Re}\left[\partial_z S\right]\right) P(x, y; t) + \partial_y \mathrm{Im}\left[\partial_z S\right] P(x, y; t)$$

Criteria for the CLM to be justified

For holomorphic observables $O(x+iy)$,
the expectation values is given in the CLM as

$$\langle O(x+iy)\rangle = \int dxdy\, P_{\mathrm{eq}}(x,y)\, O(x+iy)$$

- The crucial point is that there exists a real and non-negative weight $P(x,y)$ such that

$$\int dx\, \underline{\rho(x)}\, O(x) = \int dxdy\, \underline{P_{\mathrm{eq}}(x,y)}\, O(x+iy)$$

$\sim e^{-S(x)}$: complex

$P_{eq}(x,y)$ needs to damp rapidly
- in the imaginary direction. ⟶ gauge cooling
- around singularities of the drift term.

[Aarts, James, Seiler, Stamateseu '11] [Nishimura, Shimasaki '15]

- In particular, fermion determinant will cause the singular-drift problem.

⇨ We propose a new technique to resolve this problem in such a case.

Equivalence to the path integral

- We assume

$$P(x,t) = \psi(x,t)\, e^{-S/2}$$

Then, the FP eq. becomes

$$\frac{d}{dt}\psi(x,t) = -2H\psi(x,t)$$

$$H = \frac{1}{2}\left(-\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial S}{\partial x}\right)\left(\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial S}{\partial x}\right)$$

: self-adjoint op.

mode expansion

$$H\psi_n(x) = E_n\psi_n(x)$$

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n\psi_n(x)\, e^{-E_n t}$$

$t \to \infty$

$$\psi_0(x) = e^{-\frac{1}{2}S}$$

dominant

Thus, we obtain

$$\lim_{t\to\infty} P(x,t) = a_0 e^{-S}$$

up to normalization

- The expectation value

$$\lim_{t\to\infty}\left\langle O\left(x^{(\eta)}(t)\right)\right\rangle_\eta = \lim_{t\to\infty}\int dx\, O(x)\, P(x;t) = \langle O(x)\rangle\,,$$

Proof of the relation

$$\int dx\, \rho(x;t)\, O(x) = \int dxdy\, P(x,y;t)\, O(x+iy)$$

- at $t=0$, we can choose

$$P(x,y;0) = \rho(x;0)\,\delta(y) \longrightarrow$$ The relation holds.

- for an arbitrary t, we need to show the below relations

$$\int dx\, O(x)\, \rho(x;t) = \int dx\, O(x;t)\, \rho(x;0)$$

$$\int dxdy\, O(x+iy)\, P(x,y;t) = \int dxdy\, O(x+iy;t)\, P(x,y;0)$$

- the time-dependent observable is defined by

$$\frac{\partial}{\partial t} O(z;t) = \tilde{L} O(z;t)$$

$$\tilde{L} = \left(\frac{\partial}{\partial z} - \frac{\partial S}{\partial z} \right) \frac{\partial}{\partial z}$$

$P_{eq}(x,y)$ damps rapidly
- in the imaginary direction.
- around singularities of the drift term.

- consider time interpolating function

$$F(t,\tau) = \int dxdy\, O(x+iy;\tau)\, P(x,y;t-\tau)$$

- We show that $F(t,\tau)$ is independent of τ

$$\frac{\partial}{\partial\tau}F(t,\tau) = \int dxdy\, \frac{\partial}{\partial\tau}O(x+iy;\tau)\, P(x,y;t-\tau) + \int dxdy\, O(x+iy;\tau)\, \frac{\partial}{\partial\tau}P(x,y;t-\tau)$$

↓ FP eq.

$$= \int dxdy\, \tilde{L}O(x+iy;\tau)\, P(x,y;t-\tau) - \int dxdy\, O(x+iy;\tau)\, L^{\top}P(x,y;t-\tau),$$

↓ partial derivative

$$= \int dxdy\, \tilde{L}O(x+iy;\tau)\, P(x,y;t-\tau) - \int dxdy\, LO(x+iy;\tau)\, P(x,y;t-\tau),$$

↓ $LO(z) = \tilde{L}O(z)$ for a holomorphic function

$$= 0$$

introduction of a fermion mass term in the 3rd direction

Introducing a mass term to fermions

$$D \to D' = \sum_{\mu=1}^{4} \Gamma^{\mu} \otimes (X_{\mu} + \alpha_{\mu} \mathbf{1})$$

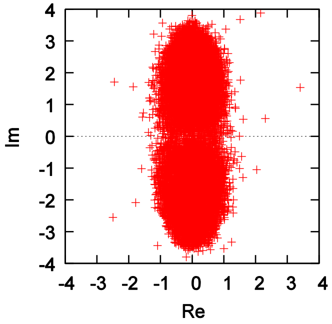
Here, we used

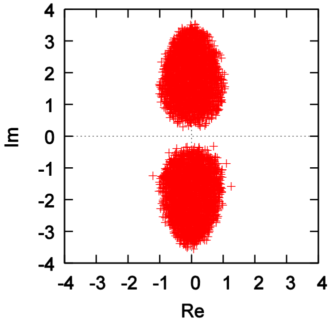
$$\alpha_{\mu} = (0, 0, m_{\mathrm{f}}, 0)$$

This term makes the eigenvalue distribution of D to avoid the pole.

