Numerical Analysis of Discretized N=(2,2) SYM on Polyhedra

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“*Anomaly and Sign problem in* *N=(2,2)* *SYM on Polyhedra : Numerical Analysis*”, arXiv:1607.01260

Lattice2016 July 26th 2016 @ Southampton Univ.

SUSY theory on Lattice

- SUSY + Lattice $\{Q, \bar{Q}\} \propto P$ boson ⇔ fermion
  - ✓ Non-perturbative analysis of SUSY theory
  - ✓ First principles calculation
  - ✓ Numerical simulation
- Motivated by
  - ✓ Gauge/Gravity correspondence
  - ✓ String theory
  - ✓ Condensed matter
  - ✓ Mathematical contexts
- Problems for preserving SUSY on Lattice
  - ✓ SUSY is quantumly broken by lattice spacing.
  - ✓ Fine-tuning
  - ✓ Locality (SLAC type differential op.) **[Dondi et.al., 1977] [Kato et.al., 2013]**
  - ✓ The sign problem

2 dim. N=(2,2) SYM on discretized spacetime

- N=(2,2) Sugino model on flat spacetime [Sugino, 2004]
  - ✓ Gauge symmetry
  - ✓ (Discrete) Translation and Rotation
  - ✓ Internal symmetry ($U(1)_V$,$U(1)_A$)
  - ✓ Locality (Ultra local)
  - ✓ No doublers
  - ✓ Exact SUSYs on Lattice
- Regular lattice

  ⇒ discretized spacetime with non-trivial topology.

  [Matsuura et.al., 2014]
- Preserving one exact (0-form) SUSY
- Field contents are defined on sites, links, and faces.
- $U(1)_A$ anomaly ⇒ This effect is Hidden in the Pfaffian Phase.

2 dim. N=(2,2) SYM on discretized spacetime

[Sugino, 2004]

- Action on flat spacetime (4 dim. N=1 ⇒ 2 dim. N=(2,2))

$$
\begin{aligned}
S &= S_b + S_f, \\
S_b &= \frac{1}{2g^2}\int d^2x \ \mathrm{Tr}\left[\frac{1}{2}F_{\mu\nu}^2 + (D_\mu\Phi)(D_\mu\bar{\Phi}) + \frac{1}{4}[\Phi,\bar{\Phi}]^2\right], \\
S_f &= \frac{1}{2g^2}\int d^2x \ \mathrm{Tr}\left[i\bar{\Psi}\Gamma_\mu D_\mu\Psi - \frac{1}{2}\bar{\Psi}\Gamma_+[\bar{\Phi},\Psi] - \frac{1}{2}\bar{\Psi}\Gamma_-[\Phi,\Psi]\right],
\end{aligned}
$$

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \quad D_\mu = \partial_\mu + i[A_\mu, \bullet].
$$

$$
C = i\Gamma_4, \quad \bar{\Psi} = -\Psi^T C,
$$

- SUSY transform

$$
\delta\Phi = -i\bar{\xi}\Gamma_+\Psi, \quad \delta\bar{\Phi} = -i\bar{\xi}\Gamma_-\Psi, \quad \delta A_\mu = -i\bar{\xi}\Gamma_\mu\Psi,
$$

$$
\delta\Psi = -F_{12}\Gamma_{12}\xi - \frac{1}{2}D_\mu\bar{\Phi}\Gamma_{\mu+}\xi - \frac{1}{2}D_\mu\Phi\Gamma_{\mu-}\xi - \frac{i}{4}[\bar{\Phi},\Phi]\Gamma_{+-}\xi,
$$

2 dim. N=(2,2) SYM on discretized spacetime

[Sugino, 2004]

