# Stochastic approaches to extract spectral functions from Euclidean correlators

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# Outline

- Introduction & Motivation
- Two stochastic approaches
- Mock data tests
- Summary & Conclusion

# Difficulties in extracting spectral functions

**Spectral functions are crucial to understand in-medium hadron properties and transport properties of QGP.**

$q(\tau,\vec{x})$
$\Gamma_H$
$\bar{q}(\tau,\vec{x})$

$\bar{q}(0{,}0)$
$\Gamma_H$
$q(0{,}0)$

$$J_H(\tau,\vec{x}) = \vec{q}(\tau,\vec{x})\Gamma_H q(\tau,\vec{x})$$

$$\mathrm{G}(\tau,\vec{p},T)=\sum_{\vec{x}} \exp(-i\cdot\vec{p}\cdot\vec{x})\langle J_H(0,0) {J_H}^{+}(\tau,\vec{x})\rangle$$

Relation between correlators and spectral functions:

$$K(\tau,\omega,\mathrm{T}) = \frac{\cosh(\omega(\tau-1/2T))}{\sinh(\omega/2T)}$$

$$\mathrm{G}(\tau,\vec{p},T)=\int_0^\infty \frac{d\omega}{2\pi}\rho(\omega,\vec{p},T)\, K(\tau,\omega,\mathrm{T})$$

***Discretized ~O(10)***

***Continuous >=O($10^3$ )***

***$\chi^2$ fitting fails without prior information***

# Maximum Entropy Method(MEM)

- A method based on Bayesian theorem:

$$\langle\rho\rangle = \int d\alpha P[\alpha|\bar{G}] \int \boldsymbol{\mathcal{D}\rho\, P[\rho|\bar{G},\alpha]\rho}$$
$$\approx \int d\alpha\, P[\alpha|\bar{G}]\hat{\boldsymbol{\rho}}_{\boldsymbol{\alpha}} \text{ (sharp-peak assumption)}$$

Goal: obtain the most probable solution

- $P[\alpha|\bar{G}] = \frac{P[\alpha]}{P[\bar{G}]} \int \mathcal{D}\rho P[\bar{G}|\rho,\alpha] \cdot P[\rho|\alpha]$

sharp-peak assumption

$P[\bar{G}|\rho,\alpha] \sim \exp(-\chi^2[\rho]/2)$: likelihood function

$P[\rho|\alpha] \sim \exp(\alpha S[\rho])$: prior probability

minimize $F = \frac{\chi^2}{2} - \alpha S$

- Shanon-Jaynes entropy:

$$S[\rho] = \int d\omega \left[\rho(\omega) - D(\omega) - \rho(\omega) \ln \frac{\rho(\omega)}{D(\omega)}\right]$$

- Default Model $D(\omega)$: parameter needed as input

[M. Jarrell, J. E. Gubernatis, Phy. Rep.269,133(1996)]
[M. Asakawa et al., Prog.Part.Nucl.Phys. 46(2001) 459-508]

# Recent developed new methods

☐ A novel Bayesian approach

New axiom: requirement of **smoothness** of the reconstructed spectra **with constant default model.**

[Y. Burnier, A. Rothkopf, Phys. Rev. Lett. 111, 182003 (2013)]

$$\Longrightarrow S = \alpha \int d\omega (1 - \frac{\rho}{m} + \ln \frac{\rho}{m})$$

☐ The Backus-Gilbert method(BGM)

Manipulate in the **local vicinity** of some $\omega$ in a **model independent** way.

[B. B. Brandt, A. Francis, B. Jaeger, H. B. Meyer, Phys. Rev. D 93, 054510 (2016)]

Filtered spectral function:

$$\hat{\rho}(\omega) = f(\omega/T) \cdot \sum g_i(\omega, [W^{-1}]^{reg}, R) \cdot G(\tau_i)$$

$$[W^{-1}]_{ij}^{reg}(\omega) = \lambda W_{ij}(\omega) + (1-\lambda) Cov_{ij}, \qquad 0 \le \lambda \le 1$$

Contribution of local vicinity enters !

$$W_{ij} = \int d\omega' K(\tau_i, \omega') K(\tau_j, \omega') (\omega - \omega')^2$$

# Stochastic Analytic Inference(SAI)

- A stochastic method based on Bayesian theorem :
  $\langle n \rangle = \int d\alpha \, \langle n \rangle_\alpha \, P[\alpha|\bar{G}]$
- $n(x) = \frac{\rho(\omega)}{D(\omega)}, \qquad x \equiv \phi(\omega) = \int_0^\omega D(\nu) d\nu$ [H. Ohno, PoS(LATTICE 2015)175]

Goal: find the distribution of $P[\alpha|\bar{G}]$

- Field treatment of $n(x)$ gives:
  $\langle n \rangle_\alpha = \int \mathcal{D}n \, n \, \boldsymbol{P[n|\alpha, \bar{G}]} = \int \mathcal{D}n \, n \frac{\mathbf{1}}{\boldsymbol{Z(\alpha)}} \boldsymbol{e^{-\chi^2/2\alpha}}$
- Posterior probability: $P[n|\alpha, \bar{G}] = \frac{1}{P[\bar{G}|\alpha]} P[\bar{G}|\alpha, n] P[n|\alpha]$