$\longrightarrow$ we can extrapolate the values of $\langle \lambda_i \rangle$ using much smaller ϵ.

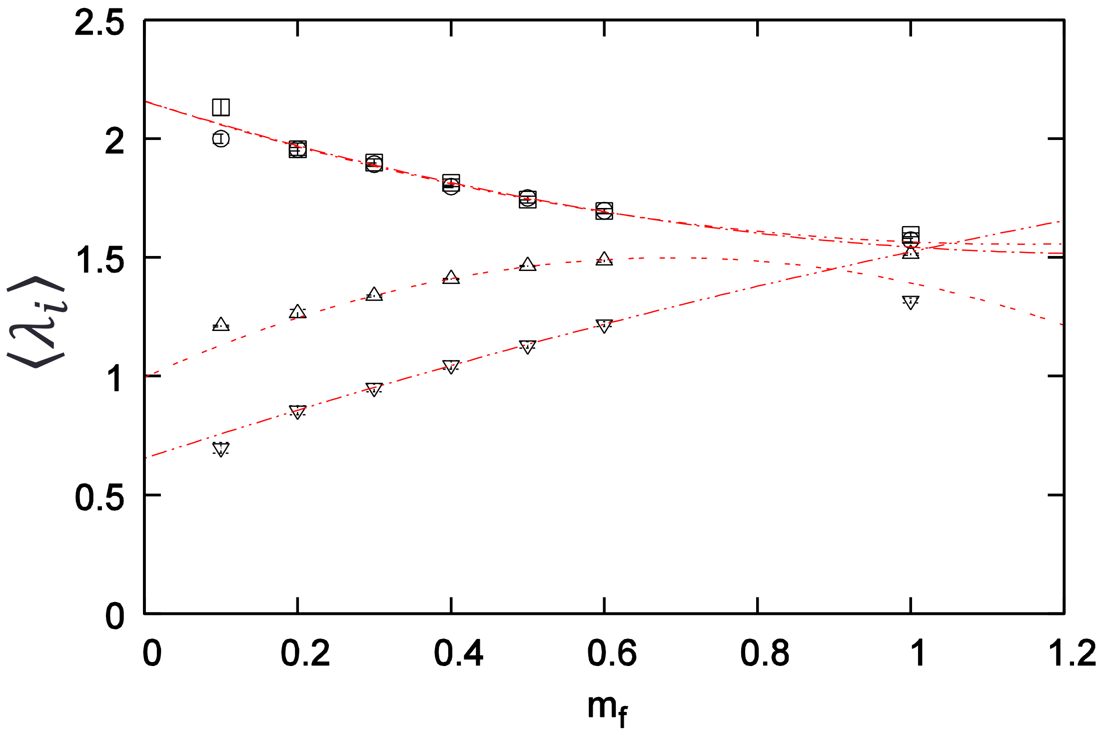
for $\epsilon = 0.1$ $m_f = 0$ $m_f = 0.5$





Improvement by means of fermion mass terms

1. taking the $\epsilon \to 0$ limit with m_f fixed.
2. taking the $m_f \to 0$ limit.



$\langle \lambda_1 \rangle = 2.15(6)$

$\langle \lambda_2 \rangle = 2.15(7)$

SO(2)