- Action on flat spacetime (Q-exact form)

$$S = \hat{Q}\frac{1}{2g^2}\int d^2x \ \mathrm{Tr}\left[\frac{1}{4}\eta[\Phi,\bar{\Phi}] - ig^{\mu\nu}\lambda_\mu D_\nu\bar{\Phi} + \chi(Y - 2iF_{12})\right]$$

$$\Psi = (\lambda_1, \lambda_2, \chi, \eta/2)^T, \quad \bar{\Psi} = (-i\eta/2, i\chi, -i\lambda_2, i\lambda_1).$$

$$Q \equiv Q^1 + \bar{Q}_2.$$

- SUSY transform $\boxed{\hat{Q}^2 = 0 \text{ up to gauge trf.}}$

$$\hat{Q}\Phi = 0, \quad \hat{Q}\bar{\Phi} = \eta, \quad \hat{Q}A_\mu = \lambda_\mu, \quad \hat{Q}Y = [\Phi, \chi],$$
$$\hat{Q}\eta = [\Phi, \bar{\Phi}], \quad \hat{Q}\lambda_\mu = iD_\mu\Phi, \quad \hat{Q}\chi = Y,$$

2 dim. N=(2,2) SYM on discretized spacetime

[Matsuura et.al., 2014]

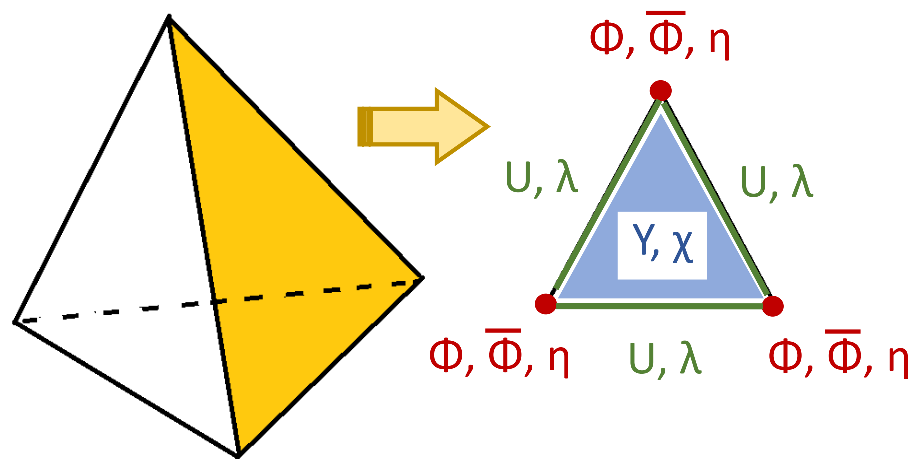
- Action on discretized curved spacetime

$$S = S_S + S_L + S_F = Q\left[\sum_{s\in S}\alpha_s\mathcal{V}_s + \sum_{\langle st\rangle\in L}\alpha_{\langle st\rangle}\mathcal{V}_{\langle st\rangle} + \sum_{f\in F}\alpha_f\mathcal{V}_f\right],$$

$$\mathcal{V}_s = \frac{1}{2g^2}\mathrm{Tr}\left[\frac{1}{4}\eta_s[\Phi_s,\bar{\Phi}_s]\right],$$

$$\mathcal{V}_{\langle st\rangle} = \frac{1}{2g^2}\mathrm{Tr}\left[-i\lambda_{st}(U_{st}\bar{\Phi}_t U_{st}^{-1} - \bar{\Phi}_s)\right]$$

$$\mathcal{V}_f = \frac{1}{2g^2}\mathrm{Tr}\left[\chi_f(Y_f - i\beta_f\Omega(U_f))\right].$$



- SUSY transform

$$\begin{aligned}
&Q\Phi_s = 0, && \\
&Q\bar{\Phi}_s = \eta_s, && Q\eta_s = [\Phi_s,\bar{\Phi}_s], \\
&QU_{st} = i\lambda_{st}U_{st}, && Q\lambda_{st} = i(U_{st}\Phi_t U_{st}^{-1} - \Phi_s + \lambda_{st}\lambda_{st}), \\
&QY_f = [\Phi_f,\chi_f], && Q\chi_f = Y_f.
\end{aligned}$$

Site → Site
Link → Link
Face → Face

U(1)A anomaly and Pfaffian phase

- U(1)A symmetry is broken by quantum effect (fermion zero-modes) on general curved background.

