Towards a determination of the ratio of the kaon to pion decay constants

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– QCDSF-UKQCD Collaboration –

Vladivostok – Edinburgh – RIKEN (Kobe) – Leipzig – FZ (Jülich) – Liverpool – DESY – Hamburg – Adelaide

Lattice 2016, Southampton, UK

Thursday 28/7/16 15:00 (67 1003)

QCDSF related talks with $2+1$ flavours:

- James Zanotti — Monday 25/7/16 15:35 (B2a 2077)
  Transverse spin densities of octet baryons
- Alexander Chambers — Monday 25/7/16 17:30 (B2a 2077)
  Hadron structure from the Feynman-Hellmann theorem
- Gerrit Schierholz — Monday 25/7/16 18:25 (67 1007)
  Running coupling from Wilson flow for three quark flavors
- Holger Perlt — Tuesday 26/7/16 17:50 (B2a 2077)
  Partially conserved axial vector current and applications
- Paul Rakow — Wednesday 27/7/16 9:20 (67 1003)
  Finite size and infra-red effects in QCD plus QED
- Ross Young — Wednesday 27/7/16 9:40 (67 1003)
  Infrared features of dynamical QED+QCD simulations

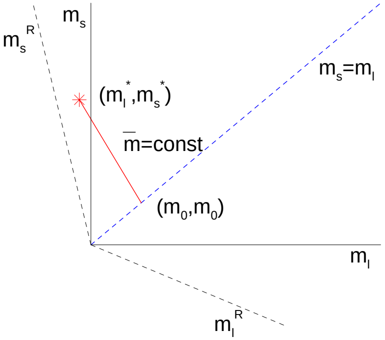
Introduction

- strategy
- $SU(3)$ flavour symmetry breaking expansions
- determination of coefficients and tuning
- results

QCDSF strategy:

[arXiv:1102.5300]

$2+1$ simulations: many paths to approach the physical point [$m_u = m_d \equiv m_l$ case]



QCDSF: extrapolate from a point on the $SU(3)_F$ flavour symmetry line to the physical point

$$(m_0, m_0) \longrightarrow (m_l^*, m_s^*)$$

Choice here: keep the singlet quark mass $\overline{m}$ constant

$$\overline{m} = m_0 = \frac{1}{3}\,(2m_l + m_s)$$

QCDSF strategy

[arXiv:1102.5300]

- develop $SU(3)_F$ flavour symmetry breaking expansion for hadron masses
- expansion in: $SU(3)$ flavour symmetric point $\delta m_q = 0$

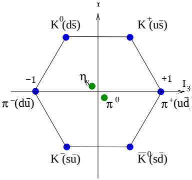
$$\delta m_q = m_q - \overline{m}\,, \quad \overline{m} = \tfrac{1}{3}(m_u + m_d + m_s) = m_0$$

  - expansion coefficients are functions of $\overline{m}$
  - trivial constraint

$$\delta m_u + \delta m_d + \delta m_s = 0$$

- path called 'unitary line' as expand in both sea and valence quarks

$SU(3)$ flavour symmetry breaking expansions



- octet pseudoscalar meson masses:

$$
\begin{aligned}
M^2(a\overline{b}) \quad = \quad & M^2_{0\pi} + \alpha(\delta m_a + \delta m_b) \\
& + \beta_0 \tfrac{1}{6}(\delta m_u^2 + \delta m_d^2 + \delta m_s^2) \\
& + \beta_1(\delta m_a^2 + \delta m_b^2) + \beta_2(\delta m_a - \delta m_b)^2 \\
& + \ldots \qquad [a, b = u, d, s \text{ (outer ring)}]
\end{aligned}
$$