$\langle \lambda_3 \rangle = 0.99(4)$

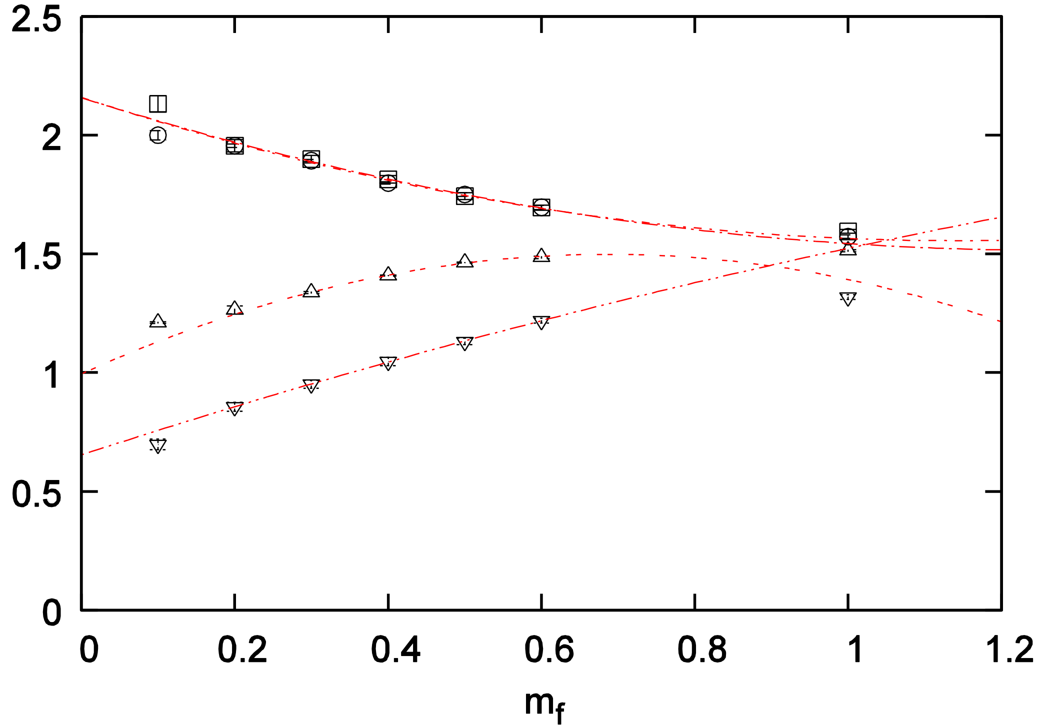
$\langle \lambda_4 \rangle = 0.654(9)$

previous result

$\langle \lambda_1 \rangle = \langle \lambda_2 \rangle \simeq 2.1$

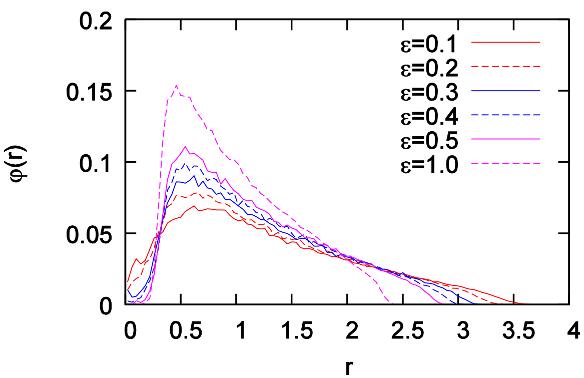
$\langle \lambda_3 \rangle \simeq 1.0 \quad \langle \lambda_4 \rangle \simeq 0.8$

⇨ The result clearly shows the SSB from SO(4) to SO(2).

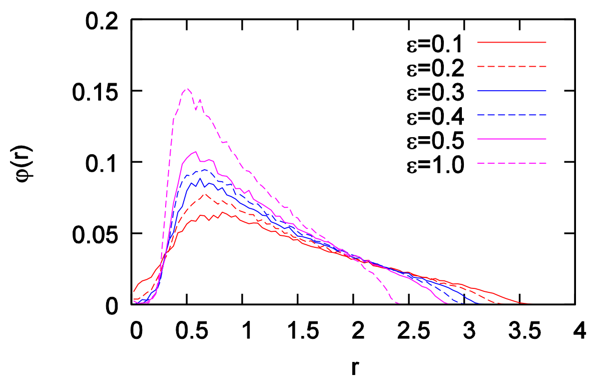


$$\varphi\left(r\right)=\frac{1}{2\pi r}\int dz\,P\left(z,t=\infty\right)\delta\left(\left|z\right|-r\right)$$

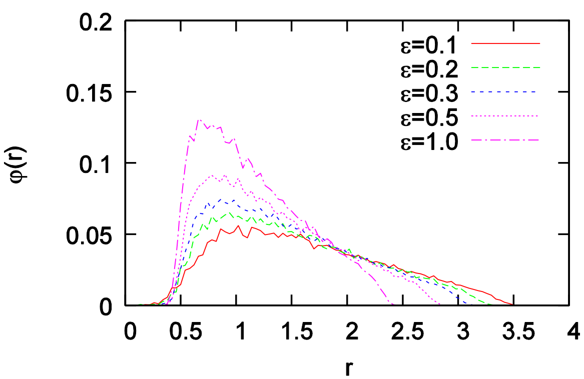
$\epsilon = 0.1$ N=48



$\epsilon = 0.2$ N=48



$\epsilon = 0.5$ N=48



Idea of “gauge cooling”

For lattice gauge theory,

Link variables $U_{x,\mu}$

$$SU(N) \rightarrow SL(N,\mathbb{C})$$

Considering unitarity norm.

$$\frac{1}{N}\mathrm{tr}\left(UU^{\dagger}-\mathbf{1}\right)$$

It is no longer zero.

- It is necessary to control the norm to be small.

⟶ “gauge cooling”

$$U_{x,\mu} \rightarrow \Omega_x U_{x,\mu} \Omega_{x+\hat{\mu}}^{-1} \qquad \Omega_x \in SL(N,\mathbb{C})$$

- Gauge inv. observables are independent of the gauge cooling.