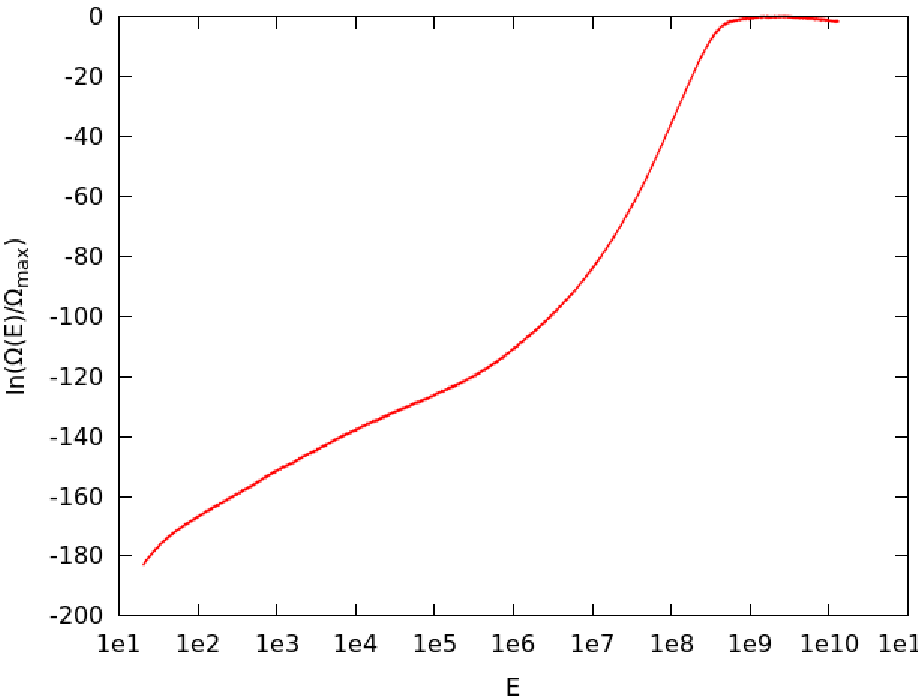
$$\Longrightarrow \begin{cases} P[\bar{G}|\alpha, n] = \frac{1}{Z'} e^{-\chi^2[n]/2\alpha}, \text{ likelihood function} \\ P[n|\alpha] = \Theta(n)\delta\left(\int_0^{x_{max}} dx \, n(x) - 1\right), \text{ prior probability} \\ P[\bar{G}|\alpha] = Z/Z', \qquad \text{normalization factor} \end{cases}$$

- $P[\alpha|\bar{G}] = \frac{P[\alpha]}{P[\bar{G}]} \int \mathcal{D}n \; P[\bar{G}|\alpha, n] P[n|\alpha] \sim P[\alpha] \alpha^{-\frac{N}{2}} Z(\alpha)$

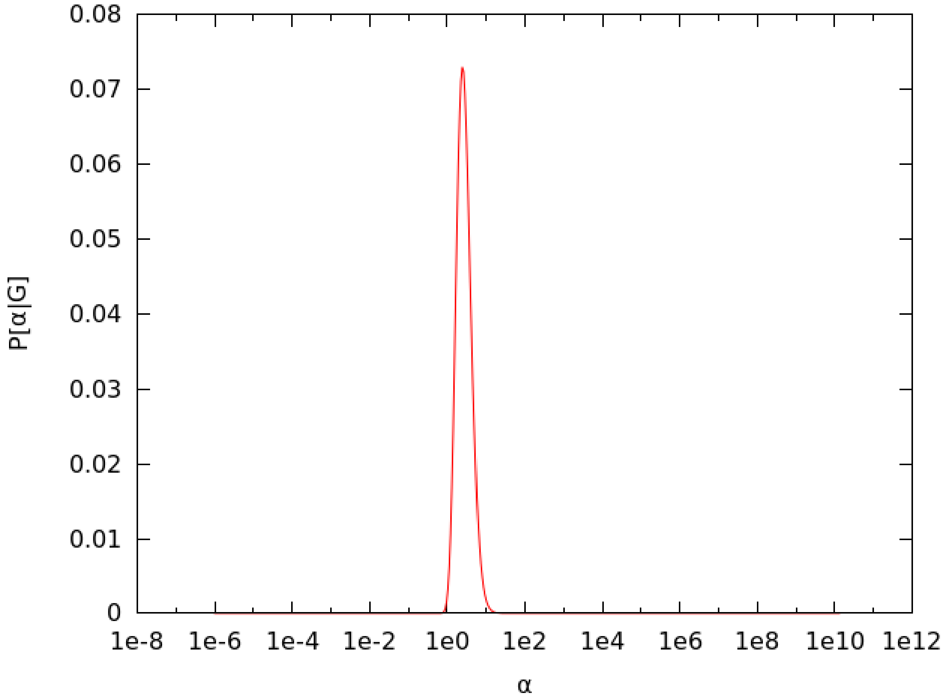
# Stochastic Analytic Inference

- Density of States(DoS): $\Omega(E) = \int \mathcal{D}n\, \delta(\chi^2[n]/2 - E)$
- $P[\alpha|\bar{G}] = P[\alpha]\alpha^{-\frac{N}{2}} \int dE\Omega(E)e^{-E/\alpha}$

F.-G. Wang, D.P. Landau arXiv:cond-mat/0107006




Density of States.




$P[\alpha|\bar{G}]$.