$$\boxed{\begin{array}{l}\Phi \to e^{2i\theta}\Phi, \quad \bar{\Phi} \to e^{-2i\theta}\bar{\Phi}, \quad A_\nu \to A_\nu, \quad Y \to Y, \\ \eta \to e^{-i\theta}\eta, \quad \lambda_\nu \to e^{i\theta}\lambda_\nu, \quad \chi \to e^{-i\theta}\chi,\end{array}}$$

- Pfaffian has two kinds of phases:

$$\begin{aligned}\langle \mathcal{O} \rangle &= \int \mathcal{D}B\mathcal{D}F \; \mathcal{O} \; e^{-S_b - S_f} \\ &= \int \mathcal{D}B \; \mathrm{Pf}(D) \, \mathcal{O} \; e^{-S_b},\end{aligned}$$

$$\mathrm{Pf}(D) = |\mathrm{Pf}(D)| e^{i\theta_{\mathrm{Pf}}} \qquad \theta_{\mathrm{Pf}} = \underline{\theta} + \underline{\theta_A}$$

Dirac op. (vanishes in the cont. lim.(?))

U(1)A anomaly

$$\mathcal{D}X = \mathcal{D}\Phi\mathcal{D}\bar{\Phi}\mathcal{D}U\mathcal{D}Y\underline{\mathcal{D}\eta\mathcal{D}\lambda_\mu\mathcal{D}\chi}$$

$$\boxed{[\mathcal{D}X]_A = (N_c^2 - 1)\chi}$$

✓The partition function is ill-defined due to θ_A rotation

✓Naïve phase quenched method ⇒ the anomaly is ignored.

How to define observables? How to take into account anomaly?

Anomaly-Phase-Quenched method

[S.K. et.al., 2016]

- Introduce a compensate operator $\mathcal{A}$ which cancels the U(1)$_A$ phase from the fermion measure.

$$\boxed{[\mathcal{A}]_A = -(N_c^2-1)\chi\,, \quad \mathcal{A}\ :\ \text{gauge invariant}\,, \quad Q\mathcal{A}=0\,,} \qquad \mathcal{A} = |\mathcal{A}|e^{-i\theta_A}$$

Examples:

$$\mathcal{A}_{\mathrm{tr}} = \frac{1}{N_S}\sum_{s=1}^{N_S}\left(\frac{1}{N_c}\mathrm{Tr}\,(\Phi_s)^2\right)^{-\frac{N_c^2-1}{4}\chi_h} \qquad \mathcal{A}_{\mathrm{IZ}} = \frac{1}{N_l}\sum_{l=1}^{N_l}\left(\frac{1}{N_c}\mathrm{Tr}\left(2\Phi_{\mathrm{org}(l)}U_l\Phi_{\mathrm{tip}(l)}U_l^\dagger + \lambda_l\lambda_l(U_l\Phi_{\mathrm{tip}(l)}U_l^\dagger + \Phi_{\mathrm{org}(l)})\right)\right)^{-\frac{N_c^2-1}{4}\chi_h}$$

- Definition of observables : the APQ method.

$$\boxed{\langle\mathcal{O}\rangle^{\hat{q}} \equiv \langle\mathcal{O}\,e^{i\theta_A}\rangle^q = \langle\mathcal{O}\frac{\mathcal{A}^*}{|\mathcal{A}|}\rangle^q\,,}$$

$[\mathcal{O}_0]_A = 0$

Insert A

$$\langle\mathcal{O}_0\,\mathcal{A}\rangle^{\hat{q}} = \langle\mathcal{O}_0\,\mathcal{A}\,e^{i\theta_A}\rangle^q = \langle\mathcal{O}_0\,|\mathcal{A}|\rangle^q\,,$$

- The PCSC relation of the exact SUSY.

$$\boxed{\langle\tilde{S}_b|\mathcal{A}|\rangle^q_\mu + \langle\frac{\mu^2}{2}\mathcal{V}\sum_s\mathrm{Tr}(\Phi_s\eta_s)|\mathcal{A}|\rangle^q_\mu = \frac{N_c^2-1}{2}(\#\mathrm{site}+\#\mathrm{link})\langle|\mathcal{A}|\rangle^q_\mu\,.}$$