- octet pseudoscalar meson decay constants:

$$
\begin{aligned}
f(a\overline{b}) \quad = \quad & F_{0\pi} + G(\delta m_a + \delta m_b) \\
& + H_0 \tfrac{1}{6}(\delta m_u^2 + \delta m_d^2 + \delta m_s^2) \\
& + H_1(\delta m_a^2 + \delta m_b^2) + H_2(\delta m_a - \delta m_b)^2 \\
& + \ldots \qquad [a, b = u, d, s \text{ (outer ring)}]
\end{aligned}
$$

- octet baryons – equivalent expansions

Another useful ingredient:

- Consider a flavour singlet quantity

$$X_S(m_u, m_d, m_s)$$

- Simple property:

$$X_S(\overline{m} + \delta m_u, \overline{m} + \delta m_d, \overline{m} + \delta m_s) = X_S(\overline{m}, \overline{m}, \overline{m}) + O((\delta m_q)^2)$$

- Already encoded in the $SU(3)$ flavour symmetric breaking expansions (together with $\delta m_u + \delta m_d + \delta m_s = 0$)
- More general:
  X_S (a flavour singlet quantity) has a stationary point about the $SU(3)$ flavour symmetric line
  - X_S invariant under u, d, s permutations (by definition)
  - Expand X_S about a point on the $SU(3)$-flavour line

$$X_S(\overline{m} + \delta m_u, \overline{m} + \delta m_d, \overline{m} + \delta m_s) = X_S(\overline{m}, \overline{m}, \overline{m}) + \left.\frac{\partial X_S}{\partial m_u}\right|_0 \delta m_u + \left.\frac{\partial X_S}{\partial m_d}\right|_0 \delta m_d + \left.\frac{\partial X_S}{\partial m_s}\right|_0 \delta m_s + O((\delta m_q)^2)$$

  - On the symmetric line:

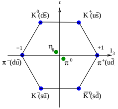
$$\left.\frac{\partial X_S}{\partial m_u}\right|_0 = \left.\frac{\partial X_S}{\partial m_d}\right|_0 = \left.\frac{\partial X_S}{\partial m_s}\right|_0$$

  together with $\delta m_u + \delta m_d + \delta m_s = 0$ implies the result

Singlet quantities – many possibilities

- Pseudoscalar mesons: (centre of mass)

$$X_\pi^2 = \tfrac{1}{6}\left(M_{K^+}^2 + M_{K^0}^2 + M_{\pi^+}^2 + M_{\pi^-}^2 + M_{\overline{K}^0}^2 + M_{K^-}^2\right)$$
$$\sim (410\,\mathrm{MeV})^2$$



- Pseudoscalar decay constants: (centre of mass)

stable under QCD

$$X_{f_\pi} = \tfrac{1}{6}\left(f_{K^+} + f_{K^0} + f_{\pi^+} + f_{\pi^-} + f_{\overline{K}^0} + f_{K^-}\right)$$

- Many other possibilities

$$X_S^2 = \begin{cases} \frac{1}{6}(M_p^2 + M_n^2 + M_{\Sigma^+}^2 + M_{\Sigma^-}^2 + M_{\Xi^0}^2 + M_{\Xi^-}^2) & S = N & \text{baryon octet} \\ \frac{1}{2}(M_\Sigma^2 + M_\Lambda^2) & S = \Lambda & \text{baryon octet} \\ M_{\Sigma^*}^2\,,\ \frac{1}{2}(M_\Delta^2 + M_{\Xi^*}^2) & S = \Sigma^*\,,\ \Delta & \text{baryon decuplet, unstable under QCD} \\ \frac{1}{6}(M_{K^{*+}}^2 + M_{K^{*0}}^2 + M_{\rho^+}^2 + M_{\rho^-}^2 + M_{\overline{K}^{*0}}^2 + M_{K^{*-}}^2) & S = \rho & \text{vector octet} \\ 1/r_0^2\,,\ 1/t_0\,,\ 1/w_0^2 & S = r_0\,,\ t_0\,,\ w_0 & \text{force, Wilson flow scales} \end{cases}$$