# SAI v.s. MEM

| | SAI | MEM |
|---|---|---|
| Average | $\langle n\rangle = \int d\alpha\, P[\alpha\|\bar{G}]\langle n\rangle_\alpha$ | $\langle \rho\rangle \approx \int d\alpha\, P[\alpha\|\bar{G}]\hat{\rho}_\alpha$ |
| Likelihood function | $P[\bar{G}\|\alpha, n] \sim e^{-\chi^2[n]/2\alpha}$ | $P[\bar{G}\|\alpha, \rho] \sim e^{-\chi^2[\rho]/2}$ |
| Prior probability | $P[n\|\alpha] = \Theta(n)\delta(\sum n(x) - 1)$ | $P[\rho\|\alpha] \sim \exp(\alpha S[\rho])$ |
| $P[\alpha\|\bar{G}]$ | Integrate $\int \mathcal{D}n$ | Minimize $\mathrm{F} = \frac{\chi^2}{2} - \alpha S$ |

# SAI to MEM

- Mean-field treatment in SAI:

$$
\begin{aligned}
S_{SAI}[n] &\equiv \int \mathcal{D}n \ln \Omega(n) \\
&\approx \ln \Omega(\bar{n}) \\
&= -\int_0^1 dx\, \bar{n}(x) \ln \bar{n}(x) \\
&= -\int d\phi \frac{\bar{\rho}}{D} \ln \frac{\bar{\rho}}{D} \\
&= S_{MEM}[\bar{\rho}]
\end{aligned}
$$

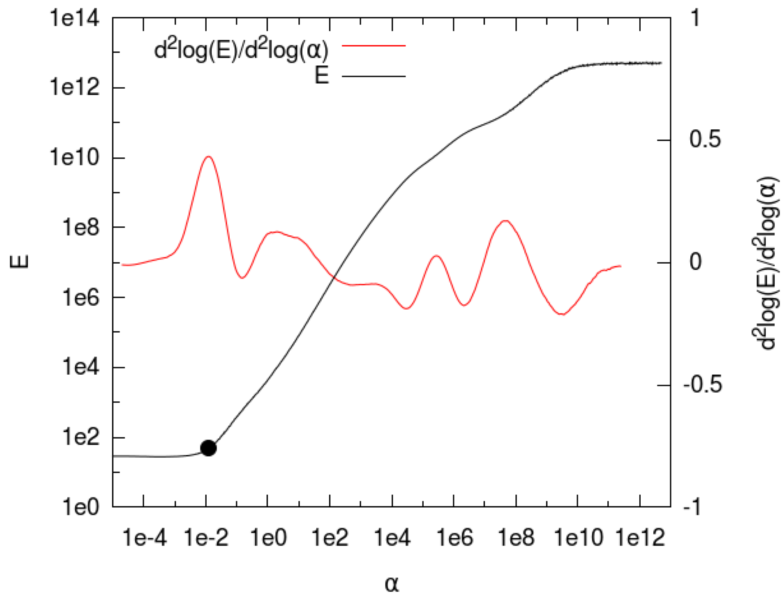
- SAI reduces to MEM at the **mean-field level** !

# Stochastic Optimization Method(SOM)

[H.-T. Shu, H.-T. Ding, O. Kaczmarek, S. Mukherjee, H. Ohno, PoS(LATTICE 2015)180 ]

- Based on Central Limit Theorem.
- **No prior information** is needed.
  All information comes from correlators:
  $E = \chi^2[\rho]/2$ (Fictitious energy)
- Field treatment of $\rho$. Evolves with fictitious temperature $\alpha$.
- Possible solution obtained when **phase transition** occurs.

[K.S.D. Beach arXiv:cond-mat/0403055]




Phase transition occurs at the *kink(black spot)*.

## SAI to SOM

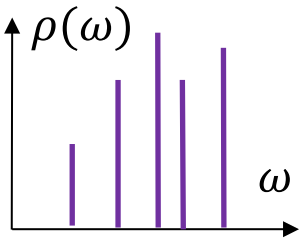
- SAI reduces to SOM when using **constant default model**!
  $x = \omega, n(x) = \rho(\omega)$

# Basis of Stochastic approaches

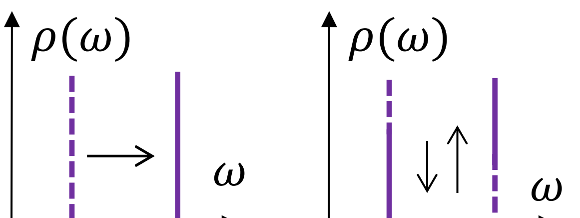
## SAI

- Ingredients: $\delta$ functions

$$\rho(\omega) = \sum_t^K r_i \delta(\omega - a_i),$$




- Elementary updates




1.Shift 2. Change residues

- Normalization

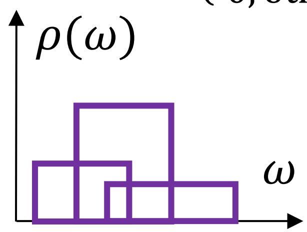
$$\sum r_i = \bar{G}(\tau_0)$$

## SOM

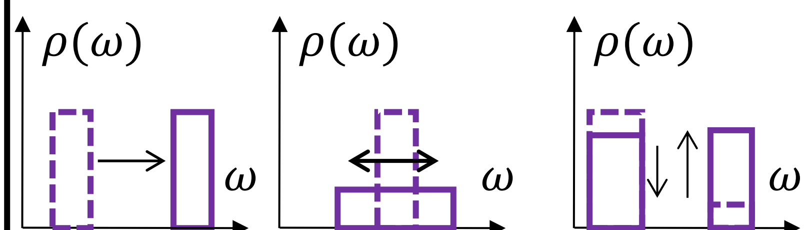
- Ingredients: boxes

$$\rho(\omega) = \sum_t^K \eta\,(P_t)(\omega),$$

$$\eta(P_t)(\omega) = \begin{cases} h_t, \omega \in [c_t - w_t/2, c_t + w_t/2,] \\ 0, \text{otherwise} \end{cases}$$