Set-up for numerical simulations

- Background topology : S^2 (tetra,octa,...), $T^2(3^2, 4^2, 5^2)$, F_2^2
- Gauge group : SU(2)
- 't Hooft coupling and surface area : $\lambda_{\text{phys.}} \equiv g^2_{\text{phys.}}/N_c = 1, \quad S_{\text{Area}} = 1$
- Boson mass term (for lifting flat direction) : $S_\mu = \frac{\mu^2}{2}\sum_s \text{Tr}(\Phi_s\bar{\Phi}_s)\,,$
- Pseudo-fermion method and rational approximation.
- Lattice action without the admissibility condition for avoiding unphysical degenerate vacua of link variables.

[Matsuura et.al., 2014]

$$\Omega(U_f) = \begin{cases} 2i[\mathcal{S}^{-1}(U_f)\mathcal{C}(U_f) + \mathcal{C}(U_f)\mathcal{S}^{-1}(U_f)] & \text{for } G = U(N) \\ \frac{2i}{M}[\mathcal{S}^{-1}(U_f^M)\mathcal{C}(U_f^M) + \mathcal{C}(U_f^M)\mathcal{S}^{-1}(U_f^M)] & \text{for } G = SU(N) \end{cases},$$

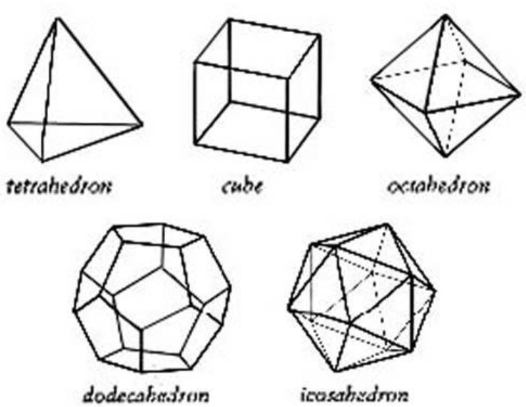
$$\text{with} \quad \mathcal{S}(U_f) = U_f - U_f^{-1}, \quad \mathcal{C}(U_f) = 2 - U_f - U_f^{-1}.$$

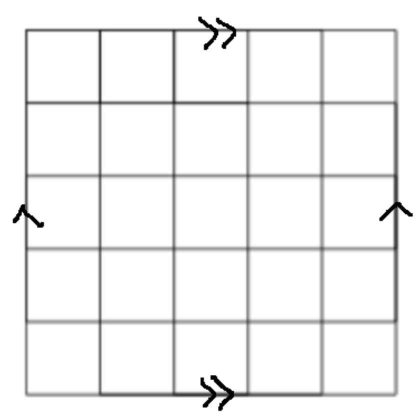
Can we separate θ_{Pf} into θ and θ_A ?
The sign problem ?

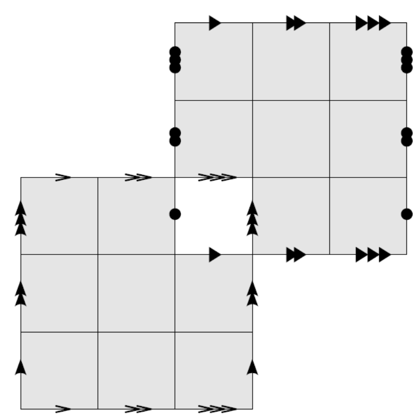
- Measure the PCSC relation and the Pfaffian phase.