Lattice

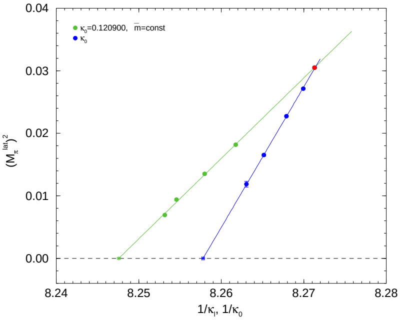
- $O(a)$ NP improved clover action
  - tree level Symanzik glue
  - mildly stout smeared $2+1$ clover fermion
  - $\beta = 5.40, 5.50, 5.65, 5.80$ $[24^3 \times 48,\ 32^3 \times 64,\ 48^3 \times 96]$

- $$m_q = \frac{1}{2}\left(\frac{1}{\kappa_q} - \frac{1}{\kappa_{0c}}\right)$$

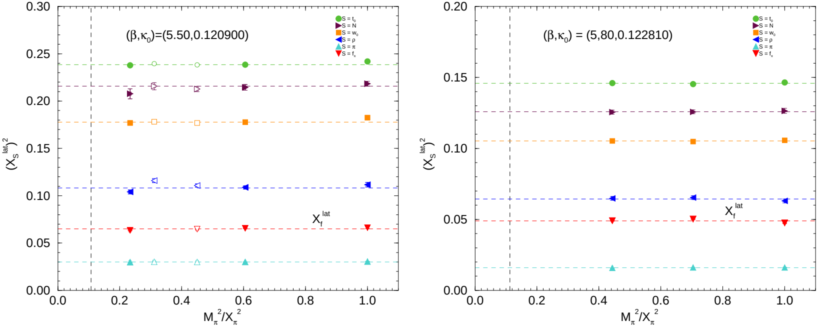
  κ_{0c} is chiral limit along symmetric line

- $$\delta m_q = m_q - m_0 = \frac{1}{2}\left(\frac{1}{\kappa_q} - \frac{1}{\kappa_0}\right)$$

- typical $M_\pi^{lat\,2}$ values



X_S^2 determination:



- $(\beta, \kappa_0) = (5.50, 0.120900), (5.80, 0.122810)$
- $X_{t_0}^2$, $X_{w_0}^2$, X_π^2, X_ρ^2, $X_N^2 \approx X_\Lambda^2$, X_{f_π} along the $\overline{m} = \text{const.}$ line

[in LH plot M_π $\sim$ 430 MeV - 280 MeV; RH $\sim$ 465 MeV - 220 MeV]

Alternatively:

- we have

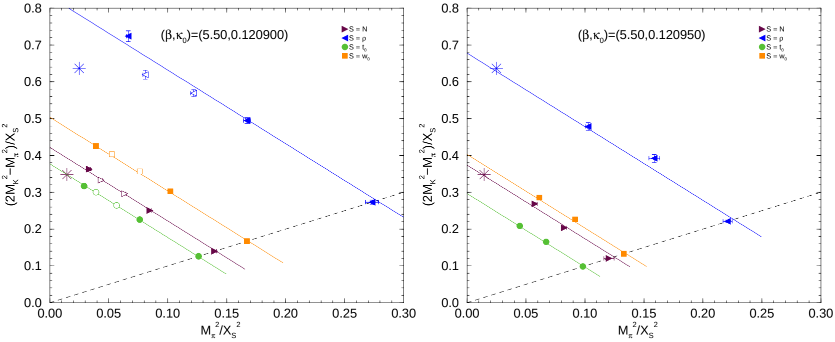
$$\frac{X_\pi^2}{X_s^2} = \frac{(2M_K^2 + M_\pi^2)/3}{X_s^2}$$