- Elementary updates

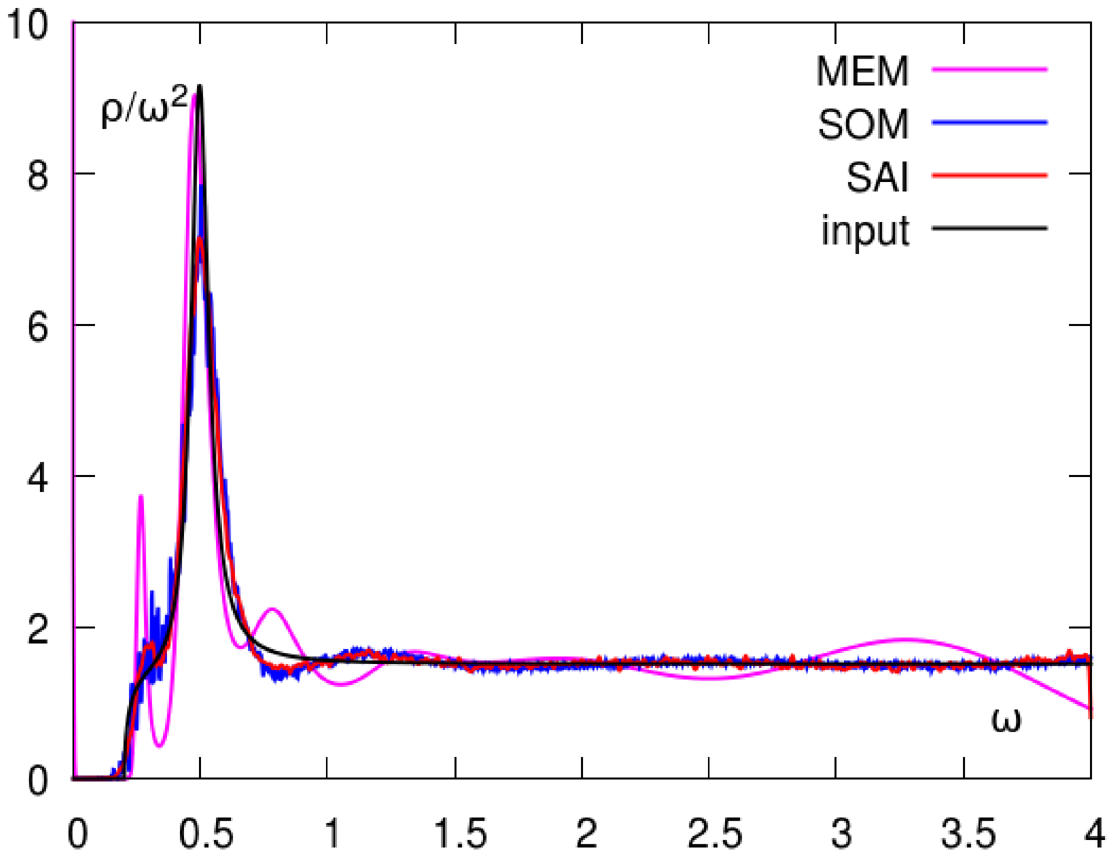



1.Shift 2. Change width 3. Change heights

- Normalization

$$\sum h_i w_i = \bar{G}(\tau_0)$$

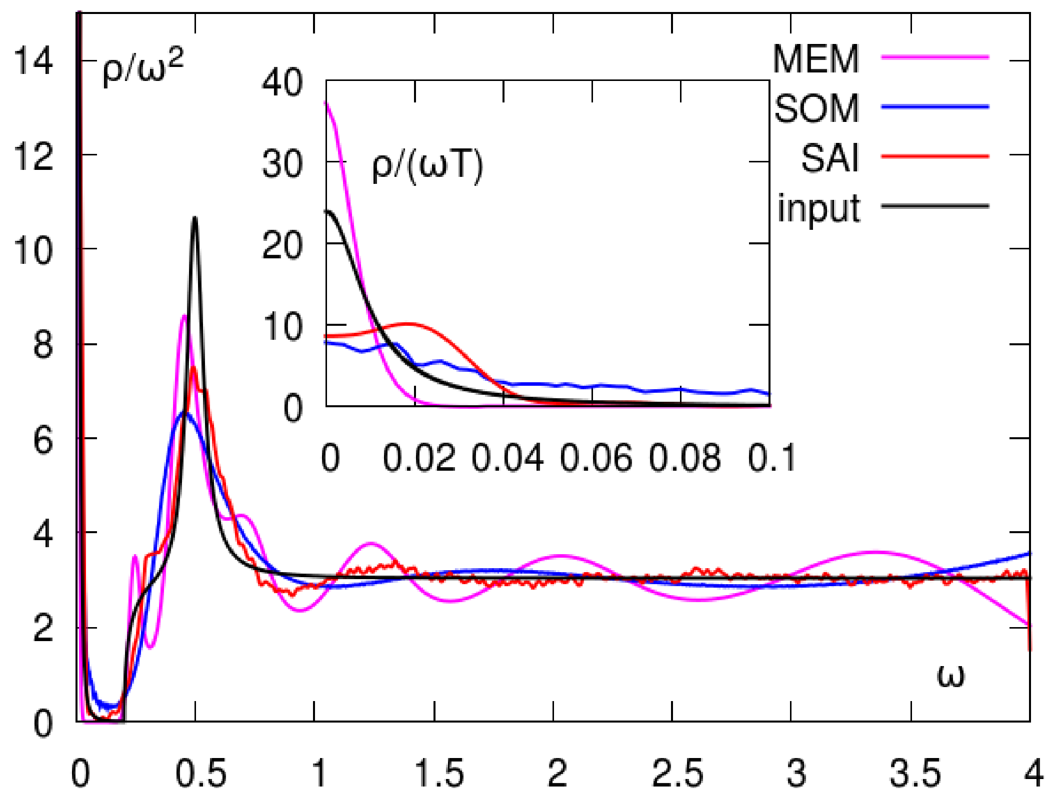
# Mock data test:Different spectral functions