Set-up for numerical simulations

genus	Euler ch.	geometry	N_S	N_L	N_F	shape of face	lattice spacing
0	2	tetra	4	6	4	T	0.7598
		octa	6	12	8	T	0.5373
		cube	8	12	6	S	0.4082
		icosa	12	30	20	T	0.3398
		dodeca	20	30	12	P	0.2201
1	0	3×3 reg.lat.	9	18	9	S	0.3333
		4×4 reg.lat.	16	32	16	S	0.2500
		5×5 reg.lat.	25	50	25	S	0.2000
2	-2	Fig.2	14	32	16	S	0.2500

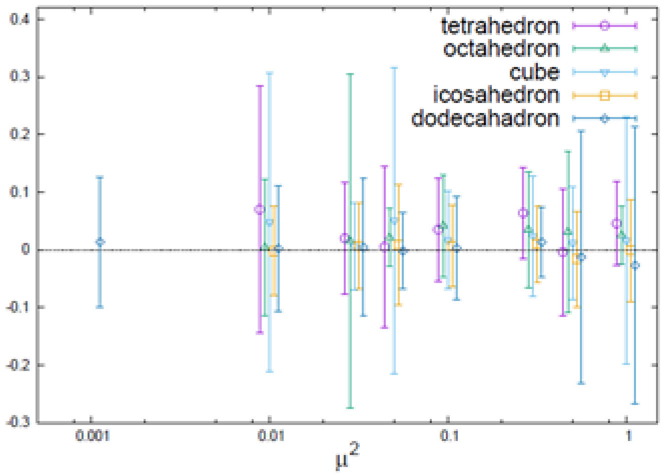




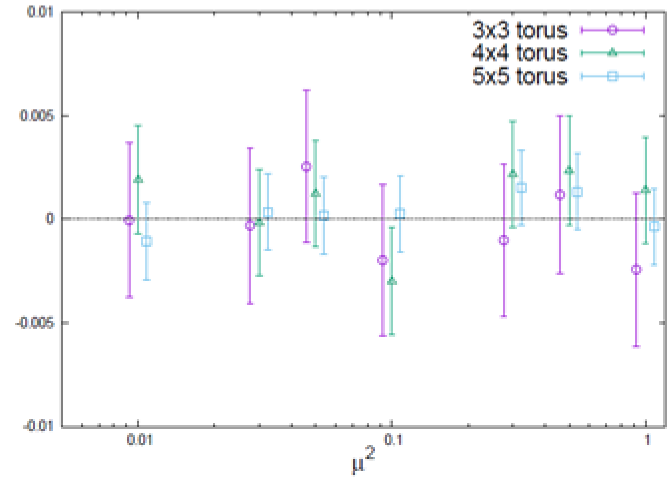


PCSC relation

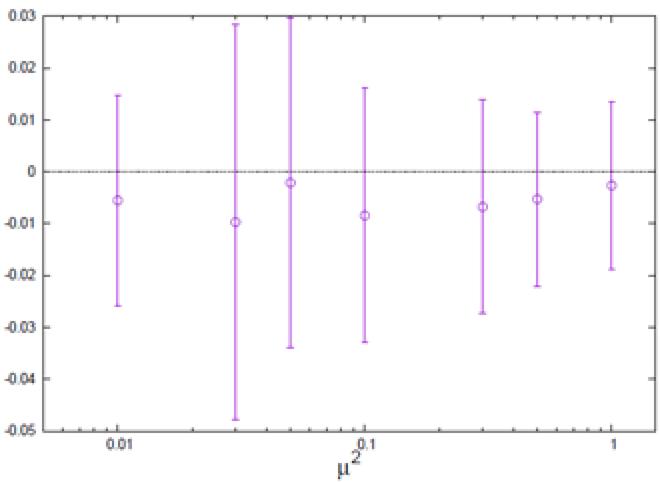
$$\boxed{\langle \tilde{S}_b|\mathcal{A}|\rangle^q_\mu + \langle \frac{\mu^2}{2}\mathcal{V}\sum_s \mathrm{Tr}(\Phi_s\eta_s)|\mathcal{A}|\rangle^q_\mu = \frac{N_c^2-1}{2}(\#\text{site}+\#\text{link})\langle|\mathcal{A}|\rangle^q_\mu .}$$