- giving

$$\frac{2M_K^2 - M_\pi^2}{X_S^2} = 3\underbrace{\frac{X_\pi^2}{X_s^2}}_{\text{const}} - \underbrace{2}_{\text{const}}\frac{M_\pi^2}{X_s^2}$$

for $S = N, \rho, t_0, w_0, \ldots$

Path in quark mass plane

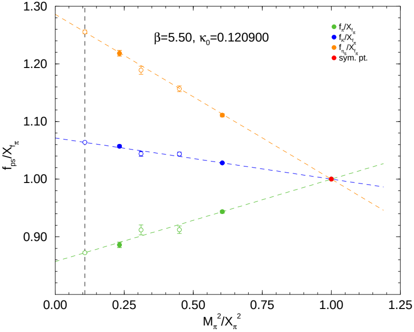


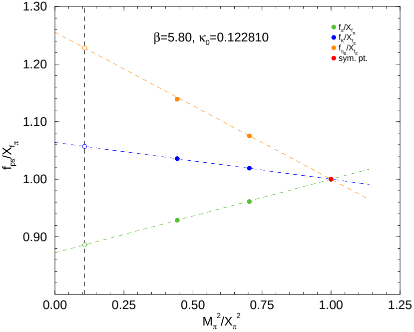
- $S = N, \rho, t_0, w_0, \ldots$
- $(\beta, \kappa_0) = (5.50, 0.120900) \rightarrow (5.50, 0.120950 \approx \kappa_0^*)$

Main observations:

- X_S appears constant over a large range from the $SU(3)$ flavour symmetric line
- Paths from a point on the $SU(3)$ flavour symmetric line are linear
- Need to find this point: m_0 (ie κ_0)

Unitary line





- linear behaviour in $SU(3)$ flavour symmetry breaking expansions

Programme

- Determine κ_0
- Determine expansion coefficients: $\alpha, \ldots$
  [function of $\overline{m} \equiv m_0$ only]
- Use $M_\pi^{*\,2}$, $M_K^{*\,2}$ to determine δm_l^*, δm_s^*
- Estimate (in this talk) decay constants at physical point

Partially quenching

- Unitary range rather small so introduce PQ partially quenching (ie valence quark masses $\neq$ sea quark masses) and (N)NLO
- Furthermore can generalise to different valence quark masses, μ_q to sea quark masses m_q without increasing number of expansion coefficients

$$\delta\mu_q = \mu_q - \overline{m}$$

- pseudoscalar meson octet

Useful: $\tilde{M}^2 = M^2 / X_\pi^2$; $\tilde{\alpha}(\overline{m}) = \alpha / M_{0\pi}^2, \ldots$

$$\begin{aligned}\tilde{M}^2(a\overline{b}) &= 1 + \tilde{\alpha}(\delta\mu_a + \delta\mu_b) \\ &\quad -(\tfrac{2}{3}\tilde{\beta}_1 + \tilde{\beta}_2)(\delta m_u^2 + \delta m_d^2 + \delta m_s^2) + \tilde{\beta}_1(\delta\mu_a^2 + \delta\mu_b^2) + \tilde{\beta}_2(\delta\mu_a - \delta\mu_b)^2\end{aligned}$$

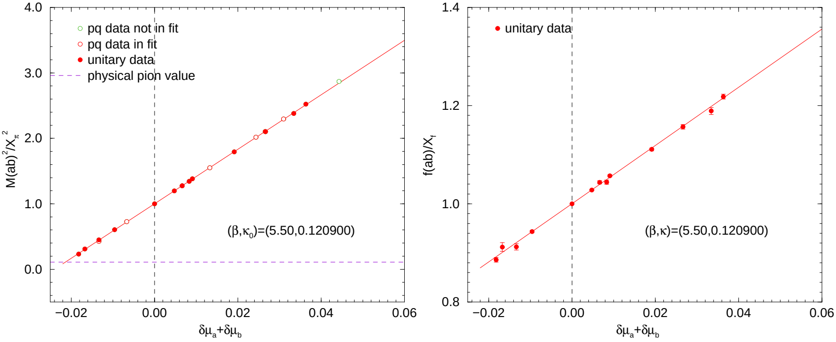
- pseudoscalar decay constants

Useful: $\tilde{f} = f / X_{f_\pi}$; $\tilde{G}(\overline{m}) = G / F_{0\pi}^2, \ldots$