$\rho(\omega) = \rho_{res} + \rho_{cont}$,
corresponding to $T < T_c$
Error in mock data:
$\sigma(\tau) = \epsilon G(\tau)\tau$
$N_\tau = 48, \epsilon = 10^{-4}$

1. MEM gives fake resonance peaks.
2. SOM gives similar results to SAI with constant DM.
3. SAI&SOM reconstruct the input well.

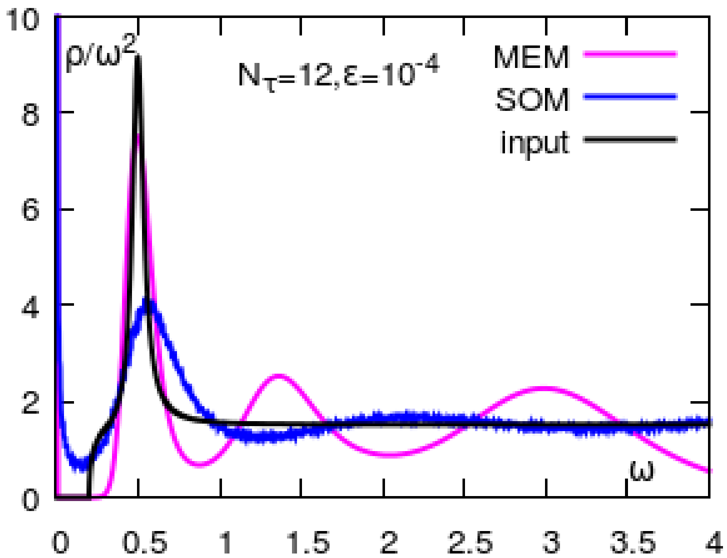
# Mock data test: Different spectral functions

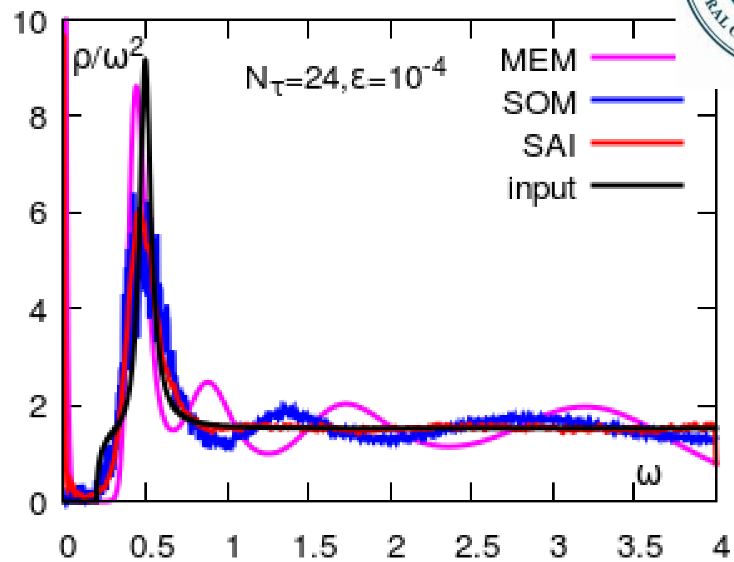


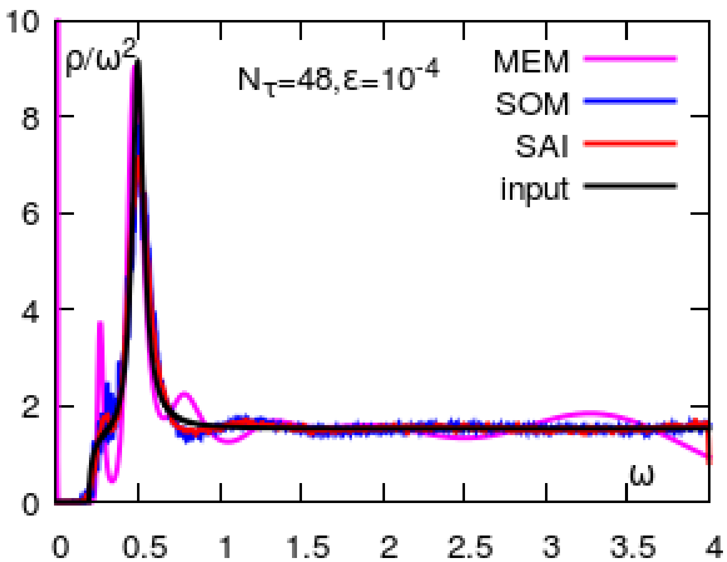

$\rho(\omega) = \rho_{trans} + \rho_{res} + \rho_{cont}$,
corresponding to $T > T_c$
$N_\tau = 48, \epsilon = 10^{-4}$

1. MEM&SAI with constant DM and SOM can not reconstruct the transport peak precisely.
2. MEM&SAI&SOM reconstruct the resonance peak, but the width and peak-location differ from the input.
3. MEM generates waggles in continuum part.
4. SAI&SOM can reconstruct the continuum part well.