(1) WT identity for $h = 0$



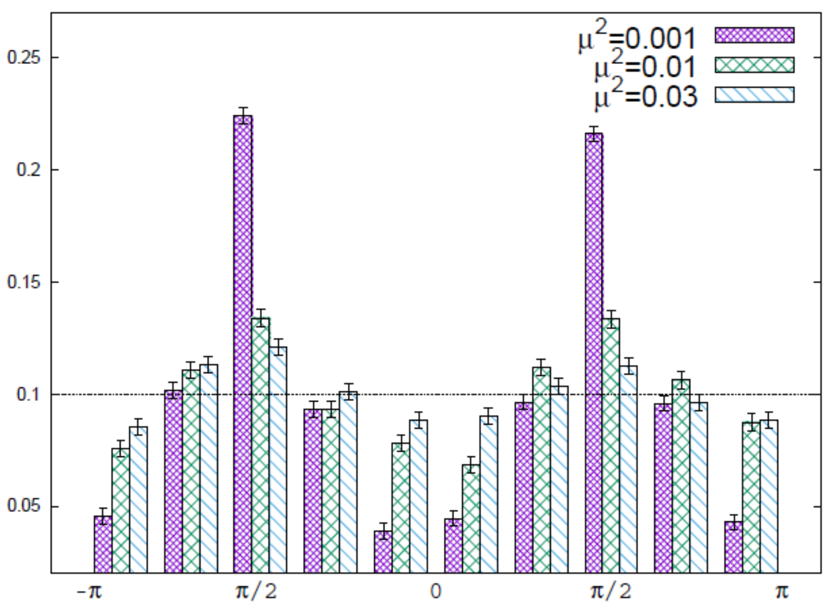
(2) WT identity for $h = 1$



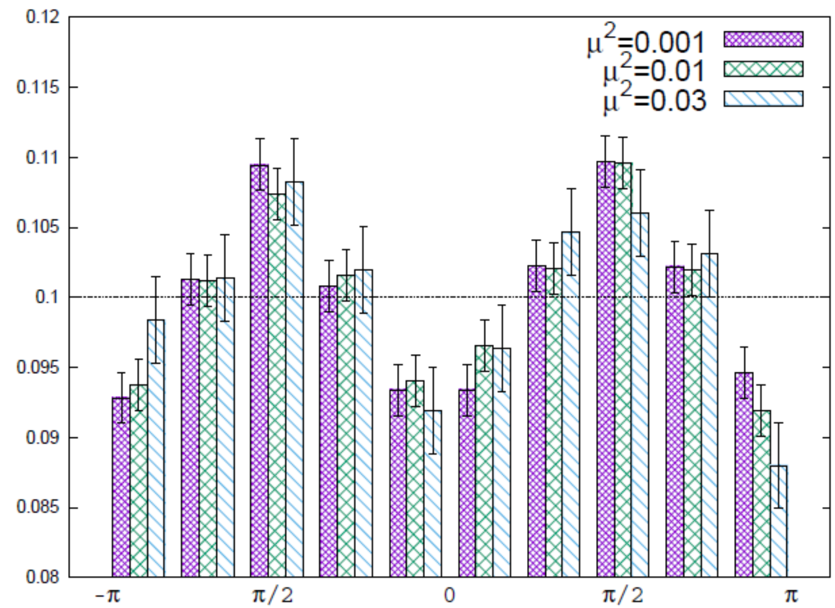
(3) WT identity for $h = 2$

- Consistent with the theoretical prediction.
- APQ method works well.

Histgram of $\mathrm{Pf}(D)\mathcal{A}$ phase ($\theta_{\mathrm{Pf}} = \boxed{\theta} + \theta_{\mathcal{A}}$)