$$\begin{aligned}\tilde{f}_{\mathrm{ps}}(a\overline{b}) &= 1 + \tilde{G}(\delta\mu_a + \delta\mu_b) \\ &\quad -(\tfrac{2}{3}\tilde{H}_1 + \tilde{H}_2)(\delta m_u^2 + \delta m_d^2 + \delta m_s^2) + \tilde{H}_1(\delta\mu_a^2 + \delta\mu_b^2) + \tilde{H}_2(\delta\mu_a - \delta\mu_b)^2\end{aligned}$$

- mixed sea/valence mass terms
- unitary limit: $\delta\mu_q \to \delta m_q$

PQ results



- $(\beta, \kappa_0) = (5.50, 0.120900)$
- Linear behaviour for $M_\pi \lesssim \sqrt{3} \times 410\,\text{MeV} \sim 700\,\text{MeV}$

κ_0 (fine) tuning

- If miss (slightly) starting point on $SU(3)$ flavour symmetric line
- Tune using PQ results so that get physical values of (say)

$$M_\pi^* \qquad X_N^* \qquad M_K^*$$

correct

- Gives

$$\kappa_0 \qquad \delta\mu_l^* \qquad \delta\mu_s^*$$

- Philosophy: most change is due to change in valence quark mass, rather than sea quark mass $\delta\overline{\mu} \equiv (2\delta\mu_l^* + \delta\mu_s^*)/3 \neq 0$ necessarily
- eg $(\beta, \kappa_0) = (5.50, 0.120950)$, $\delta\overline{\mu} = 0.00007$, $\kappa_0^{\mathrm{val}} = 0.120948$

The axial current I

[Bhattacharya et al., hep-lat/0511014]

- The renormalised and $O(a)$ improved axial current is given by

$$\mathcal{A}_\mu^{ab;R} = Z_A \mathcal{A}_\mu^{ab;IMP}$$

with

$$\mathcal{A}_\mu^{ab;IMP} = \left(1 + \mathrm{a}\left[\overline{b}_A \overline{m} + \tfrac{1}{2} b_A (m_a + m_b)\right]\right) \mathcal{A}_\mu^{ab} \qquad \mathcal{A}_\mu^{ab} = A_\mu^{ab} + \mathrm{a} c_A \partial_\mu P^{ab}$$

and

$$A_\mu^{ab} = \overline{q}_a \gamma_\mu \gamma_5 q_b\,, \qquad P^{ab} = \overline{q}_a \gamma_5 q_b$$

- Matrix elements

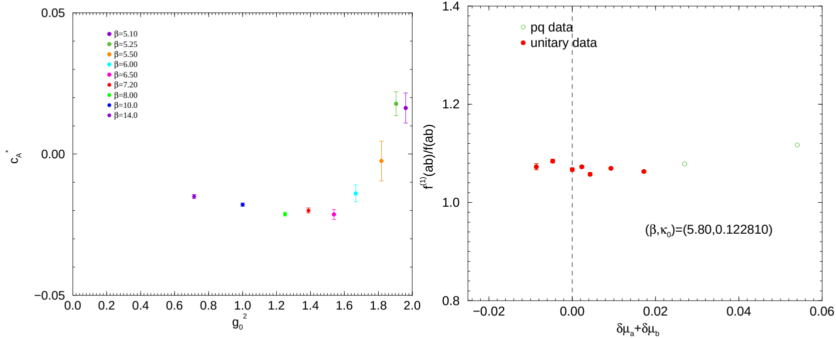
$$\langle 0 | A_4^{ab} | M \rangle = M\, f \qquad \langle 0 | \partial_4 P^{ab} | M \rangle = M\, f^{(1)}$$

giving

$$f^R = Z_A \left(1 + \mathrm{a} c_A \frac{f^{(1)}}{f}\right) \left(1 + \mathrm{a}\left((\overline{b}_A + b_A)\overline{m} + \tfrac{1}{2} b_A (\delta m_a + \delta m_b)\right)\right) f$$