# Mock data test: Dependence on $N_\tau$

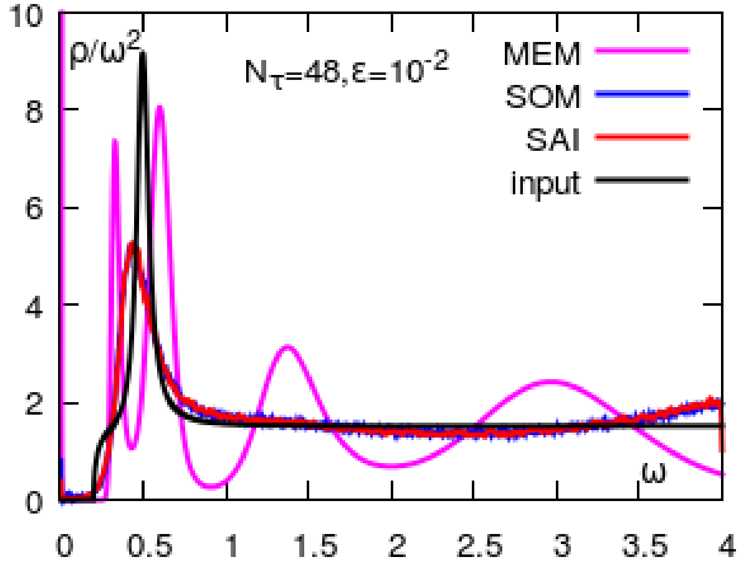


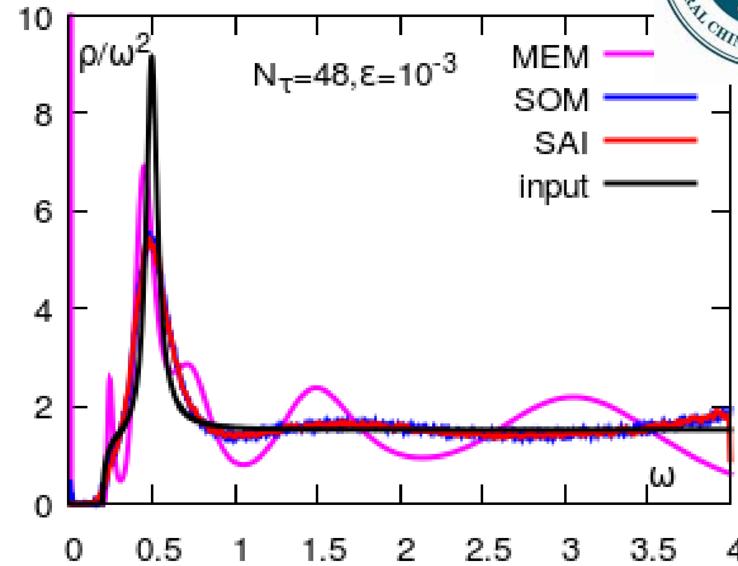



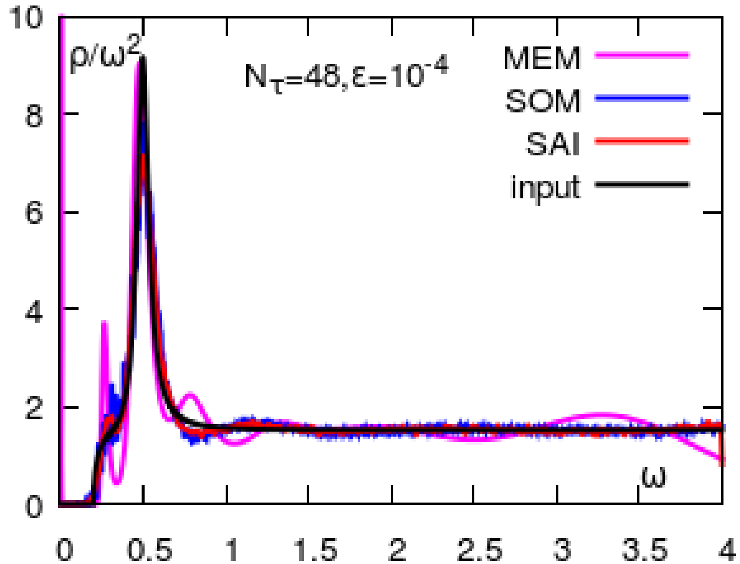




1. MEM reconstructs the resonance peak well even at small $N_\tau$.
2. SAI&SOM give fake transport peak at small $N_\tau$ .
3. SAI&SOM reconstruct the continuum part well.
4. As $N_\tau$ increases, the output approaches to the input for all methods.