(1) dodecahedron



(2) double torus

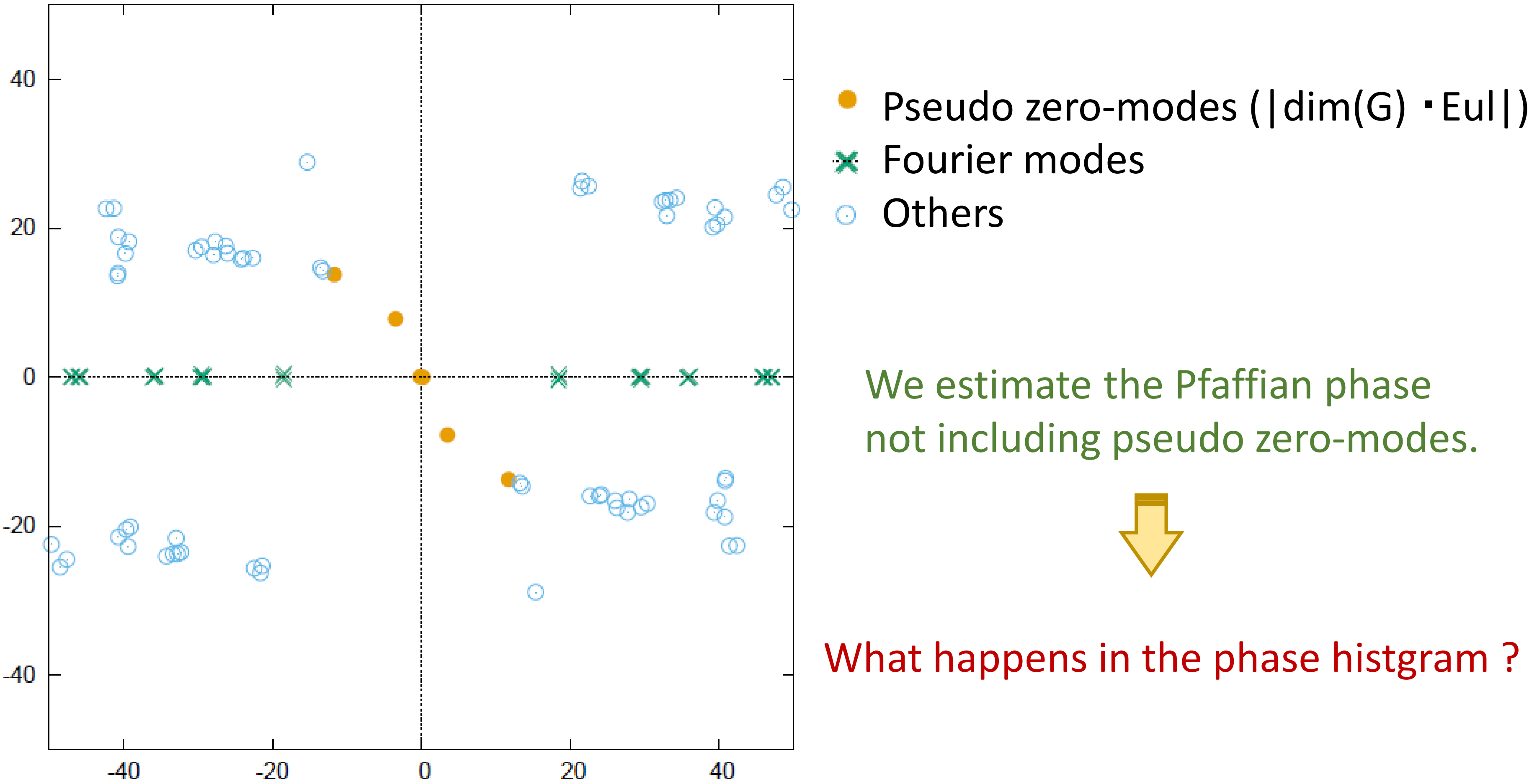
$$\mathcal{A}_{\mathrm{tr}} = \frac{1}{N_S}\sum_{s=1}^{N_S}\left(\frac{1}{N_c}\mathrm{Tr}\,(\Phi_s)^2\right)^{-\frac{N_c^2-1}{4}\chi_h}$$

$$\mathcal{A}_{\mathrm{IZ}} = \frac{1}{N_l}\sum_{l=1}^{N_l}\left(\frac{1}{N_c}\mathrm{Tr}\left(2\Phi_{\mathrm{org}(l)}U_l\Phi_{\mathrm{tip}(l)}U_l^\dagger + \lambda_l\lambda_l(U_l\Phi_{\mathrm{tip}(l)}U_l^\dagger + \Phi_{\mathrm{org}(l)})\right)\right)^{-\frac{N_c^2-1}{4}\chi_h}$$

Massless limit (Scalar SUSY restored)

$\Rightarrow$ peaks appear !!