The axial current II

- c_A small, $f^{(1)}/f$ constant

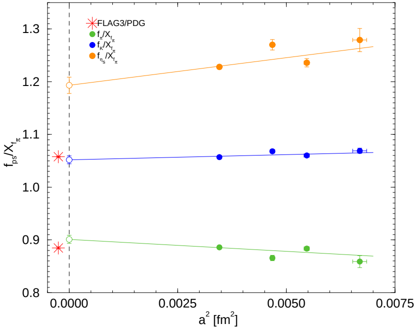


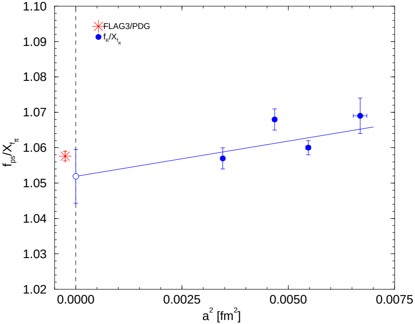
- For constant $\overline{m}$ absorb some terms to give

$$\tilde{f}^R \equiv \frac{f^R}{X_f^R} = 1 + \left(\tilde{G} + \tfrac{1}{2} a b_A\right)(\delta m_a + \delta m_b) + \dots$$

presently take $b_A = 1 + O(g_0^2)$, tree level

Unitary line





- Converting

$$\frac{f_K}{f_\pi} = 1.17(3)$$

FLAG3/PDG 1.195(3)

$SU(2)$ isospin breaking effects

- Provided $\overline{m}$ kept constant, then expansion coefficients $\alpha(\overline{m}), \ldots$ remain unaltered whether
  - $1+1+1$
  - $2+1$
- Parameterise $SU(2)$ isospin breaking effects by

$$\frac{f_{K^+}}{f_{\pi^+}} = \frac{f_K}{f_\pi}\left(1+\tfrac{1}{2}\delta_{SU(2)}\right)$$

- Expanding about average light quark mass $\delta m_l = (\delta m_u + \delta_d)/2$ gives in LO (which appears to work quite well)

$$\begin{aligned}\delta_{SU(2)} &= \frac{2}{3}\left(1-\left(\frac{f_K}{f_\pi}\right)^{-1}\right)\frac{\delta m_d - \delta m_u}{\delta m_u + \delta m_d} \\ &= \left(1-\left(\frac{f_K}{f_\pi}\right)^{-1}\right)\frac{M^2_{K^0} - M^2_{K^+}}{M^2_{\pi^+} - \frac{1}{2}\left(M^2_{K^0}+M^2_{K^+}\right)}\end{aligned}$$

- At the physical point this gives

$$\delta_{SU(2)} \approx -0.0042$$

Conclusions

- Programme:
  Tune strange and light quark masses to their physical values simultaneously by keeping

$$\overline{m} = \frac{1}{3}\left(2m_l + m_s\right) = \text{const.}$$

  starting from a point on the $SU(3)$ flavour symmetric line
- Expansion coefficients determined using both unitary and pq data
- $SU(3)$ flavour symmetry breaking expansion works well even at leading order — Gell-Mann–Okubo expansion
  - $X_S(\kappa_0)$ (singlet quantities) remain constant from $SU(3)$ flavour symmetric line
  - path to physical point linear
- Have extended formalism from pseudoscalar meson masses to pseudoscalar decay constants