# Mock data test:Dependence on noise level



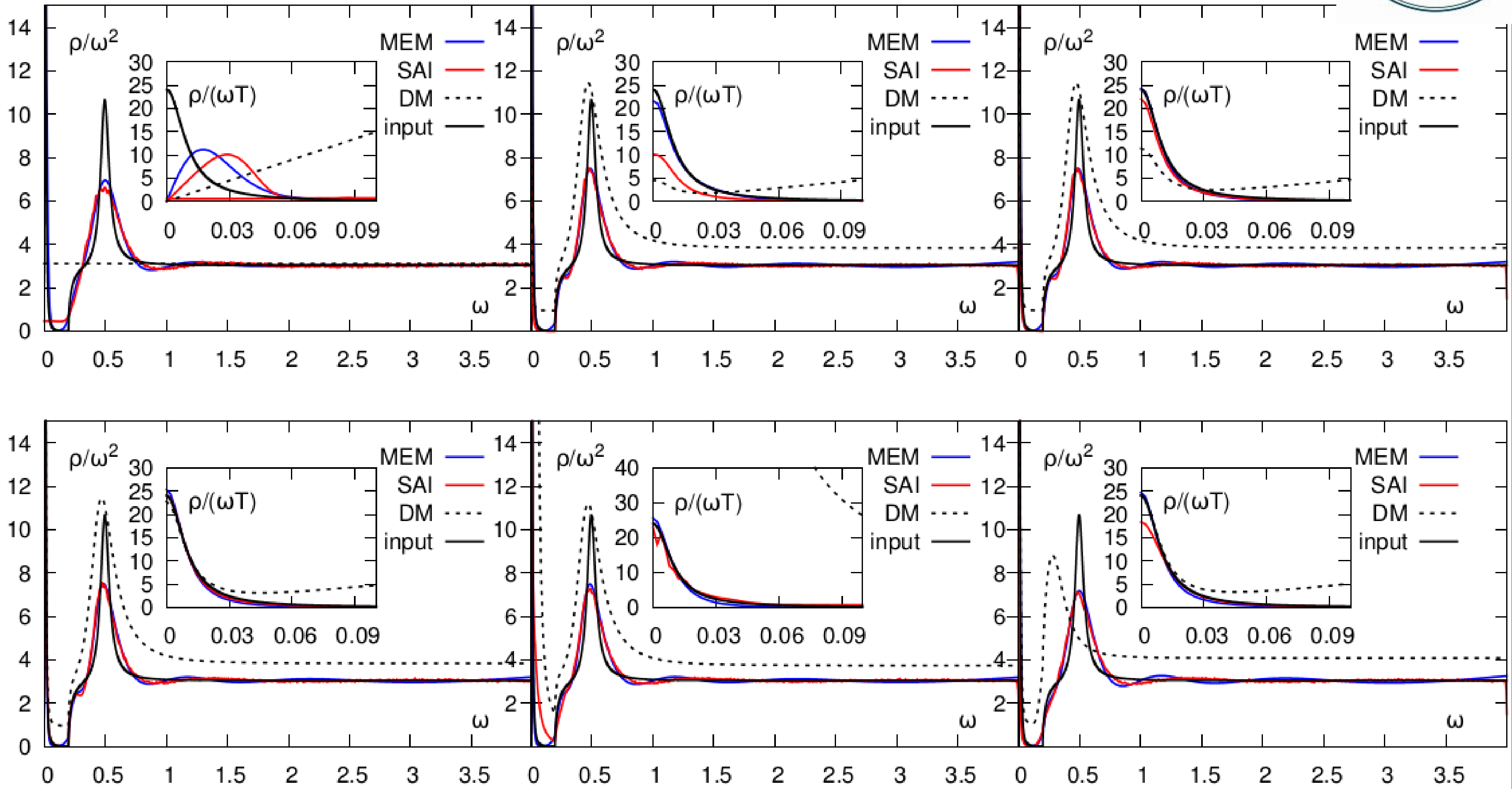




1. MEM gives bad output with noisy data.
2. SAI&SOM give a rough resonance peak with noisy data.
3. SAI&SOM reconstruct the continuum part well.
4. As noise becomes weak, the output approaches to the input for all methods.

# Mock data test: Dependence on $DM$

$N_\tau = 48, \epsilon = 10^{-4}$ for all.




1. Dependence is weak for resonance peak and continuum.
2. There is dependence for transport peak. But upper bound exists in this case.

# Summary & Outlook

- SOM&SAI are introduced to study the uncertainties of spectral functions besides MEM.
- SAI is a generalization of MEM and reduces to MEM in mean field limit. SAI reduces to SOM when using constant default model.
- From mock data tests, we found SAI&SOM work well for resonance peaks and continuum parts.
- For small transport peak, we need check the dependence on DM carefully.
- Apply SOM&SAI into real lattice data to investigate in-medium hadron properties.

  [**Dr. Hiroshi OHNO** on **28 Jul 2016** at **15:00**]