Eigenvalue distribution of Dirac op.

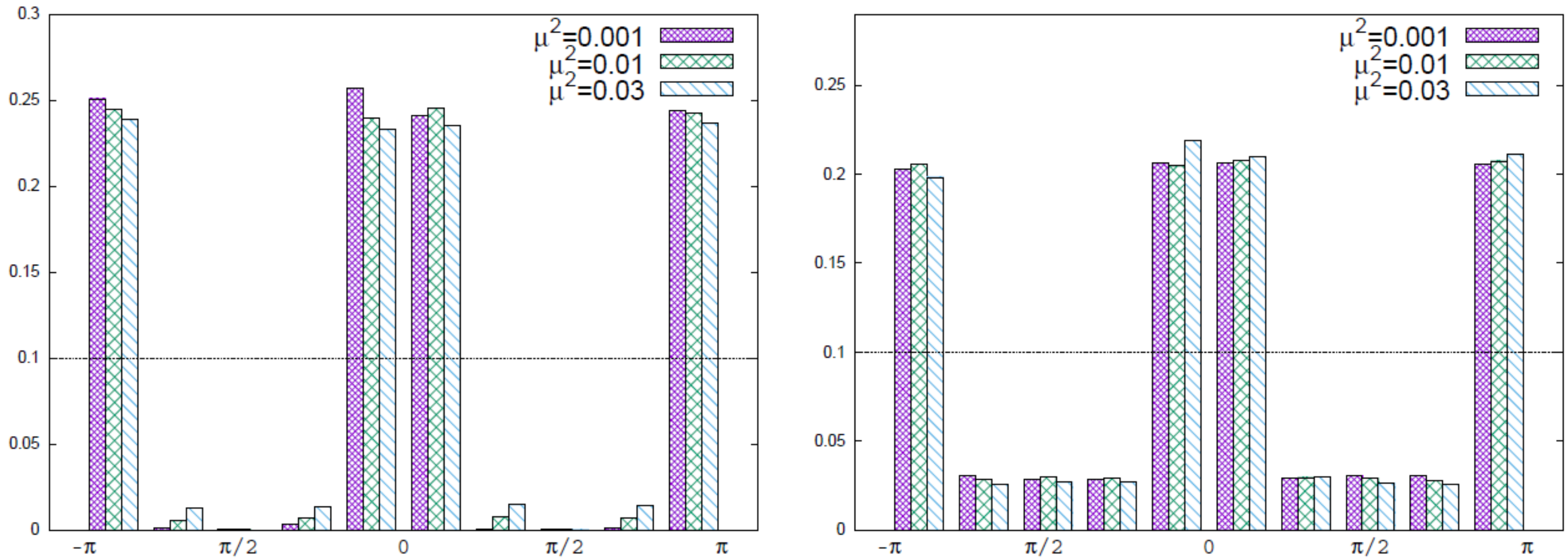


We estimate the Pfaffian phase not including pseudo zero-modes.

What happens in the phase histgram ?

Histgram of phase of $\mathrm{Pf}(D)$

(not including pseudo-zero modes)



(1) dodecahedron, $\mu^2 = 0.01, 0.03, 0.05$

(2) double torus, $\mu^2 = 0.001, 0.01, 0.03$

Sharp peaks appear for both h=0 and h=2.

Summary and Conclusion

- We performed the numerical simulation of the N=(2,2) SYM on discretized spacetime with non-trivial toplogy.
- $U(1)_A$ symmetry is generally broken by anomaly.
- We numerically calculated the PSCS relation for the exact SUSY using the anomaly-phase-quenched method.
- The results are consistent with the theoretical prediction, and the APQ method works.
- The residual phase θ has peaks
  $\Rightarrow$ the sign problem vanishes.

✓ Continuum limit
✓ More general background
✓ SQCD (fundamental matter contents)