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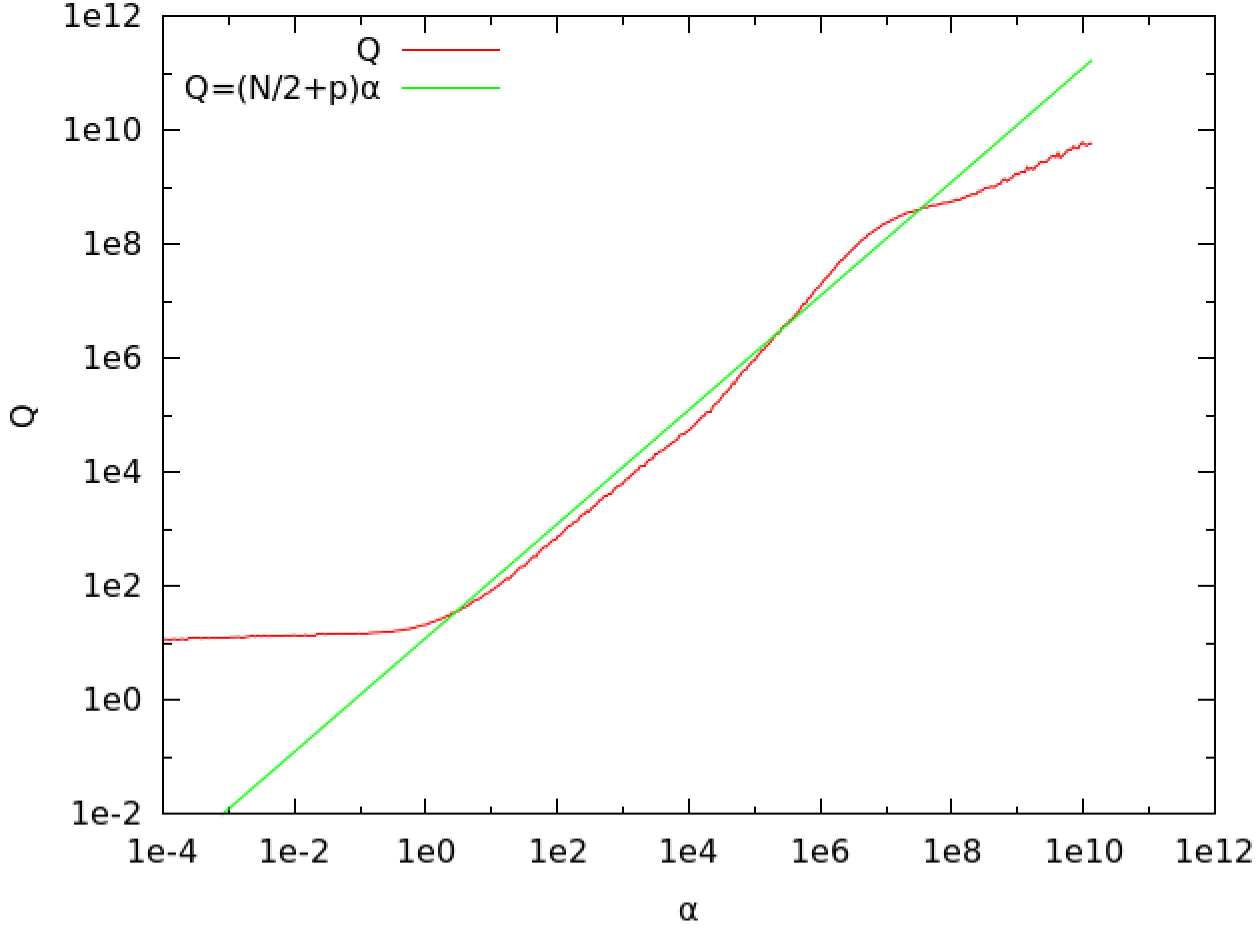

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## Mock SPFs

- Resonance peak : $\rho_{res} = c_{res} \frac{\Gamma(\omega,\omega_0,\gamma_0)M}{(\omega^2-M^2)^2+M^2\Gamma^2(\omega,\omega_0,\gamma_0)} \frac{\omega^2}{\pi}$

  Where $\Gamma(\omega,\omega_0,\gamma_0) = \theta(\omega-\omega_0)\gamma_0(1-\frac{\omega^2{}_0}{\omega^2})^5$

- Transport peak:

  $$\rho_{trans} = c_{trans} \frac{\eta\omega}{(\omega^2-\eta^2)^2}$$

- Free continuum:

  $$\rho_{cont} = c_{cont} \frac{N_c}{8\pi^2}\Theta(\omega^2-4m^2)\omega^2 \tanh\left(\frac{\omega}{4T}\right) \times \sqrt{1-\left(\frac{2m}{\omega}\right)^2}\left[a^1+a^2\left(\frac{2m}{\omega}\right)^2\right]$$

- Free Wilson:

  $$\rho_{Wilson} = c_{Wilson} \frac{N}{L^3}\sum_k \sinh\left(\frac{\omega}{2T}\right)\left[b^1-b^2\frac{(\sum_{i=1}^3 \sin^2 k_i)}{\sinh^2 E_k(m)}\right]$$

  Where $\cosh E_k(m) = 1+\frac{K_k^2+M_k^2(m)}{2(1+M_k(m))}$, $K_k = \sum_{i=1}^3 \gamma_i \sinh k_i$

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## Mock SPFs parameters

| Elements in SPF | Parameters |
|---|---|
| $\rho^1_{res}$ | $c_{res} = 1, \omega_0 = 0.2, \gamma_0 = 0.20, M = 0.5$ |
| $\rho^2_{res}$ | $c_{res} = 4, \omega_0 = 0.2, \gamma_0 = 0.25, M = 1.2$ |
| $\rho^3_{res}$ | $c_{res} = 6, \omega_0 = 0.2, \gamma_0 = 0.20, M = 2.5$ |
| $\rho_{trans}$ | $\mathrm{c}_{trans} = 0.2, \eta = 0.01$ |
| $\rho_{cont}$ | $c_{cont} = 20, a^1 = 2, a^2 = 1, m = 0.1$ |
| $\rho_{Wilson}$ | $c_{Wilson} = 1, b^1 = 3, b^2 = 1, m = 0.112$ |

$\mathrm{N}_\tau = 48, a = 1, \tau_{min} = 1, \omega \in [0,4]$.

Error in mock data: $\sigma(\tau) = \epsilon G(\tau)\tau$